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Invariance Principles for Gaussian Sequences

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An almost sure invariance principle is proved for stationary Gaussian sequences whose covariances $r(n)$ satisfy $r(n)=O(n^{-1-\varepsilon})$ for some $\varepsilon>0$.

1. Introduction

Let $\{x_\nu, \nu\geqq 1\}$ be a Gaussian sequence centered at expectations. Suppose that for some $0<\delta\leqq 1$

$$E\left(\sum_{\nu=m}^{m+n-1} x_\nu\right)^2 = n+O(n^{1-\delta}) \qquad n\geqq 1 \tag{1.1}$$

uniformly in $m=1,2,\ldots$. We also assume that for some $-1<\varepsilon\leqq 1$

$$E\{x_m x_{m+n}\} \ll n^{-1-\varepsilon} \tag{1.2}$$

uniformly in $m\geqq 1$. The main intention here is to establish an almost sure invariance principle when $\varepsilon>0$ in (1.2).

Theorem 1. *Let $\{x_\nu, \nu\geqq 1\}$ be a Gaussian sequence which is centered at expectations and satisfies* (1.1) *and* (1.2) *with $\varepsilon>0$. Then, without loss of generality, there exists a standard Brownian motion $\{X(t), t\geqq 0\}$ such that*

$$\sum_{\nu\leqq t} x_\nu - X(t) \ll t^{\frac{1}{2}-\lambda} \quad a.s. \tag{1.3}$$

where $\lambda=\min(\varepsilon,\delta)/500$.

The phrase "without loss of generality ..." is to be understood in the sense of Strassen [8]: without changing its distribution we can redefine the sequence $\{x_\nu, \nu\geqq 1\}$, say as $\{x_\nu^*, \nu\geqq 1\}$, on a new probability space on which there exists standard Brownian motion $\{X(t), t\geqq 0\}$ satisfying (1.3) with x_ν replaced by x_ν^*.

The following corollary is an immediate consequence of Theorem 1. (For details see the proof of Corollary 5.1 of Philipp and Stout [6].)

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Corollary. *Let $\{x_\nu, \nu \geqq 1\}$ be a stationary Gaussian sequence centered at expectations. Suppose*

$$E\{x_1 x_n\} \ll n^{-1-\varepsilon}$$

for some $\varepsilon > 0$. Then the conclusion of Theorem 1 holds with $\lambda = \min(1, \varepsilon)/500$ when $E x_1^2 + 2 \sum_{n \geqq 1} E x_1 x_{n+1} = 1$.

In the sense that one can still obtain an error term of the form $t^{\frac{1}{2}-\lambda}$ in (1.3), Theorem 1 improves Theorem 5.1 of [6] which amounts to the case of $\varepsilon = 1$ in (1.2). As was observed in Sect. 1 in [6] an almost sure invariance principle of the form (1.3) implies upper and lower class results, the functional law of the iterated logarithm, the functional central limit theorem, etc. Consequently the Corollary includes a recent upper and lower class result of Lai and Stout [4] (corresponding to the case $\varepsilon = \frac{1}{2}$) and the functional law of the iterated logarithm of Deo [2]. Condition (1.2) with $\varepsilon > 0$ is an a certain sense best possible. For Deo [2] gives an example when $-\frac{1}{2} < \varepsilon < 0$ of a stationary Gaussian sequence for which the finite dimensional distributions of the random elements of $C[0, 1]$ generated by polygonal interpolation of its properly normalized partial sum process converge to those of a Gaussian process which is not Brownian motion.

In fact, (1.3) does not remain valid in general when $\varepsilon = 0$. This was shown by Robert P. Kaufman using the following argument. For every nonincreasing sequence of real numbers $\{a_n\}$ converging to zero and satisfying $\sum_{n \geqq 1} a_n = \infty$ there is an even function ψ: $\mathbb{R} \to \mathbb{R}^+$ which is convex and decreasing on $(0, \infty)$ and also has the property that $\frac{1}{2} a_{2n} \leqq \psi(n) \leqq a_n$, $n \geqq 1$. For an explicit construction of such a function take

