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A Non-Uniform Estimate of the Rate of Convergence in the Central Limit Theorem for *m*-Dependent Random Fields

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Summary. A non-uniform estimate of the rate of convergence in the central limit theorem for m-dependent random fields is obtained extending the work of Maejima (1978) for m-dependent random variables.

1. Introduction

Let $Z^k=\{\mathbf{z}: \mathbf{z}=(z_1, z_2, \dots, z_k), z_i \in Z$, the set of integers$\}$ and for any $\mathbf{z}\in Z^k$, define $\|\mathbf{z}\|=\max\{|z_i|, 1\leqq i\leqq k\}$. For any subsets V_1 and V_2 in Z^k, define $d(V_1, V_2) =\inf\{\|\mathbf{z}-\mathbf{w}\|: \mathbf{z}\in V_1, \mathbf{w}\in V_2\}$. Consider the random field $\{\xi(\mathbf{z})\}$ with index $\mathbf{z}\in Z^k$. Suppose that $E[\xi(\mathbf{z})]=0$ and $\operatorname{Var}[\xi(\mathbf{z})]<\infty$ for all $\mathbf{z}\in Z^k$. Let $m(V)$ be the smallest σ-algebra with respect to which $\{\xi(\mathbf{z}): \mathbf{z}\in V\}$ are measurable for any subset $V\subset Z^k$. The random field $\{\xi(\mathbf{z}): \mathbf{z}\in Z^k\}$ is said to be m-dependent if the σ-algebras $m(V_1)$ and $m(V_2)$ are independent whenever $d(V_1, V_2)>m$.

Central limit theorems for m-dependent random fields were obtained in Rosen (1969) and Zolotukhina and Chugueva (1973). An estimate of the rate of convergence in the central limit theorem for m-dependent random fields is obtained by Leonenko (1975). Our aim in this paper is to obtain a nonuniform estimate for the rate of convergence in the central limit theorem for m-dependent random fields. Recently Maejima (1978) obtained a non-uniform estimate in the central limit theorem for m-dependent random variables and Shergin (1976) obtained a uniform estimate for the same problem.

2. Some Lemmas

Lemma 1. *For any two random variables X and Y such that $0<E(X^2)<\infty$, $0<E(X+Y)^2<\infty$, the following inequality holds:*

$$\left|\frac{E(X^2)}{E(X+Y)^2}-1\right|\leqq 2\left\{\frac{E(Y^2)}{E(X+Y)^2}\right\}^{\frac{1}{2}}.$$

Proof. Obvious.

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Lemma 2. *If, for a sequence of random variables* $X_1, X_2, \ldots, X_k$, $E|X_i|^{2+\delta}<\infty$, $EX_i=0$, $1\leqq i\leqq k$ *for some* $\delta\geqq 0$, *then*

$$E|X_1+\ldots+X_k|^{2+\delta}\leqq k^{1+\delta}\sum_{j=1}^{k} E|X_j|^{2+\delta}.$$

In particular, if $\sup_i E|X_i|^{2+\delta}\leqq M<\infty$, *then*

$$E|X_1+\ldots+X_k|^{2+\delta}\leqq M k^{2+\delta}.$$

Proof. Follows from extension of C_r-inequality (cf. Loève (1963)).

Lemma 3. *Let* $X_1, X_2, \ldots, X_n$, *be an m-dependent sequence of random variables with* $EX_i=0$ *and* $E|X_i|^{2+\delta}<\infty$ *for some* $\delta\geqq 0$. *Then there exists a constant* $C_\delta>0$ *such that*

$$E|X_1+\ldots+X_n|^{2+\delta}\leqq C_\delta(m+1)^{1+\delta/2}\, n^{\delta/2}\sum_{j=1}^{n} E|X_j|^{2+\delta}$$

for all n. In particular, if $\sup E|X_j|^{2+\delta}<\infty$, *then*

$$E|X_1+\ldots+X_n|^{2+\delta}\leqq C'_\delta n^{1+\delta/2}$$

for all $n\geqq 1$.

