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Exit Systems for Dual Markov Processes

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0. Introduction

In 1975 Maisonneuve [13] introduced the "exit system", a kernel/additive functional pair $(^*P, B)$ associated to a homogeneous random set, as a general tool for studying the excursions of Markov processes (a similar idea had been suggested by Dynkin [5] in 1971). In [6] Getoor used the exit system to analyze specific excursions, and recently exit systems were used by Getoor and Sharpe [7, 9] to study excursions under duality hypotheses. The purpose of this paper is to explore the relationship between "dual" exit systems $(^*P, B)$ and $(^*\hat{P}, \hat{B})$ of a pair of dual Markov processes, by means of an auxiliary "two-sided" Markov process with random birth and death in which the original dual processes may be simultaneously realized. The object of our study is to contribute to the theory of excursions of dual processes, using the auxiliary process as a convenient and natural tool.

Section 1 introduces the two-sided processes we will use throughout the paper, and establishes our notation. Sections 2 and 3 concern exit systems and excursions of the auxiliary process. We introduce and exploit the notion of "co-exit system", which comes from the dual exit system. The presentation of this part of the paper was influenced by Maisonneuve's recent work on regenerative sets on the real line [12] and benefited greatly from his comments.

In Sect. 4 we look again at the original dual processes. Here we generalize the result of [9] on reversing excursions from a point, eliminating the hypothesis of dual densities. This section also contains a formula (first discovered by Kaspi [11]) which expresses a duality relationship between the dual exit systems. See Sect. 4 for the precise statements.

1. The Two-sided Markov Process

Fix a Lusinian space E and let $\mathscr{E}$ denote its Borel σ-algebra. Let W be the space of paths from $\mathbb{R}$ into $E \cup \{\Delta, \hat{\Delta}\}$ which admit a "birth time" α and a

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"death time" β, and which are r.c.l.l. on (α, β). That is,

(i) $w(t)=\hat{\Delta}$ implies $w(s)=\hat{\Delta}$ all $s<t$;

(ii) $w(t)=\Delta$ implies $w(s)=\Delta$ all $s>t$;

(iii) $\alpha(w)=\sup\{t\colon w(t)=\hat{\Delta}\}$ and $\beta(w)\equiv\inf\{t\colon w(t)=\Delta\}$, with $w(\alpha(w))=\hat{\Delta}$ if $\alpha(w)\in\mathbb{R}$, $w(\beta(w))=\Delta$ if $\beta(w)\in\mathbb{R}$.

Let $Z_t(w)=w(t)$ denote the coordinate maps on W, and define

$$\hat{Z}_t\equiv Z_{(-t)-}, \qquad t\in\mathbb{R}.$$

$\hat{Z}$ allows us to view the paths of W in the reverse direction of time; its trajectories are r.c.l.l. on $(\hat{\alpha}, \hat{\beta})=(-\beta, -\alpha)$. Let $\sigma_t\colon W\to W$ denote the shift operators on W

$$Z_s\circ\sigma_t(w)=Z_{s+t}(w),$$

and set $\hat{\sigma}_t=\sigma_{-t}$. The σ-algebra on W generated by Z_t $(t\in\mathbb{R})$ is denoted $\mathscr{G}^0$; $(\mathscr{G}_t^0)$ and $(\hat{\mathscr{G}}_t^0)$ are the natural filtrations of Z and $\hat{Z}$ respectively.

The pair $(Z, \hat{Z})$ constitute a "two-sided Markov process" when governed by a measure Q on $(W, \mathscr{G}^0)$ which is σ-finite on $Z_s^{-1}(\mathscr{E})$ for every s and which satisfies:

(1.1) there is a right semigroup $(P_t)_{t\geq 0}$ on $(E, \mathscr{E})$ so that $Z=(W, \mathscr{G}^0, (\mathscr{G}_t^0), Z_t, \sigma_t, \alpha, \beta, Q)$ is Markov with respect to (P_t);

(1.2) there is a right semigroup $(\hat{P}_t)_{t\geq 0}$ on $(E, \mathscr{E})$ so that $\hat{Z}=(W, \mathscr{G}^0, (\hat{\mathscr{G}}_t^0), \hat{Z}_t, \hat{\sigma}_t, \hat{\alpha}, \hat{\beta}, Q)$ is Markov with respect to $(\hat{P}_t)$.