$$\psi(x) = \sup_{n \geqq 1} \left(1 - \frac{x}{n}\right) a_n$$

for $x \in [0, \infty)$. Then by a theorem of Polya $\varphi = \psi/\psi(0)$ is a characteristic function. Therefore, a stationary Gaussian process $\{x_\nu, \nu \in \mathbb{Z}\}$ exists whose covariance function is just φ restricted to the integers. Moreover, because φ is even and $\sum_{n \geqq 1} \varphi(n) = \infty$ it is easy to see that

$$\frac{1}{n} E \left(\sum_{\nu=1}^{n} x_\nu\right)^2 = \frac{1}{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \varphi(j-k) \to \infty$$

as $n \to \infty$. However, in this situation,

$$n^{-\frac{1}{2}} \sum_{\nu=1}^{n} x_\nu \overset{D}{\not\to} N(0, 1),$$

thus indicating that (1.3) is impossible. In particular, if

$$n^{-1} \ll a_n \ll n^{-1},$$

one has a stationary Gaussian sequence $\{x_\nu, \nu \geqq 1\}$ such that

$$n^{-1} \ll E\{x_1 x_{n+1}\} \ll n^{-1}$$

and

$$n \log n \ll E\left(\sum_{\nu=1}^{n} x_\nu\right)^2 \ll n \log n.$$

In the next theorem we relax the conditions of Theorem 1 and obtain (1.3) with a weaker error term.

Theorem 2. *Let $\{x_\nu, \nu \geqq 1\}$ be a Gaussian sequence centered at expectations. Suppose that for some $0 < \delta \leqq 1$*

$$E\left(\sum_{\nu=m}^{m+n-1} x_\nu\right)^2 = n + O(n(\log n)^{-\delta}) \qquad n \geqq 1$$

uniformly in $m = 1, 2, \ldots$, and for some $0 < \varepsilon \leqq 1$

$$E\{x_m x_{m+n}\} \ll n^{-1}(\log n)^{-24-\varepsilon} \tag{1.4}$$

also uniformly in $m \geqq 1$. Then without loss of generality we have (1.3) *with error term replaced by $t^{\frac{1}{2}}(\log t)^{-\lambda}$ for $\lambda = \min(\varepsilon, \delta)/50$.*

Remark. Theorem 2 *is stronger than Theorem* 1 *in as much as its conclusion still implies the functional law of the iterated logarithm and upper and lower class results.*

Let us pursue a reformulation of (1.4) in the situation where $\{x_\nu, \nu \geqq 1\}$ is a stationary Gaussian sequence with $Ex_1 = 0$. Assume

$$|E\{x_1 x_{n+1}\}| \leqq a_n, \qquad a_n \downarrow \sum_{n \geqq 1} a_n < \infty$$

and

$$\sigma^2 = Ex_1^2 + 2 \sum_{n \geqq 1} E\{x_1 x_{n+1}\} > 0.$$

An easy calculation reveals that

$$E\left(\sum_{\nu \leqq n} x_\nu\right)^2 = n\sigma^2(1 + o(1)) \qquad (n \to \infty).$$

Then by either Theorem 1 of [4] or Satz 2 of [5] one readily verifies the upper half of the law of the iterated logarithm, i.e.

$$\limsup_{n \to \infty} (2n \log\log n)^{-\frac{1}{2}} \left|\sum_{\nu \leqq n} x_\nu\right| \leqq \sigma \quad \text{a.s.}$$

This observation supports my conjecture that under the above hypotheses (1.3) holds with an error term $o((t\log\log t)^{\frac{1}{2}})$.

The proofs of Theorems 1 and 2 are similar. We prove Theorem 1 in detail and then in Sect. 5 sketch the proof of Theorem 2.

The proof of Theorem 1 is based on a recent theorem of Berkes and Philipp [1]. This is in contrast to the proof of Theorem 5.1 of [6] which was implemented by martingale approximation and Skorohod embedding. Theorem 1 is proved in Sects. 2–4. In Sect. 2 we introduce the blocks, in Sect. 3 we estimate λ_k as defined by (3.1) below and in Sect. 4 the proof of Theorem 1 is completed via an application of Theorem 1 of [1].

