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Weakly Wandering and Related Sequences

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1. Indroduction

Weakly wandering sets were introduced in [2] while studying necessary and sufficient conditions for the existence of a finite invariant and equivalent measure μ for a given measurable non-singular transformation φ defined on a measure space $(X, \mathscr{B}, m)$. In the same paper it was shown that every ergodic measure preserving transformation defined on an infinite measure space $(X, \mathscr{B}, m)$ admits weakly wandering sets of positive measure, see also [4]. In this paper, we consider the collection of all weakly wandering sequences $\mathscr{W} = \{W\}$ associated with a given measurable non-singular transformation φ and study some of the properties of such a collection $\mathscr{W}$. While investigating properties of the iterates of a measurable non-singular transformation φ in [1] we found that weakly wandering sequences have some interesting properties; see section 3 of [1]. For definitions of such sequences and other related sequences discussed in this paper we refer the reader to the text.

In Theorem 1 we list a number of the properties of weakly wandering sequences. In section 2 we introduce a collection of closely related sequences of integers; namely, the recurrent sequences $\mathscr{R} = \{R\}$ for the transformation φ. In Theorem 3 we give some of the properties of such sequences, while in Theorem 2 we give a characterization for an infinite sequence of integers $R = \{r_i\}$ to be a recurrent sequence for a transformation φ. Namely, $R = \{r_i\}$ is a recurrent sequence for the transformation φ if and only if there exists a measurable set A of finite measure such that $\overline{\lim_{i\to\infty}}\, m(\varphi^{r_i} A \cap A) > 0$. We then use this condition to give a characterization for an ergodic measure preserving transformation φ defined on an infinite measure space $(X, \mathscr{B}, m)$ to be of positive or zero type. Such transformations are discussed in detail in [3]; see also their connection with the problem of invariant measures [2]. In section 4 we mention a topology on a space consisting of infinite subsets of integers. We then give a somewhat different description of the weakly wandering and the recurrent sequences. In this connection we introduce a number of other sequences of integers that are related to the weakly wandering and recurrent ones.

2. Basic Notations and Definitions

We shall only consider σ-finite and non-atomic measure spaces $(X, \mathscr{B}, m)$. A subset A of X is said to be measurable in case $A \in \mathscr{B}$. Measurable subsets of X

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will be denoted by the letters $A, B, C, \ldots$, and often the word measurable will not be mentioned explicitly. By a measurable transformation φ defined on the measure space $(X, \mathscr{B}, m)$ we shall mean a $1-1$ mapping of the space X onto itself such that $A \in \mathscr{B}$ if and only if $\varphi A \in \mathscr{B}$. We say that φ is a non-singular transformation in case $m(A) = 0$ if and only if $m(\varphi A) = 0$, and φ is measure preserving or equivalently m is an invariant measure for φ in case $m(A) = m(\varphi A) = m(\varphi^{-1} A)$ for all measurable sets A.

A set $A \in \mathscr{B}$ is said to be a wandering set for the transformation φ in case $\varphi^i A \cap \varphi^j A = \emptyset$ for $i \neq j$; $i, j = 1, 2, \ldots$, The set A is said to be a weakly wandering set for the transformation φ under some sequence $W = \{w_i\}$ of integers in case $\varphi^{w_i} A \cap \varphi^{w_j} A = \emptyset$ for $i \neq j$. In case there exists a measurable set A of positive measure which is weakly wandering under a sequence $W = \{w_i\}$ of integers, we shall say that $W = \{w_i\}$ is a weakly wandering sequence for the transformation φ.

Let φ be a measurable non-singular transformation defined on a measure space $(X, \mathscr{B}, m)$. φ is said to be ergodic in case $\varphi A = A$ implies $m(A) = 0$ or $m(X - A) = 0$. φ is said to be recurrent in case $m(A) > 0$ implies for almost all $x \in A$ $\varphi^n x \in A$ for some positive integer $n = n(x) > 0$. It is possible to verify the following statements for a measurable non-singular transformation φ defined on a measure space $(X, \mathscr{B}, m)$:

(1) φ is recurrent if and only if φ^k is recurrent for every integer $k = 1, 2, \ldots$

(2) φ is recurrent if and only if there exists no wandering set of positive measure for φ.