Proof. Follows from a result in Shergin (1976).

Lemma 4. *Let X and Y be random variables and F(x) and H(x) be the distribution functions of X and X+Y respectively. Let* $\alpha>0$. *If*

$$|F(x)-\Phi(x)|\leqq\frac{K}{(1+|x|)^\alpha}$$

where $K>0$, *then for any* $0<\varepsilon<\frac{1}{2}$ *and for all x, there exists* $C>0$ *such that*

$$|H(x)-\Phi(x)|\leqq\frac{C}{(1+|x|)^\alpha}\left(K+\varepsilon+\frac{E|Y|^\alpha}{\varepsilon^\alpha}\right)$$

where $\Phi(\cdot)$ *is the standard normal distribution function.*

Proof. See Maejima (1978).

3. Main Theorem

Let $\{\xi(\mathbf{z})\colon \mathbf{z}\in Z^k\}$ be an m-dependent random field as defined in Sect. 1. Let

$$S_{n_1,n_2,\ldots,n_k}=\begin{Bmatrix}1\leqq z_i\leqq n_i\\ 1\leqq i\leqq k\end{Bmatrix}\Sigma\,\xi(\mathbf{z}) \tag{3.0}$$

where $\{\ldots\}$ denotes that the summation later is carried out over the set of indices contained in $\{\ldots\}$. Note that

$$E(S_{n_1,n_2,\ldots,n_k})=0. \tag{3.1}$$

Define

$$B_{n_1,\ldots,n_k}=\operatorname{Var}(S_{n_1,n_2,\ldots,n_k}) \tag{3.2}$$

and

$$F_{n_1,\ldots,n_k}(x)=P\left\{\frac{S_{n_1,n_2,\ldots,n_k}}{\sqrt{B_{n_1,n_2,\ldots,n_k}}}\leqq x\right\}. \tag{3.3}$$

Let

$$\Delta_{n_1,\ldots,n_k}(x)=|F_{n_1,\ldots,n_k}(x)-\Phi(x)| \tag{3.4}$$

and $\tilde{n}=\min(n_1,\ldots,n_k)$.

Theorem 1. *Suppose the following conditions are satisfied:*

(i)
$$E|\xi(\mathbf{z})|^{2+\delta}\leqq M<\infty \tag{3.5}$$

for some $0<\delta<\delta_0$ *and for all* $\mathbf{z}\in Z^k$ *where* δ_0 *is the positive root of* $\delta^2+2\delta-2=0$.

(ii)
$$\liminf_{\tilde{n}\to\infty}\frac{B_{n_1,\ldots,n_k}}{n_1\cdot\ldots\cdot n_k}>0. \tag{3.6}$$

Then there exists a constant $C>0$ *independent of* x *and* n_i, $1\leqq i\leqq k$ *such that*

$$\Delta_{n_1,\ldots,n_k}(x)\leqq\frac{C}{(1+|x|)^{2+\delta}}\{\min(n_i,1\leqq i\leqq k)\}^{-\nu} \tag{3.7}$$

where

$$\nu=\frac{\delta(\delta+2)}{2(\delta^2+4\delta+2)}.$$

Before we give a proof of this theorem, we shall first state and prove an extension of Lemma 3 to m-dependent random fields.

Lemma 5. *Let* $\{\xi(\mathbf{z}): \mathbf{z}\in Z^k\}$ *be an m-dependent random field as defined above and* $S_{\mathbf{n}}\equiv S_{n_1,\ldots,n_k}$ *be given by* (3.0). *Suppose that* (3.5) *holds for some* $\delta\geqq 0$. *Then there exists constant* $C_\delta>0$ *such that*

$$E|S_{\mathbf{n}}|^{2+\delta}\leqq C_\delta\left(\prod_{i=1}^{k}n_i\right)^{\frac{2+\delta}{2}}. \tag{3.8}$$

Proof. We shall prove the lemma in case $k=2$. The general proof is similar.