2. Introduction of the Blocks

As is typical in proofs of limit theorems for weakly dependent random variables we consider large blocks H_k, $k\geqq 1$ and small blocks I_k, $k\geqq 1$ of consecutive integers. The blocks are ordered as $H_1, I_1, H_2, I_2, \ldots$ and, as sets, constitute a partition of the positive integers. Here the natural order of the integers is preserved by the order of the blocks. As a means of determining the block lengths we introduce

$$\alpha=56\max\{[\varepsilon^{-1}],[\delta^{-1}]\}. \tag{2.1}$$

We define the lengths of the blocks and therefore, inductively, the blocks themselves by setting

$$\operatorname{card} H_k=k^{\alpha}, \qquad \operatorname{card} I_k=k^{\alpha-3}. \tag{2.2}$$

New random variables y_k and z_k are specified by

$$y_k=\sum_{v\in H_k} x_v, \qquad z_k=\sum_{v\in I_k} x_v. \tag{2.3}$$

The idea of the proof of Theorem 1 can now be described. Since the small blocks I_k are much shorter than the corresponding large blocks H_k, the random variables $\{z_k, k\geqq 1\}$ can be discarded without affecting the final result (this is the upshot of Lemma 2 below). Still the I_k are long enough to separate the random variables y_k until these variables become nearly pairwise independent as $k\to\infty$. In Sect. 4 Theorem 1 of [1] is then applied to $X_k=y_k(Ey_k^2)^{-\frac{1}{2}}$. In the present section we give some preliminary estimates which are all motivated by relation (4.5), which in turn leads immediately to the desired result (1.3).

Lemma 1. *We have uniformly in* $1\leqq j<k<\infty$

$$E\{y_j y_k\}\ll j^{\alpha}k^{-20}$$

and

$$E\{z_j z_k\}\ll j^{\alpha-3}k^{-30}.$$

Proof. From (1.1) and (2.1)-(2.3) we obtain

$$E\{y_j y_k\} \ll \operatorname{card} H_j \operatorname{card} H_k (\operatorname{card} I_{k-1})^{-1-\varepsilon}$$
$$\ll j^\alpha k^\alpha (k-1)^{-(\alpha-3)(1+\varepsilon)} \ll j^\alpha k^{-20}$$

uniformly in $1 \leqq j < k < \infty$. The estimate of $E\{z_j z_k\}$ is handled in the same way. □

Lemma 2. *As* $M \to \infty$

$$\sum_{j \leqq M} z_j \ll M^{\frac{1}{2}\alpha} \quad a.s.$$

Proof. By (1.1), (2.1)-(2.3) and Lemma 1,

$$E(\sum_{j \leqq M} z_j)^2 \leqq \sum_{j \leqq M} E z_j^2 + 2 \sum_{1 \leqq j < k \leqq M} |E\{z_j z_k\}|$$
$$\ll \sum_{j \leqq M} j^{\alpha-3} + \sum_{1 \leqq j < k \leqq M} j^{\alpha-3} k^{-30}$$
$$\ll M^{\alpha-2}.$$

Hence, by Čebyšev's inequality,

$$P\{|\sum_{j \leqq M} z_j| \geqq M^{\frac{1}{2}\alpha}\} \ll M^{-2}.$$

The lemma is thus a consequence of the Borel-Cantelli lemma. □

Next define

$$h_M = \sum_{k \leqq M} \operatorname{card}(H_k \cup I_k). \tag{2.4}$$

One easily checks that

$$M^{\alpha+1} \ll h_M \ll M^{\alpha+1}. \tag{2.5}$$

The next lemma shows that we can break into the blocks.

Lemma 3. *As* $M \to \infty$

$$\max_{h_M < n \leqq h_{M+1}} \left| \sum_{v=h_M+1}^{n} x_v \right| \ll M^{\frac{1}{2}(\alpha+\frac{3}{4})} \quad a.s.$$

Proof. From (2.4), we have $(h_{M+1} - h_M) \sim M^\alpha$. Hence, we obtain the lemma upon applying Corollary B1 of Serfling [7], Markov's inequality, and the Borel-Cantelli lemma. For

$$E\left(\sum_{v=r+1}^{r+n} x_v\right)^2 \leqq \text{const.}\, n^2 \qquad r \geqq 0,\ n \geqq 1$$

so that also, by the corollary just mentioned,

$$E \max_{1 \leqq m \leqq n} \left| \sum_{v=r+1}^{r+m} x_v \right|^4 \leqq \text{const.}\, n^2 \qquad r \geqq 0,\ n \geqq 1. \quad □$$