(3) «φ is ergodic» implies «φ is recurrent».

(4) «φ ergodic and measure preserving with $m(X) = \infty$» implies «there exist weakly wandering sequences for φ».

For a proof of the above statements see [5], [4], and [2].

Let φ be an ergodic measure preserving transformation defined on the measure space $(X, \mathscr{B}, m)$ with $m(X) = \infty$. The notion of a transformation of positive type and that of zero type was introduced in [2] as follows:

Definition. i) φ is said to be a transformation of *positive type* if

$$m(A) > 0 \quad \text{implies} \quad \varlimsup_{n\to\infty} m(\varphi^n A \cap A) > 0 .$$

ii) φ is said to be of *zero type* if

$$m(A) < \infty \quad \text{implies} \quad \lim_{n\to\infty} m(\varphi^n A \cap A) = 0 .$$

In section 4 we need to mention such transformations, and we note that in [3] ergodic measure preserving transformations of positive type and those of zero type were constructed on a measure space $(X, \mathscr{B}, m)$ with $m(X) = \infty$.

In what follows we study mainly ergodic measure preserving transformations φ defined on a measure space $(X, \mathscr{B}, m)$ with $m(X) = \infty$. A number of the new concepts connected with a transformation φ can be introduced without requiring the transformation φ be ergodic, measure preserving, or $1-1$. Moreover, many of the subsequent properties can be shown to be true under a more general setting.

Howevere, in order to emphasize the significant steps and make the arguments simpler we restrict ourselves to transformations of the above type.

Since different subsets of the integers are used throughout this paper extensively, we establish some further notation.

We shall denote by $I = \{n \mid n = 0, \pm 1, \pm 2, \ldots\}$ the set of all integers. The letters $R, S, V, W, \ldots$ will be used to indicate infinite subsets of the integers, and we shall often refer to them as sequences $R = \{r_i \mid i = 1, 2, \ldots\}$. In particular, in what follows, all sequences $R = \{r_i\}$ mentioned are assumed to have the property that $r_i \neq r_j$ for $i \neq j$; $i, j = 1, 2, \ldots$. For any integer $k \neq 0$ we let $N_k = \{nk \mid n = 1, 2, \ldots\}$, and we denote by $\{k\}$ the subset of I consisting of the single element $k \in I$. Let V and W be two subsets of I. We shall have occasion to use the following additional subsets of I:

$$
\begin{aligned}
W^c &= \{n \in I \mid n \notin W\} \\
V + W &= \{n \in I \mid n = v + w \quad \text{for} \quad v \in V \quad \text{and} \quad w \in W\} \\
-W &= \{n \in I \mid n = -w \quad \text{for} \quad w \in W\}\,. \\
W - W &= \{n \in I \mid n = w - w' \quad \text{for} \quad w, w' \in W\}\,.
\end{aligned}
$$

3. Weakly Wandering and Recurrent Sequences

For a measurable and non-singular transformation φ defined on a measure space $(X, \mathscr{B}, m)$ we denote by $\mathscr{W} = \{W\}$ the collection of all weakly wandering sequences $W = \{w_i \mid i = 1, 2, \ldots\}$ for the transformation φ. We introduce next the following:

Definition. We say that a sequence $R = \{r_i \mid i = 1, 2, \ldots\}$ of integers is a *recurrent sequence* for the transformation φ in case $R \cap W$ has only a finite number of elements for any $W \in \mathscr{W}$. We denote by $\mathscr{R} = \{R\}$ the collection of all recurrent sequences $R = \{r_i \mid i = 1, 2, \ldots\}$ for the transformation φ.