Let $k_0>m$. Applying Lemma 3, we can find a constant $B>0$ such that

$$E|S_{\mathbf{n}}|^{2+\delta}\leqq B(n_1 n_2)^{1+\delta/2} \tag{3.9}$$

for all $n_1\geqq 1$ and $n_2\leqq 2^{k_0}$. Suppose that (3.9) holds for some $n_2\geqq 2^{k_0}$. We shall show that it holds for $2n_2$. Let

$$T_1=S_{n_1,n_2},$$
$$T_2=S_{n_1,2n_2+k_0}-S_{n_1,n_2+k_0},$$
$$R_1=S_{n_1,n_2+k_0}-S_{n_1,n_2},$$

and

$$R_2 = S_{n_1, 2n_2} - S_{n_1, 2n_2+k_0}.$$

Then $S_{n_1, 2n_2} = T_1 + T_2 + R_1 + R_2$. Hence, by Lemma 2,

$$E|S_{n_1, 2n_2}|^{2+\delta} \leqq 4^{1+\delta}\{E|T_1|^{2+\delta} + E|T_2|^{2+\delta} + E|R_1|^{2+\delta} + E|R_2|^{2+\delta}\}. \quad (3.10)$$

In view of (3.9) and the fact that $k_0 \leqq 2^{k_0}$ and $n_2 \geqq 2^{k_0}$, it follows that

$$E|R_1|^{2+\delta} \leqq B(n_1 k_0)^{1+\delta/2},$$
$$E|R_2|^{2+\delta} \leqq B(n_1 k_0)^{1+\delta/2},$$
$$E|T_1|^{2+\delta} \leqq B(n_1 n_2)^{1+\delta/2},$$

and

$$E|T_2|^{2+\delta} \leqq B(n_1 n_2)^{1+\delta/2}.$$

Hence

$$E|S_{n_1, 2n_2}|^{2+\delta} \leqq B^*(n_1(2n_2))^{1+\delta/2}.$$

Thus, (3.9) holds for all $n_1 \geqq 1$ and all n_2 of the form 2^r, $1 \leqq r < \infty$. The general result follows now by writing n_2 as sums of powers of 2 and applying Lemma 2.

Remark. The argument given above is akin to the proof of a similar lemma in Deo (1975).

4. Proof of Theorem 1

We shall now prove the main theorem in case $k=2$. The general proof is similar but more complex in notation.

Let $k_j = [n_j^\alpha]$, $0 < \alpha < \frac{1}{2}$, $j = 1, 2$. Define h_j and r_j by the relations $n_j = k_j h_j + r_j$, $0 \leqq r_j < k_j$, $j = 1, 2$. Let

$$\Delta_{i_j}^{(j)} = \{z_j\colon (i_j - 1)k_j + 1 \leqq z_j \leqq i_j k_j - m\}, \quad 1 \leqq i_j \leqq h_j,\ j = 1, 2,$$
$$\bar{\Delta}_{i_j}^{(j)} = \{z_j\colon (i_j k_j - m + 1 \leqq z_j \leqq i_j k_j\}, \quad 1 \leqq i_j \leqq h_j,\ j = 1, 2,$$

and

$$\Delta_{h_j+1}^{(j)} = \{z_j\colon k_j h_j + 1 \leqq z_j \leqq k_j h_j + r_j\}, \quad j = 1, 2.$$

For large $n = (n_1, n_2)$, define

$$Y_{i_1, i_2} = \{z_j \in \Delta_{i_j}^{(j)}, j = 1, 2\} \sum \xi(\mathbf{z}), \quad (4.1)$$
$$W_{i_1, i_2}^{(1)} = \{z_1 \in \Delta_{i_1}^{(1)}, z_2 \in \bar{\Delta}_{i_2}^{(2)}\} \sum \xi(\mathbf{z}), \quad (4.2)$$
$$W_{i_1, i_2}^{(2)} = \{z_1 \in \bar{\Delta}_{i_1}^{(1)}, z_2 \in \Delta_{i_2}^{(2)}\} \sum \xi(\mathbf{z}), \quad (4.3)$$
$$W_{i_1, i_2}^{(3)} = \{z_j \in \bar{\Delta}_{i_j}^{(j)}, j = 1, 2\} \sum \xi(\mathbf{z}), \quad (4.4)$$