In this paper we will be concerned with the case

(1.3) (P_t) and $(\hat{P}_t)$ are in duality relative to a σ-finite measure ξ on $(E, \mathscr{E})$.

For any realization $X=(\Omega, \mathscr{F}, (\mathscr{F}_t), X_t, \theta_t, (P^x, x\in E\cup\Delta); t\geqq 0)$ of the semigroup (P_t) – i.e., X is a right process with semigroup (P_t) and cemetery point Δ – define projections $\tau_t\colon W\to\Omega$ $(t\in\mathbb{R})$ by

(1.4)
$$\begin{aligned} X_s\circ\tau_t(w)&=Z_{s+t}(w) \quad &&\text{if } Z_t(w)\in E\\ &=\Delta &&\text{otherwise.}\end{aligned}$$

It follows from (1.1) that for any $(\mathscr{G}_{t+}^0)$ stopping time T and $f\in\mathscr{F}$ we have

(1.5)
$$Q(f\circ\tau_T|\mathscr{G}_{T+}^0)=P^{Z_T}(f) \quad \text{on } \{Z_T\in E\}.$$

Naturally the duals of (1.4) and (1.5) hold, relative to a realization $\hat{X}$ of $(\hat{P}_t)$. In addition, we define another projection $\tilde{\tau}_t\colon W\to\hat{\Omega}$

(1.6)
$$\tilde{\tau}_t\equiv\hat{\tau}_{-t}$$

which will be useful in the sequel.

(1.7) *Remark.* We showed in [14] how to construct the measure Q from a given pair of dual processes/semigroups. This measure is stationary

(1.8) $$\sigma_s Q = Q, \quad s \in \mathbb{R},$$

with one-dimensional distributions equal to ξ, and Z was called the *auxiliary process* associated to the dual pair $X, \hat{X}$. The present notation, and some terminology, is different from that of [14, 15] where only Z was used. Results for $\hat{Z}$ were stated in terms of Z; for instance what was called a "Z-starting-time" (or co-stopping time) in [14] would here be the negative of a $(\hat{\mathscr{G}}_t)$-stopping time.

Let $\mathscr{G}$ denote the Q-completion of $\mathscr{G}^0$ and let $\mathscr{N}$ be the ideal of Q null sets of $\mathscr{G}$. We complete the filtrations of Z by adjoining $\mathscr{N}$: $\mathscr{G}_t \equiv \mathscr{G}_t^0 \vee \mathscr{N}$ and $\hat{\mathscr{G}}_t = \hat{\mathscr{G}}_t^0 \vee \mathscr{N}$. Using the fact that the duality measure ξ is excessive, one can show that $Q(\alpha = t) = Q(\beta = t) = 0$ for any $t \in \mathbb{R}$. From this observation and the strong Markov properties of Z and $\hat{Z}$, it follows that the completed filtrations $(\mathscr{G}_t)$ and $(\hat{\mathscr{G}}_t)$ are right continuous, i.e., satisfy the "usual hypotheses".

In the following, $\mathbb{R}^+$ stands for the non-negative real numbers, $\mathbb{R}^{++} = \mathbb{R}^+ - \{0\}$, and $\mathscr{B}(\cdot)$ denotes the Borel sets of whatever space appears inside the parentheses.

2. Exit Systems for the Two-sided Process

Let $M \subset \mathbb{R}^{++} \times \Omega$ be a perfectly homogeneous optional subset of $]\!]0, \zeta[\![$, where ζ denotes the lifetime of X, with $M(\omega)$ closed in $]\!]0, \zeta[\![$ for each ω. "Perfectly homogeneous" means $t + s \in M(\omega)$ if and only if $t \in M(\theta_s \omega)$ for every $t > 0$ and $s \geqq 0$. Assume $M([\Delta]) = \emptyset$. We associate to M the random subset N of $]\!]\alpha, \beta[\![\subset \mathbb{R} \times W$ which satisfies

(2.1) $$(N - t) \cap^{R++} = M \circ \tau_t$$

on $\{\alpha < t < \beta\}$. The left-hand side of (2.1) means $\{s - t : s \in N(w), s > t\}$. Because $\tau_{s+t}(w) = \theta_s \circ \tau_t(w)$ $(t \in (\alpha, \beta),\ s \geqq 0)$, the homogeneity of M guarantees that N is welldefined. (In the terminology of [15], N is the "homogeneous embedding" of M.)

