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Local Times and Random Time Changes *

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Summary. The purpose of this paper is to prove an integral representation theorem for continuous additive functionals (of a Hunt process satisfying hypothesis (F)) as integrals of local times (when they exist) with respect to certain measures. The effect of random time changes on the local times and on the integral representation is investigated.

Preliminaries

See [4] for all definitions and notation employed below. Let $X = (\Omega, X_t, \Theta_t, P_x)$ be a *Hunt process* with state space E. That is, E is a locally compact separable metric space with $\bar{E} = E \cup \{\Delta\}$ where Δ is the point adjoined to E in the one-point compactification if E is not compact, otherwise Δ is an isolated point. Let $\mathscr{B}$ and $\overline{\mathscr{B}}$ be the Borel sets of E and $\bar{E}$ respectively. Ω is the sample space of paths ω which are maps $\omega\colon [0,\infty] \to \bar{E}$ that are right continuous and have left hand limits such that $\omega(\infty) = \Delta$ and if $\omega(t) = \Delta$ then $\omega(s) = \Delta$ for all $s \geqq t$. We write $X_t(\omega) = X(t, \omega) = \omega(t)$. The shift operators $\Theta_t\colon \Omega \to \Omega$ are defined by $X_s(\Theta_t \omega) = X_{s+t}(\omega)$. Let $\mathscr{F}^0$ be the smallest σ-algebra with respect to which the maps $X_t\colon \Omega \to E$ are measurable for all $t \geqq 0$ and $\mathscr{F}_t^0$ the smallest σ-algebra with respect to which $X_s\colon \Omega \to E$ are measurable for $0 \leqq s \leqq t$. For each x in $\bar{E}$, P_x is a probability measure on $\mathscr{F}^0$ such that $P_x[X(0) = x] = 1$ and $x \to$ $\to P_x[\Lambda]$ is $\overline{\mathscr{B}}$ measurable for each Λ in $\mathscr{F}^0$. For each finite measure μ on $(\bar{E}, \overline{\mathscr{B}})$ we define a measure P_μ on $\mathscr{F}^0$ by $P_\mu(\Lambda) = \int P_x(\Lambda)\, d\mu(x)$. We define the σ-algebra $\mathscr{F}_t$ $(\mathscr{F})$ as the intersection of the P_μ-completions of the σ-algebra $\mathscr{F}_t^0$ $(\mathscr{F}^0)$ taken over all finite measures μ on $(\bar{E}, \overline{\mathscr{B}})$. A *stopping time* is a function $T\colon \Omega \to [0,\infty]$ such that $\{T < t\} \in \mathscr{F}_t$ for each $t > 0$. For each stopping time T let $\mathscr{F}_T$ be the σ-algebra of sets Λ in $\mathscr{F}$ such that $\Lambda \cap \{T < t\} \in \mathscr{F}_t$ for all $t > 0$. We define $\Theta_T \omega$ by $X_t(\Theta_T \omega) = X_{t+T(\omega)}(\omega)$. We assume X is a *strong Markov process*: For each stopping time T and each bounded real random variable F on $(\Omega, \mathscr{F})$ we have

$$E_x\{F(\Theta_T \omega);\ \Lambda\} = E_x\{E_{X(T)}(F);\ \Lambda\}$$

for all x in $\bar{E}$ and $\Lambda \in \mathscr{F}_T$. We denote $E_x(F; \Lambda) = \int F\, dP_x$. Finally, X is *quasi-left continuous*: If $\{T_n\}$ is an increasing sequence of stopping times with limit T, then $X(T_n) \to X(T)$ almost surely on $\{T < \infty\}$. (An expression is said to be true almost surely if it is true almost everywhere P_x for each x; almost surely is abbreviated a.s.) This completes the definition of a Hunt process. The reader is referred to [4] for the properties of Hunt processes and terms not defined here.

The *lifetime*, ζ, of a Hunt process is defined by $\zeta(\omega) = \inf\{t > 0\colon X_t(\omega) = \Delta\}$.

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Given a set $B \in \overline{\mathscr{B}}$ we define the *hitting time*, T_B, of B as

$$T_B(\omega) = \inf\{t > 0:\ X_t(\omega) \in B\}.$$

T_B can be shown to be a stopping time. If D is an arbitrary set in $\bar{E}$, then a point x is said to be *irregular* for D if there exists a set $B \in \overline{\mathscr{B}}$ such that $D \subset B$ and $P_x[T_B > 0] = 1$. A set G is *finely open* if each x in G is irregular for $G^c = \bar{E} - G$. The finely open sets form a topology on $\bar{E}$ called the *fine topology*.