for $1 \leqq i_1 \leqq h_1$, $1 \leqq i_2 \leqq h_2$ and

$$b_{h_1, h_2} = E\left(\sum_{i_1=1}^{h_1} \sum_{i_2=1}^{h_2} Y_{i_1, i_2}\right)^2, \quad (4.5)$$

$$X_{h_1,h_2}=\frac{1}{\sqrt{b_{h_1,h_2}}}\sum_{i_1=1}^{h_1}\sum_{i_2=1}^{h_2}Y_{i_1,i_2}, \tag{4.6}$$

$$\eta^{(1)}_{h_1,h_2}=\frac{1}{\sqrt{B_{n_1,n_2}}}\sum_{i_1=1}^{h_1}\sum_{i_2=1}^{h_2}\sum_{j=1}^{3}W^{(j)}_{i_1,i_2}, \tag{4.7}$$

$$\eta^{(2)}_{h_1,h_2}=\frac{1}{\sqrt{B_{n_1,n_2}}}(S_{n_1,n_2}-S_{h_1k_1,h_2k_2}) \tag{4.8}$$

and

$$\eta^{(3)}_{h_1,h_2}=\frac{1}{\sqrt{B_{n_1,n_2}}}-\frac{1}{\sqrt{b_{h_1,h_2}}}\sum_{i_1=1}^{h_1}\sum_{i_2=1}^{h_2}Y_{i_1,i_2}. \tag{4.9}$$

Let

$$\eta_{h_1,h_2}=\eta^{(1)}_{h_1,h_2}+\eta^{(2)}_{h_1,h_2}+\eta^{(3)}_{h_1,h_2}. \tag{4.10}$$

Observe that

$$\frac{1}{\sqrt{B_{n_1,n_2}}}S_{n_1,n_2}=X_{h_1,h_2}+\eta_{h_1,h_2}.$$

Let $U_{h_1,h_2}(x)$ be the distribution function of X_{h_1,h_2}. By a theorem of Bikelis (1966),

$$|U_{h1,h_2}(x)-\Phi(x)|\leqq\frac{C_1}{(1+|x|)^{2+\delta}}\frac{\sum_{i_1=1}^{h_1}\sum_{i_2=1}^{h_2}E|Y_{i_1,i_2}|^{2+\delta}}{(b_{h_1,h_2})^{1+\delta/2}} \tag{4.11}$$

where C_1 is a constant independent of x, h_1 and h_2 since Y_{i_1,i_2}, $1\leqq i_1\leqq h_1$ and $1\leqq i_2\leqq h_2$ are independent random variables for large n_1 and n_2. Let

$$W_{i_1,i_2}=W^{(1)}_{i_1,i_2}+W^{(2)}_{i_1,i_2}+W^{(3)}_{i_1,i_2}. \tag{4.12}$$

By independence of W_{i_1,i_2}, $1\leqq i_1\leqq h_1$, $1\leqq i_2\leqq h_2$ and the fact that $E(W_{i_1,i_2})=0$, it follows that

$$E\left(\sum_{i_1=1}^{h_1}\sum_{i_2=1}^{h_2}W_{i_1,i_2}\right)^2=\sum_{i_1=1}^{h_1}\sum_{i_2=1}^{h_2}E\{W_{i_1,i_2}\}^2$$
$$\leqq 3\sum_{i_1=1}^{h_1}\sum_{i_2=1}^{h_2}\sum_{j=1}^{3}E(W^{(j)}_{i_1,i_2})^2$$

by Lemma 2. But

$$E(W^{(1)}_{i_1,i_2})^2\leqq C_2(k_1-m)m, \qquad E(W^{(2)}_{i_1,i_2})^2\leqq C_2 m(k_2-m)$$