Now let $({}^*P, B)$ denote the exit system for (X, M), as defined in [13]. We identify the additive functional B with the homogeneous random measure having B as distribution function. From now on, the letter B will denote this *measure*. As in [15], B has a homogeneous embedding into W, that is, a random measure $A(w, \cdot)$ carried by $(\alpha(w), \beta(w))$ which satisfies

(2.2) $$A(w, t + A) = B(\tau_t w, A)$$

if $Z_t(w) \in E$, $A \in \mathscr{B}(\mathbb{R}^+)$. With this definition we have

(2.3) $$Q \sum_{t \in N_\ell} Y_t \cdot H_t \circ \tau_t = Q \int Y_t\, {}^*P^{Z_t}(H_t)\, A(dt),$$

where $Y \geqq 0$ is $(\mathscr{G}_t)$-optional, H is positive and $\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}^*$-measurable, and N_ℓ denotes the set of left endpoints of intervals contiguous to N. The pair $({}^*P, A)$ is called the *exit system for* (Z, N).

Of course, we can carry out analogous embeddings to arrive at exit systems for $\hat{Z}$. Note that an exit system for $\hat{Z}$ can be interpreted as a kind of "co-exit" system for Z. Namely, if $\hat{N}$ is a homogeneous random subset of $]\!]\hat{\alpha}, \hat{\beta}[\![$ and $(*\hat{P}, \hat{\Lambda})$ is the exit system for $(\hat{Z}, \hat{N})$, the set $\tilde{N} \equiv -\hat{N}$ is a homogeneous random subset of $]\!]\alpha, \beta[\![$. The "dual" of (2.3) for $\hat{N}$, rewritten in terms of $\tilde{N}$, becomes

$$(2.4) \qquad Q\Big(\sum_{s\in\tilde{N}_t} V_s \cdot K_s \circ \tilde{\tau}_s\Big) = Q \int V_s {}^*\hat{P}^{Z_{s-}}(K_s)\,\tilde{\Lambda}(ds),$$

where V_{-t} is $(\hat{\mathscr{G}}_t)$-optional, K is positive and $\mathscr{B}(\mathbb{R})\otimes\hat{\mathscr{F}}$-measurable, $\tilde{\Lambda}(ds) = \hat{\Lambda}(-ds)$, and $\tilde{N}_t$ denotes the set of right endpoints of intervals contiguous to $\tilde{N}$.

(2.5) *Remark.* The process V which appears in (2.4) might be called Z-"co-optional" (see [1] or [4], where such processes are called "left"): the σ-field of processes V such that V_{-t} is $(\hat{\mathscr{G}}_t)$-optional is generated by processes adapted to $(\tilde{\mathscr{G}}_t) \equiv (\hat{\mathscr{G}}_{-t})$ which are Q-a.e. left continuous. (Note: $\tilde{\mathscr{G}}_t = \sigma(Z_s\colon s \geqq t\} \vee \mathscr{N}$.)

Via the notions of exit and co-exit systems we can study *dual* exit systems. Under duality hypotheses, the structure of closed, homogeneous, optional sets is completely known [7]: if M is a closed, homogeneous, optional subset of $]\!]0, \zeta[\![$ then there exists a Borel set $\Gamma \subset E \times E$ such that $\{(t, \omega)\colon t > 0, (X_{t-}(\omega), X_t(\omega)) \in \Gamma\}$ is indistinguishable from M. Assuming M has this representation *exactly*, the set N corresponding to M is $\{(t, w)\colon (Z_{t-}(w), Z_t(w)) \in \Gamma\}$. Choosing $\hat{M} = \{(t, \omega)\colon \hat{X}_{t-}(\omega), \hat{X}_t(\omega)) \in \hat{\Gamma}\}$, where $\hat{\Gamma} \equiv \{(x, y)\colon (y, x) \in \Gamma\}$, the embedded set $\hat{N}$ then satisfies $-\hat{N} = N$. In an obvious sense M and $\hat{M}$ are *dual* random sets (their debuts R and $\hat{R}$ are dual exact terminal times in the classical sense [7]), and their respective exit systems $(*P, B)$ and $(*\hat{P}, \hat{B})$ are what we called "dual exit systems" in the introduction. *In general*, B and $\hat{B}$ are not what are commonly called dual additive functionals (when this is the case, $\Lambda = \hat{\Lambda}$ [15]). However, (2.3) and (2.4) (with $\tilde{N} = N$) indicate how the auxiliary process can be used to relate dual exit systems: the corresponding exit and co-exit systems for Z share their underlying homogeneous random set.