Theorem 1. *Let φ be an ergodic measure preserving transformation defined on the measure space $(X, \mathscr{B}, m)$ with $m(X) = \infty$, and let $\mathscr{W} = \{W\}$ be the collection of all weakly wandering sequences $W = \{w_i \mid i = 1, 2, \ldots\}$ for the transformation φ. Then*

i) $W \in \mathscr{W}$ *implies* $-W \in \mathscr{W}$.

ii) $W \in \mathscr{W}$, $V \subset W$ *implies* $V \in \mathscr{W}$.

iii) $W \in \mathscr{W}$ *implies* $W + \{k\} \in \mathscr{W}$ *for any integer* $k = 0, \pm 1, \pm 2, \ldots$.

iv) $W \in \mathscr{W}$ *implies there exists an integer $k \neq 0$ such that* $W \cup (W + \{k\}) \in \mathscr{W}$.

v) $W \in \mathscr{W}$ implies $(W - W)^c \cap N_k$ *is an infinite subset of I for any integer $k \neq 0$.*

vi) $W = \{w_i \mid i = 1, 2, \ldots\} \in \mathscr{W}$ *implies* $\lim\limits_{i \to \infty} i/w_i = 0$.

Proof. i) Follows from the fact that

$$\varphi^{w_i} A \cap \varphi^{w_j} A = \emptyset \quad \text{if and only if} \quad \varphi^{-w_i} A \cap \varphi^{-w_j} A = \emptyset \quad \text{for} \quad i \neq j\,.$$

ii) and iii) are obvious.

To show iv) let A be a weakly wandering set under the sequence $W = \{w_i\}$ with $m(A) > 0$. Since φ is ergodic, there exists an integer $k \neq 0$ such that $m(\varphi^{-k} A \cap A) > 0$. Let $B = \varphi^{-k} A \cap A$. By possibly considering a subset of B

we may assume without loss of generality that $0 < m(B) < \infty$. Let $C = B - \varphi^k B$. Since $m(X) = \infty$ and φ is ergodic, it follows that $m(C) > 0$. Now $C \subset B \subset A$; therefore,

$$\varphi^{w_i} C \cap \varphi^{w_j} C = \emptyset \quad \text{for} \quad i \neq j. \tag{5}$$

Furthermore, $C \subset A$ and $\varphi^k C \subset A$ imply

$$\varphi^{w_i} C \cap \varphi^{w_j + k} C = \emptyset \quad \text{for} \quad i \neq j; \tag{6}$$

and $C \subset B$, $\varphi^k C \subset \varphi^k B$ and $C \cap \varphi^k B = \emptyset$ imply

$$C \cap \varphi^k C = \emptyset. \tag{7}$$

Combining (5), (6), and (7) we obtain

$$\varphi^{w'_i} C \cap \varphi^{w'_j} C = \emptyset \quad \text{for} \quad w'_i \neq w'_j;\ w'_i, w'_j \in W \cup (W + \{k\}), \quad \text{and} \quad i, j = 1, 2, \ldots$$

This shows that $W \cup (W + \{k\}) \in \mathscr{W}$ for some $k \neq 0$.

To show v) we note that if A is a set of positive measure weakly wandering under the sequence $W = \{w_i\}$ and $p \in W - W$ with $p \neq 0$, then

$$\varphi^p A \cap A = \emptyset. \tag{8}$$

If $(W - W)^c \cap N_k$ has only a finite number of elements, assuming that $k > 0$, then by (8) there exists an integer $q > 0$ such that

$$\varphi^{kn} A \cap A = \emptyset \quad \text{for} \quad n = q, q+1, q+2, \ldots. \tag{9}$$

(9) implies that A is a wandering set of positive measure for the transformation φ^{kq} for some $k > 0$; together with (3) we obtain a contradiction to (1). A similar argument for $k < 0$ establishes v).

We next show vi). For any measurable set A with $0 < m(A) < \infty$ let

$$A_n = \bigcup_{i=0}^{n} \varphi^i A \quad \text{for} \quad n = 0, 1, 2, \ldots$$

and

$$B_n = A_n - A_{n-1} \quad \text{for} \quad n = 1, 2, \ldots.$$

We then have

$$A \supset \varphi^{-1} B_1 \supset \varphi^{-2} B_2 \supset \cdots,$$

and

$$B_i \cap B_j = \emptyset \quad \text{for} \quad i \neq j; \quad i, j = 1, 2, \ldots.$$