and

$$E(W^{(3)}_{i_1,i_2})^2\leqq C_2 m^2$$

for some constant $C_2>0$ by Lemma 5 (note that the lemma holds for $\delta=0$). Hence

$$E(W^{(j)}_{i_1,i_2})^2\leqq C_2'\max(k_1,k_2), \qquad 1\leqq j\leqq 3 \tag{4.13}$$

for sufficiently large n_1 and n_2. In view of Lemma 5 and condition (i)

$$E|Y_{i_1,i_2}|^{2+\delta} \leqq C_3(k_1 k_2)^{1+\delta/2} \tag{4.14}$$

uniformly in i_1 and i_2 where C_3 is a positive constant. Therefore

$$|U_{h_1,h_2}(x)-\Phi(x)| \leqq \frac{C_4 h_1 h_2 (k_1 k_2)^{1+\delta/2}}{(1+|x|)^{2+\delta}(b_{h_1,h_2})^{1+\delta/2}} \tag{4.15}$$

where C_4 is independent of x, h_1, h_2, k_1 and k_2 by (4.11). Now

$$\begin{aligned} b_{h_1,h_2} &= E\left[S_{h_1k_1,h_2k_2} - \sum_{i_1=1}^{h_1}\sum_{i_2=1}^{h_2} W_{i_1,i_2}\right]^2 \\ &= B_{h_1k_1,h_2k_2}\left[1+O\left(\left\{\frac{E\left(\sum_{i_1=1}^{h_1}\sum_{i_2=1}^{h_2} W_{i_1,i_2}\right)^2}{B_{h_1k_1,h_2k_2}}\right\}^{\frac{1}{2}}\right)\right] \end{aligned} \tag{4.16}$$

by Lemma 1. Condition (ii) implies that

$$B_{n_1,n_2} \geqq a^2 n_1 n_2 \qquad \text{for some } a>0 \tag{4.16a}$$

for sufficiently large n_1 and n_2. Therefore, by (4.13), it follows that

$$\begin{aligned} b_{h_1,h_2} &= B_{h_1k_1,h_2k_2}\left[1+O\left(\frac{h_1 h_2 \max(k_1,k_2)}{h_1 h_2 k_1 k_2}\right)^{\frac{1}{2}}\right] \\ &= B_{h_1k_1,h_2k_2}\left[1+O\left(\frac{1}{\min(\sqrt{k_1},\sqrt{k_2})}\right)\right]. \end{aligned} \tag{4.17}$$

Hence, there exists a constant $C_5>0$ such that

$$b_{h_1,h_2} \geqq C_5 h_1 k_1 h_2 k_2 \tag{4.18}$$

for sufficiently large n_1 and n_2. Clearly

$$\begin{aligned} E(\eta^{(1)}_{h_1,h_2})^2 &= \frac{1}{B_{n_1,n_2}} E\left(\sum_{i_1=1}^{h_1}\sum_{i_2=1}^{h_2} W_{i_1,i_2}\right)^2 \\ &\leqq C_6 \frac{h_1 h_2 \max(k_1,k_2)}{n_1 n_2} \qquad \text{(by (4.12) and (4.13))} \\ &\leqq C_6 \frac{\max(k_1,k_2)}{k_1 k_2} \end{aligned} \tag{4.19}$$

and

$$E(\eta^{(2)}_{h_1,h_2})^2 = \frac{1}{B_{n_1,n_2}} E(S_{n_1,n_2} - S_{h_1k_1,h_2k_2})^2 \tag{4.20}$$

and the last term is bounded by

$$C_7\left(\frac{k_1}{n_1}+\frac{k_2}{n_2}\right) \tag{4.21}$$

for some constant $C_7>0$ by Lemma 5 and (4.16a). This can be seen by the following argument. Note that