In the remainder of the paper we assume that X and $\hat{X}$ are *Borel* right processes, and for convenience we assume both are realized as the coordinate process on the canonical space $(\Omega, \mathscr{F}^0)$ of r.c.l.l. paths from $\mathbb{R}^+$ into $E \cup \{\Delta\}$. Now $X_t(\omega) = \hat{X}_t(\omega) = \omega(t)$, and the processes are distinguished by their laws (P^x) and $(\hat{P}^x)$ on $(\Omega, \mathscr{F}^0)$.

3. Exit and Co-Exit Systems of Z

We begin with a formula which contains (2.3) and (2.4). In the following N always denotes a closed, homogeneous random set indistinguishable from $\{(Z_{t-}, Z_t) \in \Gamma\}$.

(3.1) *Definition.* Let $\tilde{\mathscr{O}}$ denote the σ-algebra on $\mathbb{R} \times \mathbb{R} \times W$ generated by real-valued functions $U(s, t, w) \in \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{G}$ which satisfy

(i) $\lim\limits_{\substack{r \to s\ r<s \\ u \to t\ u>t}} U(r,u,w) = U(s,t,w)$ for a.a. w;

(ii) $w \to U(s,t,w) \in \mathscr{G}[s,t] \equiv \hat{\mathscr{G}}_{-s} \cap \mathscr{G}_t$ for $s \leqq t$;

(iii) $U(s,t,w) = 0$ if $s > t$.

Remark. The family $\tilde{\mathcal{O}}$ is related to the "*bi-optional*" *σ-field* $\mathcal{O}$ for two-parameter processes with index set $I = \{(s,t)\colon -s<t\}$, relative to the filtrations $(\mathscr{F}_t^1) = (\mathscr{G}_t)$ and $(\mathscr{F}_s^2) = (\hat{\mathscr{G}}_s)$ (see Bakry [3]): $\mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2$ where $\mathcal{O}_1$ is the σ-field of processes $V(s,t,w)\colon \mathbb{R} \times \mathbb{R} \times W \to \mathbb{R}$ which are $\mathscr{B}(\mathbb{R}) \otimes \mathcal{O}(\mathscr{F}^1_{\cdot})$-measurable and which vanish off I, and $\mathcal{O}_2$ is defined similarly in terms of $\mathcal{O}(\mathscr{F}^2_{\cdot})$, when the second real variable plays the role of parameter. Here, $\mathcal{O}(\mathscr{F}^i_{\cdot})$ is the optional σ-field relative to $(\mathscr{F}_t^i)$. A process U is $\tilde{\mathcal{O}}$-measurable provided $(s,t,w) \to U(-s,t,w)$ is bi-optional.

(3.2) **Proposition.** *For* H, $K \geqq 0$ *and* $\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}^*$*-measurable, and* $U \geqq 0$ *and* $\tilde{\mathcal{O}}$*-measurable,*

$$(3.3) \qquad Q\Big(\sum_{\substack{s \leqq t \\ s \in N_{\hat{\ell}} \\ t \in N_\ell}} U(s,t) \cdot K_s \circ \tilde{\tau}_s \cdot H_t \circ \tau_t\Big) = Q \iint_{s \leqq t} U(s,t) * \hat{P}^{Z_{s-}}(K_s) * P^{Z_t}(H_t)\, \tilde{A}(ds)\, A(dt).$$

Proof. Because $N_{\hat{\ell}}$ is optional with countable sections,

$$N_{\hat{\ell}} = \bigcup_n [\![S_n]\!],$$

where $\{S_n\}$ are pairwise disjoint $(\mathscr{G}_t)$-stopping times in $]\!]\alpha, \infty]\!]$. Therefore the process

$$\begin{aligned} Y_t &= \sum_{s \in N_{\hat{\ell}}} U(s,t) \cdot K_s \circ \tilde{\tau}_s \cdot 1_{\{s \leqq t\}} \\ &= \sum_n U(S_n, t) \cdot K_{S_n} \circ \tilde{\tau}_{S_n} \cdot 1_{\{S_n \leqq t\}} \end{aligned}$$

is optional if $U(s,t) \in \tilde{\mathcal{O}}^+$ is right continuous in t; this extends to general $U \in \tilde{\mathcal{O}}^+$ by a monotone class argument. Now applying the exit system (2.3) to the left side of (3.3) we obtain