3. The Conditional Characteristic Function

Normalize the random variables y_k by putting

$$X_k = y_k(E\,y_k^2)^{-\frac{1}{2}} \qquad k > a.$$

Let $a \geqq 0$ be a large integer to be chosen suitably later. We shall apply Theorem 1 of [1] to the sequence $\{X_{a+k}, k \geqq 1\}$. In order to fulfill this aim we will require an estimate of

$$\lambda_k(u) \overset{\text{def}}{=} E|E\{\exp(i u X_{a+k})|X_{a+k-1}, \ldots, X_{a+1}\} - \exp(-\tfrac{1}{2}u^2)|. \tag{3.1}$$

Let $f_k = f_k(x_{a+1}, \ldots, x_{a+k})$ be the joint density of $X_{a+1}, X_{a+2}, \ldots, X_{a+k}$ for every $k \geqq 1$, and let $g(x) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}x^2)$ be the standard normal density. Then

$$\begin{aligned}
\lambda_k(u) &= \int_{\mathbb{R}^{k-1}} f_{k-1} \left| \int_{\mathbb{R}} \exp(i u x_{a+k})(f_k/f_{k-1} - g(x_{a+k})\, dx_{a+k} \right| dx_{a+k-1} \ldots dx_{a+1} \\
&\leqq \int_{\mathbb{R}^k} |f_k - g(x_{a+k}) f_{k-1}|\, dx_{a+k} \ldots dx_{a+1} \\
&\leqq \int_{\{x_{a+1}^2 + \ldots + x_{a+k}^2 \leqq k^2\}} |f_k - g(x_{a+k}) f_{k-1}|\, dx_{a+k} \ldots dx_{a+1} \\
&\quad + P\{X_{a+1}^2 + \ldots + X_{a+k}^2 > k^2\} + P\{X_{a+1}^2 + \ldots + X_{a+k-1}^2 > \tfrac{1}{2}k^2\} \\
&\quad + P\{X_{a+k}^2 > \tfrac{1}{2}k^2\} = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} \quad \text{(say)}.
\end{aligned}$$

Since the estimates of the three probabilities are almost trivial we examine the integral first. The density of each f_k is given in terms of the inverse of the covariance matrix

$$A_k = E\{(X_{a+1}, \ldots, X_{a+k})^T (X_{a+1}, \ldots, X_{a+k})\}.$$

The lemmas we are now going to establish enable us to utilize this knowledge so far as to provide an estimate for $f_k - g(x_{a+k}) f_{k-1}$ over the range of integration in I.

Lemma 4. *As* $k \to \infty$

$$\det A_k - \det A_{k-1} \ll (a+k)^{-19} \tag{3.2}$$

and, if a is sufficiently large,

$$\tfrac{1}{2} \leqq \det A_k \leqq 2 \tag{3.3}$$

for all $k \geqq 1$. *In* (3.2) *the constant implied by* $\ll$ *depends only on the constants implied by* 0 *and by* $\ll$ *in* (1.1) *and* (1.2) *respectively.*

Proof. We expand $\det A_k$ by the k^{th} column of A_k to obtain

$$\det A_k = \det A_{k-1} + \sum_{j=1}^{k-1} (\pm) E\{X_{a+j} X_{a+k}\} \det(A_k|_{j,k}) \tag{3.4}$$

where, for any square matrix A, we denote by $A|_{r,s}$ the matrix obtained from A by deleting its r^{th} row and s^{th} column. By Lemma 1, (1.1) and Hadamard's lemma, which states that the square of a determinant does not exceed the product of the squares of the lengths of its row vectors, we find that

$$\det(A_k|_{j,k}) \leqq \prod_{r=a+1}^{a+k} \left(1+0\left(\sum_{i=a+1}^{r-1} (i\,r)^{-\alpha} r^{2\alpha} i^{-40} + \sum_{i=r+1}^{a+k-1} (i\,r)^{-\alpha} r^{2\alpha} i^{-40}\right)\right)^{\frac{1}{2}} \leqq \prod_{r=1}^{\infty} (1+0(r^{-39}))^{\frac{1}{2}} \ll 1, \tag{3.5}$$

uniformly in $1 \leqq j < k < \infty$. Again, by Lemma 1 and (1.1), we have

$$E\{X_{a+j} X_{a+k}\} \ll (a+j)^{-\frac{1}{2}\alpha}(a+k)^{-\frac{1}{2}\alpha}(a+j)^{\alpha}(a+k)^{-20} \ll (a+k)^{-20}.$$

Thus by (3.4)

$$\det A_k - \det A_{k-1} \ll (a+k)^{-19}.$$

This proves (3.2). Repeated application of (3.2) yields (3.3) since $\det A_1 = EX_{a+1}^2 = 1$. □

Let $a_k = A_k^{-1}$ and denote the element of a_k belonging to row i and column j by $a_k(i,j)$.