$$S_{n_1,n_2}-S_{h_1k_1,h_2k_2}\equiv A_1+A_2$$

where

$$A_1=\{1\leqq i\leqq n_1, j\in\Delta^{(2)}_{h_2+1}\}\sum\xi(i,j)$$

and

$$A_2=\{i\in\Delta^{(1)}_{h_1+1}, 1\leqq j\leqq h_2k_2\}\sum\xi(i,j).$$

Then

$$E(S_{n_1,n_2}-S_{h_1k_1,h_2k_2})^2\leqq 2(EA_1^2+EA_2^2).$$

Since $\xi(\mathbf{z})$ is an m-dependent random field, Lemma 5 implies that

$$EA_1^2\leqq C_6' n_1 r_2\leqq C_6' n_1 k_2$$

and

$$EA_2^2\leqq C_6' r_1 h_2 k_2\leqq C_6' k_1 n_2.$$

Combining these inequalities with (4.16a), we obtain the bound given in (4.21). Note that

$$\begin{aligned}\frac{B_{h_1k_1,h_2k_2}}{B_{n_1,n_2}}&=E\left(\frac{S_{n_1,n_2}}{\sqrt{B_{n_1,n_2}}}-\eta^{(2)}_{h_1,h_2}\right)^2\\&=1+E(\eta^{(2)}_{h_1,h_2})^2-2E\left(\frac{S_{n_1,n_2}}{\sqrt{B_{n_1,n_2}}}\eta^{(2)}_{h_1,h_2}\right)\\&=1+E(\eta^{(2)}_{h_1,h_2})^2+O([E(\eta^{(2)}_{h_1,h_2})^2]^{\frac{1}{2}})\\&\leqq 1+C_9\max\left(\frac{\sqrt{k_1}}{\sqrt{n_1}},\frac{\sqrt{k_2}}{\sqrt{n_2}}\right)\end{aligned}\tag{4.22}$$

by (4.21) for some constant $C_9>0$. Therefore

$$\begin{aligned}\frac{b_{h_1,h_2}}{B_{n_1,n_2}}&\leqq\left(1+C_9\max\left(\frac{\sqrt{k_1}}{\sqrt{n_1}},\frac{\sqrt{k_2}}{\sqrt{n_2}}\right)\right)\left(1+\frac{C_{10}}{\min(\sqrt{k_1},\sqrt{k_2})}\right)\\&\leqq 1+\frac{C_{11}}{\min(\sqrt{k_1},\sqrt{k_2})}\end{aligned}\tag{4.23}$$

by (4.17) and (4.22) for some constant $C_{11}>0$ since $0<\alpha<\frac{1}{2}$.

In view of (4.15) and (4.18), it follows that

$$\begin{aligned}|U_{h_1,h_2}(x)-\Phi(x)|&\leqq\frac{C_{12}}{(1+|x|)^{2+\delta}}\frac{h_1h_2(k_1k_2)^{1+\delta/2}}{(h_1k_1h_2k_2)^{1+\delta/2}}\\&=\frac{C_{12}}{(1+|x|)^{2+\delta}}\frac{1}{(h_1h_2)^{\delta/2}}.\end{aligned}\tag{4.24}$$

By Lemma 4, for every $0<\varepsilon<\frac{1}{2}$,

$$\left|P\left\{\frac{S_{n_1,n_2}}{\sqrt{B_{n_1,n_2}}}\leqq x\right\}-\Phi(x)\right|\leqq\frac{C_{13}}{(1+|x|)^{2+\delta}}\left\{C_{12}(h_1h_2)^{-\delta/2}+\varepsilon+\frac{E|\eta_{h_1,h_2}|^{2+\delta}}{\varepsilon^{2+\delta}}\right\}.\tag{4.25}$$