Let

$$C = \bigcap_{n=1}^{\infty} \varphi^{-n} B_n.$$

Since φ is $1-1$ and $\varphi^n C \subset B_n$ for $n = 1, 2, \ldots$ it follows that C is a wandering set. Since φ is recurrent and measure preserving, we conclude

$$\lim_{n\to\infty} m(B_n) = \lim_{n\to\infty} m(\varphi^{-n} B_n) = m\left(\bigcap_{n=1}^{\infty} \varphi^{-n} B_n\right) = m(C) = 0. \tag{10}$$

It follows also from the definitions of the sets A_n and B_n that

$$m(A_n) = m\left(\bigcup_{k=1}^{n} B_k\right) = \sum_{k=1}^{n} m(B_k). \tag{11}$$

Combining (10) and (11) we obtain

$$\lim_{n\to\infty} (1/n)\, m(A_n) = 0\,. \tag{12}$$

Now suppose A is a set of positive measure and weakly wandering under the sequence $W = \{w_i\}$. We note that any subset of the set A is again a weakly wandering set under the same sequence W, A is also weakly wandering under the sequence $-W$, and if $W = U \cup V$ where U and V both satisfy the conclusion of vi) then so does W. Thus, without loss of generality we may assume the following: A is a weakly wandering set under the sequence W, $0 < m(A) < \infty$, and $W = \{w_i\}$ is an increasing sequence of positive integers. We then have

$$\frac{1}{w_i}\, m\left(\bigcup_{k=1}^{w_i} \varphi^k A\right) \geqq \frac{1}{w_i}\, m\left(\bigcup_{k=1}^{i} \varphi^{w_k} A\right) = \frac{1}{w_i} \sum_{k=1}^{i} m\,(\varphi^{w_k} A) = \frac{i}{w_i}\, m(A).$$

Using (12) and the fact $m(A) > 0$ we conclude vi), and this completes the proof of the theorem.

Theorem 2. *Let φ be an ergodic measure preserving transformation defined on the measure space $(X, \mathscr{B}, m)$. A sequence $R = \{r_i \mid i = 1, 2, \ldots\}$ of integers is a recurrent sequence for the transformation φ if and only if there exists a set A of finite measure such that $\overline{\lim_{i\to\infty}}\, m(\varphi^{r_i} A \cap A) > 0$.*

Proof. Assume $R = \{r_i\}$ is not a recurrent sequence for the transformation φ. This means that there exists a weakly wandering sequence W such that $W \cap R = S$ has an infinite number of elements. By ii) of Theorem 1 we have that R contains a weakly wandering subsequence $S = \{s_i\}$. By Theorem 3 of [1] it follows that the sequence $S = \{s_i\}$ satisfies

$$\sum_{i=1}^{\infty} |f(\varphi^{s_i} x)| \quad \text{converges almost everywhere for all } f \in L^1(X)\,. \tag{13}$$

Let A be any set of finite measure, and let f_A be the characteristic function of the set A. From (13) follows

$$\lim_{i\to\infty} f_A(\varphi^{s_i} x) = 0 \quad \text{for almost all} \quad x \in X\,. \tag{14}$$

Applying the Bounded Convergence Theorem to (14) we obtain

$$\lim_{i\to\infty} m(\varphi^{s_i} A \cap A) = 0\,. \tag{15}$$

Since $R = \{r_i\}$ contains $S = \{s_i\}$, from (15) we conclude

$$\underline{\lim_{i\to\infty}}\, m(\varphi^{r_i} A \cap A) = 0 \quad \text{for any set } A \text{ of finite measure.} \tag{16}$$

To prove the converse, assume (16) is true, and let A and B be two measurable sets both of finite measure. Then

$$\underline{\lim_{i\to\infty}}\, m[\varphi^{r_i}(A \cup B) \cap (A \cup B)] \geqq \underline{\lim_{i\to\infty}}\, m(\varphi^{r_i} B \cap A) \tag{17}$$

Applying (16) to the set $A \cup B$ and using (17) we obtain

$$\underline{\lim_{i\to\infty}}\, m(\varphi^{r_i} B \cap A) = 0 \quad \text{for any pair of sets } A \text{ and } B \text{ of finite measure.} \tag{18}$$

We conclude the proof of the Theorem by proving the following Lemma; we note the similariry of this Lemma and the method of its proof to that of Lemma 4 of [2].