Lemma 5. *As* $k \to \infty$

$$a_k(i,j) - a_{k-1}(i,j) \ll k^{-19} \tag{3.6}$$

uniformly in $1 \leqq i, j < k$,

$$a_k(i,k) \ll k^{-19} \tag{3.7}$$

uniformly in $1 \leqq i < k$, *and*

$$a_k(k,k) - 1 \ll k^{-19}. \tag{3.8}$$

Proof. The proof is much like that of Lemma 4. First we note that as $k \to \infty$

$$\det(A_k|_{i,j}) - \det(A_{k-1}|_{i,j}) \ll k^{-19} \tag{3.9}$$

uniformly in $1 \leqq i, j < k$. For, expanding $\det(A_k|_{i,j})$ by the $(k-1)^{\text{th}}$ column of $A_k|_{i,j}$ one has

$$\det(A_k|_{i,j}) = \det(A_{k-1}|_{i,j}) + \sum_{r=1}^{i-1} (\pm) E\{X_{a+r} X_{a+k}\} \det((A_k|_{i,j})|_{r,k-1}) + \sum_{r=i+1}^{k-1} (\pm) E\{X_{a+r} X_{a+k}\} \det((A_k|_{i,j})|_{r-1,k-1}).$$

And, as in (3.5), Hadamard's lemma renders

$$\det((A_k|_{i,j})|_{r,k-1}) \ll 1$$

uniformly in $1 \leqq i, j, r < k < \infty$. Hence (3.9) follows from Lemma 1.

Relation (3.6) is now evident from (3.9), Lemma 4 and the formula

$$a_k(i,j)=(-1)^{i+j}\det(A_k|_{j,i})/\det A_k.$$

The proof of (3.7) is dealt with in the same fashion. Indeed, by Lemma 1, as $k\to\infty$

$$\det(A_k|_{k,i})=\sum_{j=1}^{k-1}(\pm)E\{X_{a+j}X_{a+k}\}\det((A_k|_{k,i})|_{j,k-1})$$
$$\ll k^{-19}$$

uniformly in $1\leqq i<k$, since $\det((A_k|_{k,i})|_{j,k-1})\ll 1$ uniformly in $1\leqq i,j<k<\infty$.
Finally (3.8) is derived from Lemma 4 since $a_k(k,k)=\det A_{k-1}/\det A_k$. □

Lemma 6. *As* $k\to\infty$

$$I\ll k^{-9}.$$

Proof. We write

$$f_k(x_{a+1},\ldots,x_{a+k})=(2\pi)^{-\frac{1}{2}k}(\det A_k)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}P_k)$$

and

$$g(x_{a+k})f_{k-1}(x_{a+1},\ldots,x_{a+k})=(2\pi)^{-\frac{1}{2}k}(\det A_{k-1})^{-\frac{1}{2}}\exp(-\tfrac{1}{2}(P_{k-1}+x^2_{a+k}))$$

where, for each $k\geqq 1$,

$$P_k=\sum_{i,j=1}^{k}x_{a+1}x_{a+j}a_k(i,j).$$

We conclude from Lemma 5 that over the range of integration in I

$$P_{k-1}+x^2_{a+k}-P_k\ll k^{-15}.$$

Thus, by Lemma 4, over this range we have

$$\begin{aligned}&(f_k-g(x_{a+k})f_{k-1})=(2\pi)^{-\frac{1}{2}k}(\det A_k)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}P_k)\\&\quad\cdot\{1-(\det A_k)^{\frac{1}{2}}(\det A_{k-1})^{-\frac{1}{2}}\exp(\tfrac{1}{2}(P_k-P_{k-1}-x^2_{a+k}))\}\\&\ll f_k\cdot k^{-15}.\end{aligned}$$

Since f_k is a density the proof of the lemma is completed from this last estimate. □

Lemma 7. *As* $k\to\infty$

$$\|\lambda_k\|_\infty\ll k^{-15}$$

where λ_k *is defined by* (3.1).