If a Hunt process X satisfies Hunt's *hypothesis* (F) then there exists a measure ξ on E and point kernels $U^\lambda(x, y)$ defined on $E \times E$ for $\lambda \geqq 0$ such that

$$E_x \int_0^\infty e^{-\lambda t} f(X(t))\, dt = \int U^\lambda(x, y) f(y)\, d\xi(y),$$

for all bounded measurable f.

Under hypothesis (F), given a measure μ on E and a bounded measurable function f on E, we define the *λ-potential, $U_\mu^\lambda f$, of f with respect to μ* as

$$U_\mu^\lambda f(x) = \int U^\lambda(x, y) f(y)\, d\mu(y).$$

If $f \equiv 1$ we write $U_\mu^\lambda 1 = U_\mu^\lambda$. U_μ^λ is λ-excessive and lower semi-continuous.

Continuous Additive Functionals

Let X be a Hunt process. A family $A = \{A(t),\ t \geqq 0\}$ of real valued random variables is a *continuous additive functional*, CAF, of X if

(i) The following statements hold almost surely: $A(0) = 0$; $t \to A(t)$ is continuous and non-decreasing; $A(s) = \lim_{t \nearrow \zeta} A(t)$ whenever $s \geqq \zeta$.

(ii) $A(t)$ is $\mathscr{F}_t$-measurable for each t.

(iii) For each t and s,

$$A(t + s, \omega) = A(t, \omega) + A(s, \Theta_t \omega), \quad \text{a.s.}$$

Meyer [8] has proved that every CAF A satisfies the *strong Markov property*:

$$A(T + S, \omega) = A(T, \omega) + A(S, \Theta_T \omega), \quad \text{a.s.}$$

for each stopping time T and each non-negative random variable S on $(\Omega, \mathscr{F})$.

If f is a non-negative nearly Borel measurable function on E we define the λ-potential (for $\lambda \geqq 0$) of f with respect to a CAF A by

$$U_A^\lambda f(x) = E_x \int_0^\infty e^{-\lambda t} f(X_t)\, dA(t).$$

If $f \equiv 1$, we write $U_A^\lambda 1 = U_A^\lambda$. $U_A^\lambda f$ is λ-excessive.

Given a measurable function f on E and a stopping time T, let

$$P_T^\lambda f(x) = E_x[e^{-\lambda T} f(x_T)].$$

If A is a CAF with $\varphi = U_A^\lambda < \infty$ for some $\lambda \geqq 0$ then $\lim_{n\to\infty} P_{T_n}^\lambda \varphi = 0$ for each increasing sequence T_n with $\zeta = \lim T_n$. See [4, p. 417].

The important *uniqueness theorem* of Meyer [8, p. 193] states that if A and B are CAF's with U_A^λ, $U_B^\lambda < \infty$ for some $\lambda \geqq 0$ and if $U_A^\lambda = U_B^\lambda$ then A and B

are equivalent, i.e., $P_x[A(t) \neq B(t)] = 0$ for all x and t. We do not distinguish between equivalent CAF's A and B, and we write $A = B$.

Preliminary Theorem

The following theorem, see [6] or [1, p. 139] affords a connection between CAF's and measures under hypothesis (F).

Theorem 1. *Let X be a Hunt process that satisfies hypothesis (F). If A is a* CAF *of X such that $U_A^\lambda < \infty$ for $\lambda > 0$ then there exists a unique measure μ on E such that $U_A^\lambda f = U_\mu^\lambda f$ for each non-negative $\mathscr{B}$-measurable function f on E.*

Proof. φ is λ-excessive and finite. Since $\lambda > 0$ Hunt's hypothesis (G) holds [7, p. 170]. Thus by Theorem 18.7 of Hunt, [7, p. 177], φ can be written as $\varphi = U_\mu^\lambda + \psi$, where μ is a unique measure on E and ψ is a λ-excessive function such that $P_{T_D}^\lambda \psi = \psi$ whenever D is the complement of a compact set in E. As usual $T_D = \inf\{t > 0 \colon X_t \in D\}$.

Now let $\{G_n\}$ be a sequence of open subsets of E such that the closure $\bar{G}_n$ of G_n is a compact subset of G_{n+1} and $\bigcup_{n=1}^{\infty} G_n = E$ and $T_n = T_{D_n} \nearrow \zeta$ where $D_n = \bar{G}_n^c$. Since D_n is the complement of a compact set we have $P_{T_n}^\lambda \psi = \psi$. But $0 \leqq \psi \leqq \varphi$ since $U_\mu^\lambda \geqq 0$ and also $P_{T_n}^\lambda \varphi \to 0$ as $n \to \infty$, hence $\psi = P_{T_n}^\lambda \psi \leqq \leqq P_{T_n}^\lambda \varphi \to 0$ so $\psi = 0$. Hence $U_A^\lambda = U_\mu^\lambda$.