Lemma. *Let φ be a measurable non-singular transformation defined on the measure space $(X, \mathscr{B}, m)$, and let $R = \{r_i\}$ be a sequence of integers satisfying* (18). *Then there exists a weakly wandering subsequence $S = \{s_i\}$ of the sequence $R = \{r_i\}$.*

Proof. Let A be a measurable set with $0 < m(A) = a < \infty$, and let $\{\varepsilon_i \mid i = 1, 2, \ldots\}$ be a sequence of positive real numbers such that

$$\sum_{i=1}^{\infty} \varepsilon_i = \varepsilon < a\,.$$

Let $s_0 = 0$. Since the set A satisfies (16) we can choose an integer $s_1 \in R$ such that

$$m(\varphi^{s_1} A \cap A) < \varepsilon_1\,.$$

Suppose now the integers $s_1, s_2, \ldots, s_{k-1}$, all different, have been chosen from the sequence $R = \{r_i\}$. We next choose $s_k \in R$ such that s_k is different from the previously chosen $s_1, s_2, \ldots, s_{k-1}$, and

$$m\left(A \cap \bigcup_{i=0}^{k-1} \varphi^{s_k - s_i} A\right) < \varepsilon_k\,. \tag{19}$$

This is possible since (18) is true for the sets $B = \bigcup_{i=1}^{k-1} \varphi^{-s_i} A$ and A both of finite measure. Thus, we are able to choose inductively a sequence $S = \{s_i\}$ from the sequence $R = \{r_i\}$ such that (19) is satisfied for $k = 1, 2, \ldots$. It then follows that the set

$$W = A - \bigcup_{k=1}^{\infty} \bigcup_{i=0}^{k-1} \varphi^{s_k - s_i} A$$

is weakly wandering under the sequence $S = \{s_i \mid i = 1, 2, \ldots\}$. We also have

$$m(W) \geqq a - \sum_{i=1}^{\infty} \varepsilon_i = a - \varepsilon > 0\,.$$

Theorem 3. *Let φ be an ergodic measure preserving transformation defined on the measure space $(X, \mathscr{B}, m)$ with $m(X) = \infty$, and let $\mathscr{R} = \{R\}$ be the collection of all recurrent sequences $R = \{r_i \mid i = 1, 2, \ldots\}$ for the transformation φ. Then*

i) $R \in \mathscr{R}$ *implies* $-R \in \mathscr{R}$.
ii) $R \in \mathscr{R}$, $S \subset R$ *implies* $S \in \mathscr{R}$.
iii) $R \in \mathscr{R}$ implies $R + \{k\} \in \mathscr{R}$ *for any integer* $k = 0, \pm 1, \pm 2, \ldots$.
iv) $R \in \mathscr{R}$, $S \in \mathscr{R}$ *implies* $R \cup S \in \mathscr{R}$.
v) $R = \{r_i\} \in \mathscr{R}$ *implies* $\lim_{i \to \infty} (i/r_i) = 0$.

Proof. i), ii), iii), and iv) follow directly from the definitions and the properties of the collection $\mathscr{W}$ established in Theorem 1. We show v). Let A be a set of finite measure. Applying the Individual Ergodic Theorem to the characteristic function of the set A and applying the Bounded Convergence Theorem we obtain

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} m(\varphi^k A \cap A) = 0. \tag{20}$$

Let $R = \{r_i\} \in \mathscr{R}$. It follows from Theorem 2 that there exists a set A of finite measure such that

(21) $$\varliminf_{i\to\infty} m(\varphi^{r_i} A \cap A) > 0\,.$$

Without loss of generality, by possibly ignoring a finite number of elements of the sequence R and reindexing the remainder, we conclude from (21) that there exists a real number $\alpha > 0$ such that

(22) $$m(\varphi^{r_i} A \cap A) \geqq \alpha > 0 \quad \text{for all} \quad i = 1, 2, \ldots\,.$$

We further assume that the sequence $R = \{r_i\}$ is an increasing sequence of positive integers. Again this causes no loss of generality by i) and the fact that if $R = S \cup T$ where S and T satisfy the conclusion of v) then so does R. Combining (20) and (22) we obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} m(\varphi^k A \cap A) = \lim_{i\to\infty} \frac{1}{r_i} \sum_{k=1}^{r_i} m(\varphi^k A \cap A) \\ &\geqq \varlimsup_{i\to\infty} \frac{1}{r_i} \sum_{k=1}^{i} m(\varphi^{r_k} A \cap A) \geqq \varlimsup_{i\to\infty} \frac{i}{r_i} \alpha\,. \end{aligned}$$

Since $\alpha > 0$ we conclude v), and this completes the proof of the Theorem.