Proof. In view of Lemma 6 it suffices to estimate II, III, and IV. For the estimate of II we simply take

$$\mathrm{II} \leqq \sum_{j \leqq k} P\{X_{a+j}^2 > k\} = k P\{|N(0,1)| \geqq k^{\frac{1}{2}}\}$$
$$\ll k^{-15}.$$

The estimates of III and IV are much the same. □

4. Conclusion of the Proof of Theorem 1

We now apply Theorem 1 of [1] to the sequence $\{X_{a+k}, k \geqq 1\}$ where

$$\mathscr{F}_k = \sigma(X_{a+1}, \dots, X_{a+k}), \quad d_k = 1, \quad T_k = k^2,$$

and $g_k(u) = \exp(-\frac{1}{2}u^2)$. Then, there exists a probability space on which is defined a sequence of random variables, $\{x_\nu^0, \nu \geqq 1\}$, having the same distribution as $\{x_\nu, \nu \geqq 1\}$, and a sequence $\{Y_k, k \geqq 1\}$ of independent $N(0,1)$ random variables such that

$$P\{|(E\, y_{a+k}^2)^{-\frac{1}{2}} \sum_{\nu \in H_{a+k}} x_\nu^0 - Y_k| > \alpha_k\} < \alpha_k$$

where

$$\alpha_k \ll \|\lambda_k^{\frac{1}{2}}\|_\infty T_k + T_k^{-1} \log T_k + P\{|N(0,1)| \geqq T_k/4\} \tag{4.1}$$
$$\ll k^{-\frac{3}{2}}.$$

Set

$$n_k = h_k - h_{k-1} = k^\alpha + k^{\alpha-3} \tag{4.2}$$

where h_k is defined in (2.4). An elementary argument using Kolmogorov's existence theorem shows that without loss of generality (see the paragraph following (1.3)) there is a standard Brownian motion $\{X(t), t \geqq 0\}$ such that

$$P\{|X_{a+k} - n_{a+k}^{-\frac{1}{2}}(X(h_{a+k}) - X(h_{a+k-1}))| > \alpha_k\} < \alpha_k$$

where α_k satisfies (4.1). Therefore the Borel Cantelli lemma gives

$$X_k - n_k^{-\frac{1}{2}}(X(h_k) - X(h_{k-1})) \ll k^{-1} \quad \text{a.s.}$$

or by (1.1)

$$y_k - (E\, y_k^2)^{\frac{1}{2}} n_k^{-\frac{1}{2}}(X(h_k) - X(h_{k-1})) \ll k^{\frac{1}{2}\alpha - 1} \quad \text{a.s.} \tag{4.3}$$

But another application of the Borel-Cantelli lemma shows that

$$X(h_k) - X(h_{k-1}) \ll k\, n_k^{\frac{1}{2}} \quad \text{a.s.}$$

Thus, by (1.1), (4.2), and (4.3)

$$X(h_k) - X(h_{k-1}) - y_k \ll k^{\frac{1}{2}\alpha - 1} \quad \text{a.s.} \tag{4.4}$$

Finally, fix $t \geqq 0$ and let $h_M < t \leqq h_{M+1}$. Then with probability 1,

$$\begin{aligned} X(t) - \sum_{v \leqq t} x_v \ll & \sum_{j=a+1}^{M} |X(h_j) - X(h_{j-1}) - y_j| \\ & + \left| \sum_{j=a+1}^{M} z_j \right| + \max_{h_M < n \leqq h_{M+1}} \left| \sum_{v=h_M+1}^{n} x_v \right| \\ & + \sup_{h_M < t \leqq h_{M+1}} |X(t) - X(h_M)|. \end{aligned} \tag{4.5}$$

Thus from (4.4) and Lemmas 2 and 3

$$\begin{aligned} X(t) - \sum_{v \leqq t} x_v \ll & \sum_{j=a+1}^{M} j^{\frac{1}{2}\alpha - 1} \\ & + M^{\frac{1}{2}\alpha} + M^{(\alpha + \frac{3}{4})} + \sup_{h_M < t \leqq h_{M+1}} |X(t) - X(h_M)| \quad \text{a.s.} \end{aligned}$$

Furthermore, by a well known property of Brownian motion,

$$\sup_{h_M < t \leqq h_{M+1}} |X(t) - X(h_M)| \ll M^{\frac{1}{2}(\alpha + \frac{3}{4})} \quad \text{a.s.}$$

since, by (4.2), $h_{M+1} - h_M = n_{M+1} \ll M^{\alpha}$.