That $U_A^\lambda f = U_\mu^\lambda f$ now follows from a theorem of Meyer [8, p. 218]. This completes the proof of Theorem 1.

The Fine Support of a CAF

Let A be a CAF of a Hunt process X. Assume $U_A^\lambda < \infty$ for some fixed $\lambda \geqq 0$. We say A *vanishes* on a nearly analytic set D provided

$$U_A^\lambda I_D(x) = E_x \int_0^\infty e^{-\lambda t} I_D(X_t)\, dA(t) = 0 \quad \text{for all } x,$$

where I_D is the indicator function of D. The *fine support* of A is the smallest finely closed set on whose complement A vanishes. Getoor [5] has shown the existence of the fine support under the above conditions on A. Furthermore if $R = \inf\{t > 0 \colon A(t) > 0\}$ then the fine support of A is the set

$$F = \{x \colon P_x[R = 0] = 1\}.$$

It follows from Theorem 1 that under hypothesis (F) the set F is the fine support of the measure μ associated with the CAF A mentioned in that theorem. Also, under hypothesis (F) the fine support is a Borel set.

Definition. A CAF A of a Hunt process X is said to be strictly increasing if $t \to A(t)$ is strictly increasing a.s. on $[0, \zeta)$.

The following theorem provides a criterion in terms of the measure in Theorem 1 for a CAF to be strictly increasing. Theorem 5.4 of [1] is similar to this theorem.

Theorem 2. *Let A be as in Theorem 1 and let μ be the measure such that $U_A^\lambda f = U_\mu^\lambda f$ for all non-negative $\mathscr{B}$-measurable f. Then A is strictly increasing if and only if μ is strictly positive on non-empty finely open Borel sets.*

Proof. Suppose A is strictly increasing. Let G be a non-empty finely open Borel set. If $x \in G$ then $P_x[T_{G^c} > 0] = 1$, so

$$0 < E_x \int_0^\infty e^{-\lambda t} I_G(X_t)\, dA(t) = U_A^\lambda I_G(x) = \int_G U^\lambda(x, y)\, d\mu(y)$$

and hence $\mu(G) > 0$.

Conversely, suppose μ is strictly positive on non-empty finely open Borel sets. Let $R = \inf\{t > 0: A(t) > 0\}$ and $G = \{x: P_x[R > 0] = 1\}$. To show A is strictly increasing we need only show G is empty. Note that $G = F^c$ where F is the fine support of A and hence of μ. We then have $\mu(G) = 0$. But G is a finely open Borel set being the complement of a finely closed Borel set. Hence G is empty. Q.E.D.

Local Times

Let X be a Hunt process. A point x in E is *regular for* x with respect to X if $P_x[T_x = 0] = 1$ where $T_x = \inf\{t > 0: X(t) = x\}$ is the hitting time for x.

Let x_0 be a point of E such that x_0 is regular for itself. It is shown in [2] that then there exists a CAF A_{x_0} such that $U^1_{Ax_0} = E_x[e^{-T_{x_0}}]$. A_{x_0} is called the local time of the process X at x_0. Under hypothesis (F) A_{x_0} can be chosen so that $U^\lambda_{A_{x_0}}(x) = U^\lambda(x, x_0)$ for $\lambda > 0$. (There is a certain amount of freedom in constructing local time — but two local times at the same point differ by constant multiples.)

The following theorem which we state without proof is useful in what follows.

Theorem 3. (See [5].) *Let X be a Hunt process.*

(i) *The fine support of the local time A_{x_0} at a point x_0 regular for itself is the set $\{x_0\}$.*

(ii) *If x_0 is a point regular for itself and A is a CAF with $U_A^\lambda < \infty$ for $\lambda > 0$ and with fine support $\{x_0\}$ then $A = b A_{x_0}$ for some $b > 0$ where A_{x_0} is the local time at x_0.*

Integral Representation of CAF's

Theorem 4. *Let X satisfy hypothesis (F). Suppose each point in E is regular for itself. If A is a CAF of X with $U_A^\lambda < \infty$ for $\lambda > 0$ then there exists a unique measure μ such that $A = \int A_x\, d\mu(x)$, that is, $A(t) = \int A_x(t)\, d\mu(x)$, a.s.*

Proof. Let μ be the measure in Theorem 1 such that $U_A^\lambda = U_\mu^\lambda$. Let $B(t) = \int A_x(t)\, d\mu(x)$. It is not hard to check that $B = \{B(t), t \geqq 0\}$ is a CAF of X. Using Fubini's Theorem we have

$$\begin{aligned} U_B^\lambda(x) &= E_x \int_0^\infty e^{-\lambda t}\, dB(t) \\ &= \int \Big(E_x \int_0^\infty e^{-\lambda t}\, dA_y(t)\Big)\, d\mu(y) \\ &= \int U^\lambda(x, y)\, d\mu(y) \\ &= U_\mu^\lambda(x) \\ &= U_A^\lambda(x)\,. \end{aligned}$$

The result now follows from Meyer's uniqueness theorem.