$$\begin{aligned} (3.4) \qquad Q \sum_{t \in N_\ell} Y_t \cdot H_t \circ \tau_t &= Q \int Y_t * P^{Z_t}(H_t)\, A(dt) \\ &= Q\Big[\sum_{s \in N_{\hat{\ell}}} K_s \circ \tilde{\tau}_s \int_{[s,\infty)} U(s,t) * P^{Z_t}(H_t)\, A(dt)\Big] \\ &= Q\Big[\sum_{s \in N_{\hat{\ell}}} V_s \cdot K_s \circ \tilde{\tau}_s\Big]. \end{aligned}$$

The process V, given by the integral, is co-optional in the sense of (2.5)(i). Using (2.4), (3.4) becomes

$$Q \iint_{s \leqq t} U(s,t) * \hat{P}^{Z_{s-}}(K_s) * P^{Z_t}(H_t)\, \tilde{A}(ds)\, A(dt),$$

which verifies (3.3).

Next, we explore some special cases of (3.3). We shall use the notation:

$$D_t = \inf\{s \geqq t \colon s \in N\} \qquad (\inf \emptyset = \infty)$$
$$G_t = \sup\{s \leqq t \colon s \in N\} \qquad (\sup \emptyset = -\infty)$$
$$A_t = t - G_t, \qquad R_t = D_t - t.$$

The same letters with "$\hat{}$" refer to $\hat{N}$; note that $\hat{G}_t = D_{-t}$. When $-\infty < G_a < a$, $G_a \in N_\ell$ and $D_a > a$. For fixed $u > 0$, let (G^u, D^u) denote the first interval contiguous to N which lies to the right of 0 and exceeds u in length. (Here the choice of 0 as a reference point is convenient but arbitrary.) If no such interval exists, $G^u = D^u = \infty$. Let $({}^uG, {}^uD)$ be the last contiguous interval exceeding u in length which lies to the left of 0, i.e., the last one whose right endpoint is negative. (${}^uG = {}^uD = -\infty$ if such an interval fails to exist.) If $t \in N_\ell$ and $s \in N_\imath$ satisfy $t \geqq 0$, $s \leqq 0$, $R_t > u$, $A_s > u$ and $[s, t] \subset ({}^uG, D^u)$, then $s = {}^uD$ and $t = G^u$. In the following discussion F will be a non-negative $\tilde{\mathcal{O}}$-measurable function, and ϕ, ψ will be non-negative $\mathcal{F}^*$-measurable functions. Recall that R denotes the debut of $M \subset \mathbb{R}^{++} \times \Omega$ which corresponds to N, $\hat{R}$ is the dual object.

Applying (3.3) with

$$U(s, t) = F(s, t)\, 1_{\{s \leqq t\}}$$
$$H_t = \phi\, 1_{\{0 < b - t < R\}}, \qquad K_s = \psi\, 1_{\{0 < s - a < \hat{R}\}}$$

we obtain

$$(3.5) \quad Q[F(D_a, G_b) \cdot \psi \circ \tilde{\tau}_{D_a} \cdot \phi \circ \tau_{G_b};\ a < D_a \leqq G_b < b]$$
$$= Q \iint_{a < s \leqq t < b} F(s, t) * \hat{P}^{Z_{s-}}(\psi;\ s - a < \hat{R}) * P^{Z_t}(\phi;\ b - t < R)\, \tilde{A}(ds)\, A(dt),$$

which concerns the beginning of the "excursion straddling b" and the end of the "excursion straddling a".

Similarly, for the two excursions exceeding u in length $({}^uG, {}^uD)$ and (G^u, D^u), we choose

$$U(s, t) = F(s, t)\, 1_{({}^uG, 0]}(s)\, 1_{[0, D^u)}(t)$$
$$H = \phi\, 1_{\{R > u\}}, \qquad K = \psi\, 1_{\{\hat{R} > u\}}$$

to obtain

$$(3.6) \quad Q[F({}^uD, G^u) \cdot \psi \circ \tilde{\tau}_{{}^uD} \cdot \phi \circ \tau_{G^u};\ \alpha < {}^uD, G^u < \beta]$$
$$= Q \int_{t \in [0, D^u)} \int_{s \in ({}^uG, 0]} F(s, t) * \hat{P}^{Z_{s-}}(\psi;\ u < \hat{R}) * P^{Z_t}(\phi;\ u < R)\, \tilde{A}(ds)\, A(dt).$$