4. Some Extensions and Other Related Sequences

In this section we describe the collections $\mathscr{R} = \{R\}$ and $\mathscr{W} = \{W\}$ of all recurrent and weakly wandering sequences respectively for an ergodic measure preserving transformation φ in a different setting.

By an ultrafilter α of subsets E of the integers I we mean a non-empty collection of non-empty subsets E of I with the following two properties:

i) $E_1 \in \alpha$, $E_2 \in \alpha$ implies $E_1 \cap E_2 \in \alpha$.

ii) $E \subset I$ implies either $E \in \alpha$ or $E^c \in \alpha$.

We shall say that an ultrafilter α is free in case $\bigcap_{E\in\alpha} E = \emptyset$. We note that an ultrafilter α is free if and only if α contains no finite subsets of I.

We next let $\mathscr{A} = \{\alpha\}$ be the collection of all free ultrafilters as defined above. We consider the topology defined in $\mathscr{A}$ by the basic open sets of the form $\mathscr{O}(E)$ determined by the infinite subsets $E \subset I$ as follows; For any infinite subset $E \subset I$ we define the open set $\mathscr{O}(E)$ determined by E to be $\mathscr{O}(E) = \{\alpha \in \mathscr{A} \mid E \in \alpha\}$. It is easy to see that $\mathscr{O}(E) \subset \mathscr{O}(F)$ if and only if $E \cap F^c$ has a finite number of elements in I, and

(23) $$\mathscr{O}(E) \cap \mathscr{O}(F) = \emptyset$$

if and only if $E \cap F$ has only a finite number of elements in I.

For any open set $\mathscr{C} \subset \mathscr{A}$ we denote by $\mathscr{C}^* = (\overline{\mathscr{C}})^0$, the interior of the closure of the set $\mathscr{C}$. $\mathscr{C}^*$ is called the regularization of the open set $\mathscr{C}$, and $\mathscr{C}$ is said to be a regularly open set in case $\mathscr{C}^* = \mathscr{C}$. We also let $\mathscr{C}^\# = (\overline{\mathscr{C}})^c$, the complement of the closure of the set $\mathscr{C}$. $\mathscr{C}^\#$ is called the exterior of the set $\mathscr{C}$.

Theorem 4. *Let φ be an ergodic measure preserving transformation defined on the measure space $(X, \mathscr{B}, m)$ with $m(X) = \infty$. Let $\mathcal{O}(\mathscr{W}) = \bigcup_{W \in \mathscr{W}} \mathcal{O}(W)$ be the open set in $\mathscr{A}$ determined by the collection $\mathscr{W} = \{W\}$ of all weakly wandering sequences $W = \{w_i\}$ for φ, and let $\mathcal{O}(\mathscr{R}) = \bigcup_{R \in \mathscr{R}} \mathcal{O}(R)$ be the open set in $\mathscr{A}$ determined by the collection $\mathscr{R} = \{R\}$ of all recurrent sequences $R = \{r_i\}$ for φ; then*

i) $\mathcal{O}(\mathscr{R}) = \mathcal{O}(\mathscr{W})^{\#}$, *i.e. $\mathcal{O}(\mathscr{R})$ is the exterior of $\mathcal{O}(\mathscr{W})$.*
ii) $\mathcal{O}(\mathscr{R})^{\#} = \mathcal{O}(\mathscr{W})^{*}$, *i.e. the exterior of $\mathcal{O}(\mathscr{R})$ is the regularization of $\mathcal{O}(\mathscr{W})$.*
iii) $\mathcal{O}(\mathscr{R})^{*} = \mathcal{O}(\mathscr{R})$, *i.e. $\mathcal{O}(\mathscr{R})$ is a regularly open set.*
iv) $\mathcal{O}(\mathscr{R}) \neq \emptyset$ *if and only if φ is a transformation of positive type.*
v) $\mathcal{O}(\mathscr{W})^{*} = \bigcup_{W^* \in \mathscr{W}^*} \mathcal{O}(\mathscr{W}^*)$