Random Time Changes

Let A be a CAF of a Hunt process X. The *inverse* $\tau = \{\tau_t, t \geqq 0\}$ of A is defined by

$$\tau_t(\omega) = \inf\{s\colon A(s, \omega) > t\}$$

or $\tau_t(\omega) = \infty$ if $\{s\colon A(s, \omega) > t\}$ is empty. Each τ_t is a stopping time and for $s, t \geqq 0$ we have

$$\tau_{t+s} = \tau_t + \tau_s(\Theta_{\tau_t}), \quad \text{a.s.}$$

τ_t is right continuous and strictly increasing in t almost surely.

We define a new process $\tilde{X} = \{\tilde{\Omega}, \tilde{X}_t, \tilde{\Theta}_t, \tilde{P}_x\}$ as follows: $\tilde{\Omega} = \Omega$,

$$\tilde{X}_t(\omega) = X_{\tau_t(\omega)}(\omega), \quad \tilde{\Theta}_t = \Theta_{\tau_t}, \quad \tilde{P}_x = P_x.$$

$\tilde{X}$ is then a strong Markov process with state space E.

All of the above facts are well known and can be found in [3, p. 322].

$\tilde{X}$ is said to be obtained by a *random time change* (via A) from the process X.

If $A = \{A(t), t \geqq 0\}$ is a strictly increasing CAF then clearly $t \to \tau_t$ is continuous a.s.

In what follows let $A = \{A(t), t \geqq 0\}$ be a *strictly increasing* perfect CAF of a Hunt process X.[1] Let $\tilde{X}$ be the process obtained from X by a random time change via A.[2]

Lemma 1. *If $B = \{B(t), t \geqq 0\}$ is a CAF of X then $\tilde{B} = \{\tilde{B}(t), t \geqq 0\}$, where $\tilde{B}(t, \omega) = B(\tau_t(\omega), \omega)$, is a CAF of $\tilde{X}$.*

Proof. Now, $t \to \tilde{B}(t)$ is continuous and non-decreasing a.s. Also $\tilde{B}(t) = B(\tau_t)$ is $\tilde{\mathscr{F}}_t = \mathscr{F}_{\tau_t}$ measurable. It remains to check additivity, namely

$$\tilde{B}(t+s) = \tilde{B}(t) + \tilde{B}(s, \tilde{\Theta}_t),$$

a.s. Now,

$$\begin{aligned}\tilde{B}(t+s, \omega) &= B(\tau_{t+s}(\omega), \omega)\\ &= B(\tau_t(\omega) + \tau_s(\Theta_{\tau_t}\omega), \omega)\\ &= B(\tau_t(\omega), \omega) + B(\tau_s(\Theta_{\tau_t}\omega), \Theta_{\tau_t}\omega)\\ &\quad \text{(by the strong Markov property for CAF's)}\\ &= \tilde{B}(t, \omega) + \tilde{B}(s, \tilde{\Theta}_t\omega), \quad \text{a.s.} \qquad \text{Q.E.D.}\end{aligned}$$

Lemma 2. *A point x is regular for itself with respect to X if and only if x is regular for itself with respect to $\tilde{X}$.*

Proof. Let $T_x = \inf\{t > 0\colon X_t = x\}$ and $\tilde{T}_x = \inf\{t > 0\colon \tilde{X}_t = x\}$. Then $\tilde{T}_x = A(T_x)$, a.s. So,

$$\begin{aligned}P_x[\tilde{T}_x = 0] = 1 &\Leftrightarrow P_x(A(T_x) = 0] = 1\\ &\Leftrightarrow P_x[T_x = 0] = 1. \qquad \text{Q.E.D.}\end{aligned}$$

[1] See [3, p. 173] for the definition of a perfect CAF.

[2] Also assume hypothesis (L) [4, p. 420]; (L) is a mild restriction on X. $\tilde{X}$ is then a Hunt process. Note hypothesis (F) implies (L).