4. Reversing Excursions

In this section we use the exit and co-exit systems to investigate the effect of reversing all the excursions from a single point $b \in E$. We assume now that X and $\hat{X}$ are standard processes in duality. Our result extends work of Getoor and Sharpe [9] for dual processes possessing "dual densities". To avoid trivialities we assume $\{b\}$ is not a trap, and

(4.1) (i) b is regular for $\{b\}$ relative to the process X,

(ii) all excursions of X from $\{b\}$ begin and end at b,

(iii) $P^b(\sup\{t\colon X_t=b\}<\infty)=0=\hat{P}^b(\sup\{t\colon \hat{X}_t=b\}<\infty)$.

(Equivalently, dual versions of (i) and (ii) hold; (ii) is equivalent to the indistinguishability of the sets $\{(t,\omega)\colon t>0,\ X_{t-}(\omega)=b\}$ and $\{(t,\omega)\colon t>0,\ X_t(\omega)=b\}$. See [9] (Sect. 9) for the precise meaning of (ii).) The excursions from b of the process X are studied via the set $M=\{(t,\omega)\colon t>0,\ X_t(\omega)=b\}$; its debut is a.s. equal to $\inf\{t>0\colon X_{t-}=b\}\equiv R$. The associated random set $N=\{(t,w)\colon Z_t(w)=b\}$ is indistinguishable from $\tilde{N}=\{(t,w)\colon Z_{t-}(w)=b\}$ and we shall suppose for convenience (as is done in [9]) that Z hits and leaves b continuously everywhere on W. Thus $N=\tilde{N}$. Under our hypotheses b is not a holding point, and N is closed and perfect.

In order to state and discuss our result we introduce some notation from [9]. Let $\Phi\colon \Omega\to\Omega$ be the mapping which reverses every excursion of finite length away from b, namely

$$(4.2)\qquad (\Phi\,\omega)(t)=X_{(g_t(\omega)+d_t(\omega)-t)-}(\omega)\quad \text{if } 0<g_t(\omega)<d_t(\omega)$$
$$=X_t(\omega)\quad \text{otherwise,}$$

where $g_t=\sup\{s\leqq t\colon s\in M\}$ and $d_t=t+R\circ\theta_t$. *The operator of reversal at time* R, denoted ρ, is given by

$$(4.3)\qquad (\rho\,\omega)(t)=\omega[(R(\omega)-t)-]\quad \text{if } 0\leqq t<R(\omega)<\infty$$
$$=\Delta\quad \text{otherwise.}$$

Finally let n denote the "characteristic measure" (following Itô [10]) of the Poisson point process of excursions from b for the process X; $\hat{n}$ denotes the corresponding object for $\hat{X}$.

Assuming (4.1)(i)–(iii) and dual densities, it was shown in [9] that

$$(4.4)\qquad \Phi\,P^b=\hat{P}^b.$$

The proof of this theorem in [9] rests on their assumption of dual densities at only one point, namely in proving

$$(4.5)\qquad \rho\,\hat{n}=n\quad \text{on } (\Omega,\mathscr{F}^0).$$

In what follows we prove (4.5) under (4.1)(i)–(iii) only.

The additive functional B in the exit-system $({}^*P,B)$ for M is a local time at b [7]; the "standard" Maisonneuve exit system is normalized so that ${}^*P^b(1-e^{-R})=1$. In [9] the exit system for M was chosen so that the local time involved is the *standard* local time ℓ, i.e.,

$$E^x\int_0^\infty e^{-t}\,d\ell_t=E^x(e^{-T_b}).$$

Let $(\bar{P},\ell)$ denote this exit system. It was shown in [9] that

$$(4.6)\qquad n(\Gamma)=\bar{P}^b(1_\Gamma\circ k_R)\quad \text{for } \Gamma\in\mathscr{F}^0.$$

(Here k_t is the killing operator.) For the Maisonneuve exit system, $B = c\ell$, and $^*P^b = \frac{1}{c}\bar{P}^b$, where c is a constant, and furthermore,

$$\ell_t = m([0, t] \cap M) + B_t, \tag{4.7}$$

where m is Lebesgue measure [13]. Similar statements hold for the dual objects.