Therefore

$$\begin{aligned} X(t) - \sum_{v \leqq t} x_v & \ll M^{\frac{1}{2}(\alpha + \frac{3}{4})} \\ & \ll M^{(\alpha+1)(\frac{1}{2} - (8(\alpha+1))^{-1})} \ll t^{\frac{1}{2} - \lambda} \quad \text{a.s.} \end{aligned}$$

where $\lambda = (8(\alpha+1))^{-1}$ and, by our choice of α, $(8(\alpha+1))^{-1} \geqq \min(\varepsilon, \delta)/500$. □

5. Sketch of the Proof of Theorem 2

The proof of Theorem 2 proceeds just like that for Theorem 1 with the only major change being the choice of the lengths of the blocks H_k and I_k. In the setting of Sect. 2 we take, instead of (2.2),

$$\begin{aligned} \operatorname{card} H_k &= [k^{-\frac{2}{3} - \gamma} \exp(k^{\frac{1}{3} - \gamma})], \\ \operatorname{card} I_k &= [k^{-\frac{2}{3} - 4\gamma} \exp(k^{\frac{1}{3} - \gamma})] \quad \text{where } \gamma = \frac{\varepsilon}{150}. \end{aligned} \tag{5.1}$$

Define y_k and z_k as in (2.3) and assume the hypotheses of Theorem 2. The following lemmas are counterparts to the lemmas in Sect. 2.

Lemma 1'. *As* $k \to \infty$

$$E\{y_j y_k\} \ll \operatorname{card} H_j \cdot k^{-8-\frac{\varepsilon}{7}}$$

and

$$E\{z_j z_k\} \ll \operatorname{card} H_j \cdot k^{-8-\frac{\varepsilon}{6}}$$

uniformly in $1 \leqq j < k$.

Lemma 2′. *As* $M \to \infty$

$$\sum_{j \leqq M} z_j \ll M^{-\gamma} \exp(\tfrac{1}{2} M^{\frac{1}{3}-\gamma}) \quad a.s.$$

Proof. $\sum_{j \leqq M} z_j$ is normally distributed with mean zero and

$$E(\sum_{j \leqq M} z_j)^{2\left[\frac{2}{\gamma}\right]} \sim \text{const.}(M^{-3\gamma} \exp(M^{\frac{1}{3}-\gamma}))^{\left[\frac{2}{\gamma}\right]}. \quad \square$$

With h_M defined by (2.4) and (5.1) we have

$$h_M \sim \exp(M^{\frac{1}{3}-\gamma}).$$

Lemma 3′. *As* $M \to \infty$

$$\max_{h_M < n \leqq h_{M+1}} \left| \sum_{v=h_M+1}^{n} x_v \right| \ll M^{-\gamma} \exp(\tfrac{1}{2} M^{\frac{1}{3}-\gamma}) \quad a.s.$$

Proof. The proof is the same as that for Lemma 3 except now we use higher moments. □

Now, by the reasoning of Lemma 7,

$$P\{X_{a+1}^2 + \ldots + X_{a+k}^2 \geqq k(\log k)^2\} \ll k^{-5}.$$

Thus Lemmas 1′, 2′, and 3′ can be applied in the manner of Sect. 3 to obtain $\|\lambda_k\|_\infty \ll k^{-4-\frac{\varepsilon}{8}}$. Finally, proceeding along the lines of Sect. 4, we prove Theorem 2 by an application of Theorem 1 of [1] with $\lambda_k \ll k^{-4-\frac{\varepsilon}{8}}$, $d_k = 1$, $T_k = k^{1+\frac{\varepsilon}{32}}$, and $g_k(u) = \exp(-\frac{1}{2}u^2)$. □

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