By Lemma 2 the transformed process $\tilde{X}$ has local times at points where X has local times. The next theorem identifies the local times of the transformed process in terms of the local times of the original process.

Theorem 5. *Let x_0 be regular for itself with respect to X. If A_{x_0} is the local time of X at x_0, then the local time $\tilde{A}_{x_0}$ of $\tilde{X}$ at x_0 exists and is given by $\tilde{A}_{x_0}(t) = A_{x_0}(\tau_t)$ a.s., for $t \geqq 0$.* (Hypothesis (L) is assumed.)[3]

Proof. Let $\tilde{B} = \{\tilde{B}(t), t \geqq 0\}$ where $\tilde{B}(t) = A_{x_0}(\tau_t)$. By Lemma 1, $\tilde{B}$ is a CAF of $\tilde{X}$. Now, let $\tilde{R} = \inf\{t > 0 : \tilde{B}(t) > 0\}$. The fine support $\tilde{F}$ of $\tilde{B}$ is then $\tilde{F} = \{x : \tilde{P}_x[\tilde{R} = 0] = 1\}$ but $\tilde{R} = A(R)$, a.s., where $R = \inf\{t > 0 : A_{x_0}(t) > 0\}$. Thus,

$$P_x[\tilde{R} = 0] = 1 \Leftrightarrow P_x[A(R) = 0] = 1$$
$$\Leftrightarrow P_x[R = 0] = 1 .$$

But $\{x_0\} = \{x : P_x[R = 0] = 1\}$ is the fine support of A_{x_0}, so $\{x_0\}$ is the fine support of $\tilde{B}$. Hence by Theorem 3, $\tilde{B} = b\tilde{A}_{x_0}$ for a constant $b > 0$. In view of the uniqueness of local time up to constant multiples we can assume $b = 1$. With this modification, Theorem 5 is proved.

If X satisfies hypothesis (F) in the above considerations the transformed process $\tilde{X}$ need not satisfy (F). However, we have an analogue of Theorem 4 for $\tilde{X}$.

Theorem 6. *Let X be a Hunt process that satisfies hypothesis (F) and such that each point is regular for itself with respect to X. Let A be a strictly increasing CAF of X such that $U_A^\lambda < \infty$ for $\lambda > 0$*[4]*. Let $\tilde{B}$ be a CAF of the process $\tilde{X}$ obtained from X by random time change via A. If $U_{\tilde{B}}^\lambda < \infty$ for $\lambda > 0$ then there exists a unique measure ν on E such that $\tilde{B} = \int \tilde{A}_x d\nu(x)$, where $\tilde{A}_x$ is the local time of $\tilde{X}$ at x.*

Proof. Let $B(t) = \tilde{B}(A(t))$. It is easy to see $B = \{B(t), t \geqq 0\}$ is a CAF of X such that $U_B^\lambda < \infty$ for $\lambda > 0$ and hence by Theorem 4 there is a measure ν on E such that $B = \int A_x d\nu(x)$. Then by Theorem 5,

$$\tilde{B}(t) = B(\tau_t) = \int A_x(\tau_t)\, d\nu(x) = \int \tilde{A}_x(t)\, d\nu(x), \quad \text{a.s.}$$

i.e., $\tilde{B} = \int \tilde{A}_x d\nu(x)$.

Bibliography

1. BLUMENTHAL, R. M., and R. K. GETOOR: Additive functionals of Markov processes in duality. Trans. Amer. math. Soc. **112** 131–163 (1964).
2. — — Local times for Markov processes. Z. Wahrscheinlichkeitstheorie verw. Geb. **3**, 50–74 (1964).
3. DYNKIN, E. B.: Markov Processes, vol. 1. Berlin-Heidelberg-New York: Springer 1965.
4. GETOOR, R. K.: Additive functionals and excessive functions. Ann. math. Statistics **36**, 409–422 (1965).

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[3] This theorem generalizes a result of ITO and MCKEAN. See Diffusion Processes and their sample paths. Springer, Berlin-Heidelberg-New York 1965, p. 174.

[4] Under hypothesis (F) every CAF is equivalent to a perfect CAF.

GETOOR, R. K.: Continuous additive functionals of a Markov process with applications to processes with independent increments. J. math. Analysis Appl. **13**, 132—153 (1966).
6. GRIEGO, R. J.: Local time as a derivative of occupation times. Dissertation 1965, University of Illinois.
7. HUNT, G. A.: Markov processes and potentials, III. Illinois J. Math. **2**, 151—213 (1958).
8. MEYER, P. A.: Fonctionelles multiplicatives et additives de Markov. Ann. Inst. Fourier **12**, 125—230 (1962).

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