We shall now estimate $E|\eta_{h_1,h_2}|^{2+\delta}$. It is clear that

$$E|\eta_{h_1,h_2}|^{2+\delta} \leqq 3^{1+\delta} \sum_{j=1}^{3} E|\eta_{h_1,h_2}^{(j)}|^{2+\delta} \tag{4.26}$$

by the C_r-inequality (Lemma 2). Now

$$\begin{aligned} E|\eta_{h_1,h_2}^{(3)}|^{2+\delta} &\leqq \left|\frac{1}{\sqrt{B_{n_1,n_2}}} - \frac{1}{\sqrt{b_{h_1,h_2}}}\right|^{2+\delta} (h_1 h_2)^{\delta/2} \sum_{i_1=1}^{h_1} \sum_{i_2=1}^{h_2} E|Y_{i_1,i_2}|^{2+\delta} \\ &\quad \text{(by the Marcinkiewicz-Zygmund inequality)} \\ &\leqq \frac{C_{14}}{b_{h_1,h_2}^{\frac{2+\delta}{2}}} \left|\sqrt{\frac{b_{h_1,h_2}}{B_{n_1,n_2}}} - 1\right|^{2+\delta} (h_1 h_2)^{\delta/2} (k_1 k_2)^{1+\delta/2} h_1 h_2 \quad \text{(by (4.14))} \\ &\leqq \frac{C_{15}}{(h_1 h_2 k_1 k_2)^{\frac{2+\delta}{2}}} \left\{\frac{1}{\min(\sqrt{k_1}, \sqrt{k_2})}\right\}^{2+\delta} (h_1 h_2)^{1+\delta/2} (k_1 k_2)^{1+\delta/2} \\ &\leqq C_{16} \left\{\frac{1}{\min(\sqrt{k_1}, \sqrt{k_2})}\right\}^{2+\delta}. \end{aligned} \tag{4.27}$$

On the other hand

$$\begin{aligned} E|\eta_{h_1,h_2}^{(2)}|^{2+\delta} &= \frac{1}{B_{n_1,n_2}^{(2+\delta)/2}} E|S_{n_1,n_2} - S_{h_1k_1,h_2k_2}|^{2+\delta} \\ &\leqq C_{17} \left\{\max\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}\right)\right\}^{1+\delta/2} \leqq C_{18} \left\{\frac{1}{\min(\sqrt{k_1}, \sqrt{k_2})}\right\}^{2+\delta} \end{aligned} \tag{4.28}$$

by arguments similar to those given in deriving (4.21) since $0<\alpha<\frac{1}{2}$. Further more

$$\begin{aligned} E|\eta_{h_1,h_2}^{(1)}|^{2+\delta} &= \frac{1}{B_{n_1,n_2}^{1+\delta/2}} E\left|\sum_{i_1=1}^{h_1} \sum_{i_2=1}^{h_2} \sum_{j=1}^{3} W_{i_1,i_2}^{(j)}\right|^{2+\delta} \\ &\leqq \frac{C_{19}}{(n_1 n_2)^{1+\delta/2}} (h_1 h_2)^{\delta/2} \sum_{i_1=1}^{h_1} \sum_{i_2=1}^{h_2} \sum_{j=1}^{3} E|W_{i_1,i_2}^{(j)}|^{2+\delta} \\ &\quad \text{(by the Marcinkiewicz-Zygmund inequality and Lemma 2).} \end{aligned} \tag{4.29}$$

But, by Lemma 5,

$$E|W_{i_1,i_2}^{(1)}|^{2+\delta} \leqq C_{20}[(k_1-m)m]^{1+\delta/2},$$

$$E|W_{i_1,i_2}^{(2)}|^{2+\delta} \leqq C_{21}[m(k_2-m)]^{1+\delta/2}$$

and

$$E|W_{i_1,i_2}^{(3)}|^{2+\delta} \leqq C_{22}(m^2)^{1+\delta/2}.$$

Hence

$$\begin{aligned} E|\eta_{h_1,h_2}^{(1)}|^{2+\delta} &\leqq \frac{C_{23}}{(n_1 n_2)^{1+\delta/2}} (h_1 h_2)^{1+\delta/2} [\max(k_1, k_2)]^{1+\delta/2} \\ &\leqq C_{24} \left[\frac{1}{\min(\sqrt{k_1}, \sqrt{k_2})}\right]^{2+\delta}. \end{aligned}$$