The key step in proving (4.5) is

(4.8) **Lemma.** (i) $Q(\Lambda[0,1]) = Q(\tilde{\Lambda}[0,1])$.

(ii) *For non-negative* $H \in \mathscr{F}^*$,

$$^*P^b(H \circ k_R) = {}^*\hat{P}^b(H \circ \rho \circ k_R).$$

Proof. By (2.3),

$$Q \sum_{\substack{0<t<1 \\ t \in N_\ell}} H \circ k_R \circ \tau_t = {}^*P^b(H \circ k_R)\, Q(\Lambda[0,1]). \tag{4.9}$$

The left-hand side of (4.9) equals

$$Q \sum_{\substack{0<s<1 \\ s \in N_{\tilde{\ell}}}} H \circ \rho \circ k_R \circ \tilde{\tau}_s + \{Q[H \circ \rho \circ k_R \circ \tilde{\tau}_{D_1}] - Q[H \circ \rho \circ k_R \circ \tilde{\tau}_{D_0}]\}. \tag{4.10}$$

The term in brackets subtracts the contribution due to the excursion straddling 0 and adds the contribution due to the excursion straddling 1. These two contributions are equal (and hence cancel out) due to the shift invariance (1.8) of Q. (Because neither Λ nor $\tilde{\Lambda}$ charges points, $Q\{t \in N_\ell\} = 0 = Q\{t \in N_{\tilde{\ell}}\}$ for any t. Consequently, there is no contribution in (4.9) from excursions which *end* at 1 or in (4.10) from excursions which *begin* at 0). By (2.4), the remaining term in (4.10) equals $^*P^b(H \circ \rho \circ k_R)\, Q(\tilde{\Lambda}[0,1])$. Choosing $H = 1 - e^-$ so that $^*P^b(H \circ k_R) = {}^*\hat{P}^b(H \circ \rho \circ k_R) = 1$, the equality of (4.9) and (4.10) proves (i). Note that $Q(\Lambda[0,1]) = \nu_B(E) < \infty$, where ν_B is the Revuz measure of B (see [2] or [8]). That done, (ii) follows immediately for all H from the same equality.

Using (4.6) and its dual, (4.8)(ii) is equivalent to

$$c\, \rho\, \hat{n} = \hat{c}\, n. \tag{4.11}$$

But (4.7) and (4.8)(i) imply $c = \hat{c}$, for $Q(N \cap [0,1]) = Q(\hat{N} \cap [0,1])$ is immediate from (1.8). This proves (4.5).

Now consider the general homogeneous random set N as in Sect. 3, together with its exit and co-exit systems. Assume N is unbounded so that all its contiguous intervals are of finite length. Then the following analogue of (4.8) holds:

(4.12) **Proposition.** *For non-negative* $H \in \mathscr{F}^*$

$$Q \int_0^1 {}^*P^{Z_t}(H \circ k_R)\, \Lambda(dt) = Q \int_0^1 {}^*\hat{P}^{Z_{t-}}(H \circ \rho \circ k_R)\, \tilde{\Lambda}(dt).$$

Proof. Use the idea behind the proof of (4.8). That argument involves exchanging an excursion straddling 0 for an excursion straddling 1 and depends on

(4.13) **Lemma.** $Q\{(t\in N_\ell\}=Q\{t\in N_\imath\}=0$ *for any* t.

Proof. There are at most countably many values s with $P^\xi(\Delta B_s>0)>0$, where B is the additive functional in the exit system for (X, M) corresponding to N. Because ξ is excessive, $P^\xi(\Delta B_u>0)=0$ implies $P^\xi(\Delta B_s>0)=0$ for any $s>u$. This shows $Q(\Delta A_t>0)=0$ for any t. Since $\{t\in N_\ell\}$ has positive measure only if A jumps at t, the first part is proved. The second is proved similarly using $\tilde{A}$.

We can express (4.12) in terms of the original processes $X, \hat{X}$ and their exit systems. Define, for $x\in E$, $H\in\mathscr{F}^*$

$$n^x(H)={}^*P^x(H\circ k_R),$$

and let v, $\hat{v}$ be the Revuz measures of B and $\hat{B}$ respectively. Then (4.12) states

$$\int\rho\,\hat{n}^x(\cdot)\,\hat{v}(dx)=\int n^x(\cdot)\,v(dx). \tag{4.14}$$

This relationship was discovered by Kaspi [11] under slightly different hypotheses, using a quite different approach.

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