where $\mathscr{W}^* = \{W^*\}$ is the collection of all sequences $W^* = \{w_i^*\}$ satisfying $\lim_{i \to \infty} m(\varphi^{w_i^*} A \cap A) = 0$ for any measurable set A with $m(A) < \infty$.

Proof. i) and ii) follow from (23) and the definitions of $\mathscr{R}$ and $\mathscr{W}$.

iii) is obtained from the following:

For any open sets $\mathscr{C}, \mathscr{D} \subset \mathscr{A}$ we have

$$\mathscr{C} \subset \mathscr{C}^{\#\#}, \tag{24}$$

and

$$\mathscr{C} \subset \mathscr{D} \quad \text{implies} \quad \mathscr{D}^{\#} \subset \mathscr{C}^{\#}. \tag{25}$$

Putting (24) and (25) together we obtain

$$\text{if} \quad \mathscr{D} = \mathscr{C}^{\#} \quad \text{then} \quad (\mathscr{D}^{\#})^{\#} = \mathscr{D}. \tag{27}$$

From (27) and properties i) and ii) for $\mathcal{O}(\mathscr{R})$ we conclude iii). To show iv) we note that from Theorem 2 follows that the collection $\mathscr{R} = \{R\}$ of all recurrent sequences $R = \{r_i\}$ for φ is not empty if and only if for some sequence $R = \{r_i\}$ of integers the following condition is satisfied:

(28) There exists a set A with $m(A) < \infty$ such that $\varliminf_{i \to \infty} m(\varphi^{r_i} A \cap A) > 0$.

Also recalling the definition of a transformation φ of positive type we have

$$m(A) > 0 \quad \text{implies} \quad \varlimsup_{n \to \infty} m(\varphi^{n} A \cap A) > 0. \tag{29}$$

It is easy to conclude that the transformation φ satisfies (29) if and only if there exists a sequence $R = \{r_i\}$ for φ satisfying (28). This proves iv).

v) follows from the definition of $\mathcal{O}(\mathscr{W})^*$ and the fact that if a sequence $V = \{v_i\}$ of integers satisfies the condition

$$\varlimsup_{i \to \infty} m(\varphi^{v_i} A \cap A) > 0 \quad \text{for some set } A \text{ with} \quad m(A) < \infty,$$

then it is possible to obtain a subsequence $R = \{r_i\}$ of the sequence $V = \{v_i\}$ which satisfies (28). This completes the proof of the Theorem.

We note that the sets $\mathcal{O}(\mathscr{R})$, $\mathcal{O}(\mathscr{W})$, $\mathcal{O}(\mathscr{W})^*$ associated with a given ergodic measure preserving transformation φ defined on an infinite measure space $(X, \mathscr{B}, m)$

are invariants for the transformation φ under spatial isomorphism. In particular condition iv) of Theorem 4 gives a characterization of ergodic measure preserving transformations of positive and zero type. Stating the above condition in a different way we obtain the following

Corollary. *An ergodic measure preserving transformation φ defined on an infinite measure space $(X, \mathscr{B}, m)$ is of zero type if and only if the set $\mathcal{O}(\mathscr{W})$ in $\mathscr{A}$ associated with the transformation φ has the property $\mathcal{O}(\mathscr{W})^* = \mathscr{A}$.*

In order to have an idea of the relative sizes of the sets $\mathcal{O}(\mathscr{W})$ and $\mathcal{O}(\mathscr{W})^*$, we introduce the following collections $\mathscr{V}_n = \{V^n\}$, $n = 1, 2, \ldots, 6$, of subsets of the integers I:

For each $n = 1, 2, \ldots, 6$ let $\mathscr{V}_n = \{V^n\}$ be a collection of sequences $V^n = \{v_i\}$ of integers satisfying conditions $n = 1, 2, \ldots, 6$ respectively as listed below;

Condition 1. $0 < m(A) < \infty$ implies $\varphi^{v_i} x \in A$ for at most a finite number of i's and for almost all $x \in X$.