Combining the above inequalities, we have

$$E|\eta_{h_1,h_2}|^{2+\delta} \leqq C_{25}\left(\frac{1}{\min(k_1,k_2)}\right)^{1+\delta/2}. \tag{4.30}$$

Therefore

$$\left|P\left(\frac{S_{n_1,n_2}}{\sqrt{B_{n_1,n_2}}} \leqq x\right) - \Phi(x)\right| \leqq \frac{C_{26}}{(1+|x|)^{2+\delta}}\left\{(h_1 h_2)^{-\delta/2} + \varepsilon + \frac{1}{\varepsilon^{2+\delta}(\min(k_1,k_2))^{1+\delta/2}}\right\} \tag{4.31}$$

for some constant $C_{26}>0$ by (4.25). Since $k_i \simeq n_i^{\alpha}$, it follows that $h_i \simeq n_i^{1-\alpha}$ and the right hand side of (4.31) is of the order

$$(n_1 n_2)^{-\delta/2(1-\alpha)} + \varepsilon + \varepsilon^{-(2+\delta)} \frac{1}{[\min(n_1,n_2)]^{(1+\delta/2)\alpha}}.$$

If

$$\alpha = \frac{\delta^2+3\delta}{\delta^2+4\delta+2} \quad \text{where} \quad \alpha < \frac{1}{2} \quad \text{and} \quad \varepsilon = \tilde{n}^{-\frac{\delta}{2}\left(\frac{\delta+2}{\delta^2+4\delta+2}\right)} \equiv \tilde{n}^{-\nu},$$

then

$$\left|P\left(\frac{S_{n_1,n_2}}{\sqrt{B_{n_1,n_2}}} \leqq x\right) - \Phi(x)\right| \leqq \frac{C_{26}}{(1+|x|)^{2+\delta}} \frac{1}{[\min(n_1,n_2)]^{\nu}} \tag{4.32}$$

where $\nu = \dfrac{\delta^2+2\delta}{2(\delta^2+4\delta+2)}$.

Remarks. The rate obtained above may not be the best as for as the power ν is concerned. However, it is not possible in general to replace $\min(n_1,n_2)$ by $n_1 n_2$. This was pointed out by a referee of this paper. Leonenko (1975) proved that

$$\sup_x \Delta_{n_1,n_2}(x) \leqq C(n_1 n_2)^{-\gamma}$$

for some $\gamma>0$ under some conditions. It is not clear whether the rate obtained by Leonenko (1975) is valid or not as he did not provide the proof in detail. The following example, due to the referee, raises doubt regarding the validity of the estimate obtained in Leonenko (1975).

Let $\{\xi(i,j), i,j=1,2,\ldots\}$ be a random-field such that $\{\xi(i,j), i=1,2,\ldots\}$ be a sequence of independent random variables for each $j=1,2,\ldots$ and $\{\xi(i,j), j=1,2,\ldots\}$ be 1-dependent sequence of random-variables for each $i=1,2,\ldots$. Suppose $E\xi(i,j)=0$ and $E\xi(i,j)^2=1$. Then the random field $\xi(i,j)$, $i,j=1,2,\ldots$ is 1-dependent. It can be shown that

$$E(\eta^{(1)}_{h_1,h_2})^2 \geqq \frac{C}{k_2}$$

for some constant $C>0$ where $\eta^{(1)}_{h_1,h_2}$ is as defined by (4.7) by using the arguments given above for $m=1$. The above inequality invalidates the relation

$$E(\eta^{(1)}_{h_1h_2})^2=O\left(\frac{1}{k_1k_2}\right)$$

used by Leonenko (1975) (cf. the relation after inequality (4) in Leonenko (1975)) which is crucial in the derivation of his results on the rate. In the light of this discussion, it is conjectured that the best rate of convergence for the random fields is not of the order $O\left(\frac{1}{n_1n_2}\right)$ but of the order $O\left(\frac{1}{\min(n_1,n_2)}\right)$ both in the uniform as well as non-uniform cases.

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