Condition 2. $f_A =$ characteristic function of a set A with $m(A) < \infty$ implies $\sum_{i=1}^{\infty} f_A(\varphi^{v_i} x) < \infty$ a.e.

Condition 3. $f_A =$ characteristic function of a set A with $m(A) < \infty$ implies $f_A(\varphi^{v_i} x) \to 0$ a.e.

Condition 4. $f \in L^1(X)$ implies $f(\varphi^{v_i} x) \to 0$ a.e.

Condition 5. $f \in L^1(X)$ implies $\sum_{i=1}^{\infty} |f(\varphi^{v_i} x)|$ converges a.e.

Condition 6. There exists a set A with $m(A) > 0$ such that $f \in L^1(X)$ implies $\int_A \sum_{i=1}^{\infty} |f(\varphi^{v_i} x)| \, dm$ is finite.

We next associate with a given measurable non-singular transformation φ defined on a measure space $(X, \mathscr{B}, m)$ the following subsets $\mathcal{O}(\mathscr{V}_n)$ of $\mathscr{A}$ for $n = 1, 2, \ldots, 6$ respectively:

$$\text{For each } n = 1, 2, \ldots, 6 \text{ let } \mathcal{O}(\mathscr{V}_n) = \bigcup_{V^n \in \mathscr{V}_n} \mathcal{O}(V^n) \text{ where } \mathscr{V}_n = \{V^n\}$$

is the collection of subsets V^n of the integers I as defined above. We now state the following

Theorem 5.

$$\mathcal{O}(\mathscr{W}) \subset \mathcal{O}(\mathscr{V}_6) \subset \mathcal{O}(\mathscr{V}_5) \subset \mathcal{O}(\mathscr{V}_4) = \mathcal{O}(\mathscr{V}_3) = \mathcal{O}(\mathscr{V}_2) = \mathcal{O}(\mathscr{V}_1) \subset \mathcal{O}(\mathscr{W})^*.$$

Proof. The proof of $\mathcal{O}(\mathscr{W}) \subset \mathcal{O}(\mathscr{V}_6)$ is contained in Section 3 of [1]. Namely, in the proof of Theorem 3 of [1], in showing that every weakly wandering sequence $W = \{w_i\}$ satisfies Condition 5, we showed a stronger result; in fact, any set of positive measure which is weakly wandering under the sequence $W = \{w_i\}$ can serve as the set A in the above condition 6.

We next prove only $\mathcal{O}(\mathscr{V}_4) \supset \mathcal{O}(\mathscr{V}_3)$, the remaining relations follow from the definitions.

Suppose Condition 4 is not satisfied by a sequence $V = \{v_i\}$ of integers. Then for some $f \in L^1(X)$, $f \geqq 0$ a.e. and some real number $\alpha > 0$ there exists a set B with $m(B) > 0$ such that

$$\overline{\lim_{i\to\infty}} f(\varphi^{v_i} x) \geqq \alpha > 0 \quad \text{for all} \quad x \in B. \tag{30}$$

For any real number α' with $0 < \alpha' \leqq \alpha$ we let

$$A_{\alpha'} = \{x \,|\, f(x) \geqq \alpha' > 0\}.$$

Then $f \in L^1(X)$ implies $m(A_{\alpha'}) < \infty$, and (30) implies that for all $x \in B$, $\varphi^{v_i} x \in A_{\alpha'}$ for infinitely many i. Furthermore, (30) implies $f \not\equiv 0$ a.e.; or saying all this in different words, there exists a real number α' with $0 < \alpha' \leqq \alpha$ such that $0 < m(A_{\alpha'}) < \infty$. This shows that the sequence $V = \{v_i\}$ does not satisfy Condition 3 which in turn proves $\mathcal{O}(\mathcal{V}_4) \supset \mathcal{O}(\mathcal{V}_3)$.

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