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On a Multiplicative Functional Transformation Arising in Nonlinear Filtering Theory*

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Summary. This paper concerns the nonlinear filtering problem of calculating "estimates" $E[f(x_t)|y_s, s \leqq t]$ where $\{x_t\}$ is a Markov process with infinitesimal generator A and $\{y_t\}$ is an observation process given by $dy_t = h(x_t)\,dt + dw_t$ where $\{w_t\}$ is a Brownian motion. If $h(x_t)$ is a semimartingale then an unnormalized version of this estimate can be expressed in terms of a semigroup $T^y_{s,t}$ obtained by a certain y-dependent multiplicative functional transformation of the signal process $\{x_t\}$. The objective of this paper is to investigate this transformation and in particular to show that under very general conditions its extended generator is $A^y_t f = e^{y(t)h}(A - \frac{1}{2}h^2)(e^{-y(t)h} f)$.

Introduction

Let $\{x_t\}$ be a Markov process, h a bounded real-valued function and $\{w_t\}$ a standard Brownian motion independent of $\{x_t\}$. Now define

$$y_t = \int_0^t h(x_s)\,ds + w_t. \tag{1}$$

The real-valued[1] process $\{y_t\}$ is to be thought of as a "noisy observation" of the "signal" $\{x_t\}$, and the objective is to "estimate" functionals of the signal, i.e. compute quantities of the form $E[f(x_t)|y_s, s \leqq t]$. Further, this computation should be done *recursively*, i.e. in terms of a statistic $\{\pi_t\}$ which can be *updated* using only new observations:

$$\pi_{t+s} = \alpha(t, s, \pi_t, \{y_{t+u}, 0 \leqq u \leqq s\}), \tag{2}$$

and from which estimates can be calculated in a "pointwise" fashion:

$$E[f(x_t)|y_s, s \leqq t] = \beta(t, f, y_t, \pi_t). \tag{3}$$

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1 For notational simplicity; the results below extend easily to the case of vector observations

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Generally, π_t will be closely related to the conditional distribution of x_t given $\{y_s, s \leqq t\}$. Indeed, the main result of filtering theory as presented for example in Liptser and Shiryaev's book [9] is a nonlinear measure-valued stochastic differential equation [9, theorem 8.1] whose solution is this conditional distribution.

Our starting point in this paper is the so-called *Kallianpur-Striebel formula* [7], [17], equation (6) below. To introduce this let us describe the probabilistic set-up in more detail. The Lévy system of $\{x_t\}$ plays an essential role in what follows and our basic hypotheses are those of Watanabe [16], under which the Lévy system is well-defined. We use the original formulation of the Lévy system as introduced in [16] rather than the later, more streamlined version of, for example, Benveniste and Jacod [1] because we do not really need the extra generality this provides whereas we do make explicit use of the associated stochastic calculus developed in [16] and by Kunita and Watanabe in [8].

Let S be a locally compact Hausdorff space with countable base, with Borel σ-field $\mathscr{S}$, and let $\mathscr{B}(S)$ and $\mathscr{M}(S)$ denote respectively the set of bounded measurable functions $f\colon S \to \mathbb{R}$ and the set of positive measures on $(S, \mathscr{S})$. For $f \in \mathscr{B}(S)$, $\mu \in \mathscr{M}(S)$ we write

$$\langle f, \mu \rangle = \int_S f(x)\, \mu(dx).$$

Now let W be the set of right-continuous S-valued functions on $\mathbb{R}^+$, with coordinate functions $\{x_t,\ t \geqq 0\}$, and suppose that $\{P_x,\ x \in S\}$ is a family of probability measures on W such that $\mathfrak{x} = \{x_t, P_x\}$ is a Hunt process satisfying Meyer's hypothesis (L)[2] and having a lifetime $\zeta = \infty$ a.s. (P_x). The semigroup of operators $\{T_t\}_{t \geqq 0}$ on $\mathscr{B}(S)$ associated with $\mathfrak{x}$ is defined by

$$T_t f(x) = E_x[f(x_t)]$$

Additionally, $\mathfrak{x}$ is supposed to have an initial probability distribution $\pi \in \mathscr{M}(S)$.

$(A, \mathfrak{D}(A))$ will denote the *extended generator* of $\mathfrak{x}$ [6, (13.45)]: a function $f \in \mathscr{B}(S)$ belongs to $\mathfrak{D}(A)$ if there exists $Af \in \mathscr{B}(S)$ such that $\{M_t^f\}$ is a local martingale (P_x) for every $x \in S$, where

$$M_t^f = f(x_t) - f(x_0) - \int_0^t Af(x_s)\, ds. \tag{4}$$

$C(\mathbb{R}^+)$ is the space of real-valued continuous functions on $\mathbb{R}^+$ with coordinate functions $\{y_t, t \geqq 0\}$ and μ_w denotes Wiener measure on $C(\mathbb{R}^+)$ (with $\mu_w(y_0 = 0) = 1$). Our basic probability space is then $\Omega = W \times C(\mathbb{R}^+)$ equipped with the product measure $P^0 = P_\pi * \mu_w$. Thus, under P^0, $\mathfrak{x}$ and $\mathfrak{y} = \{y_t\}$ are independent processes and $\mathfrak{y}$ is a standard Brownian motion. We denote $\mathscr{Y}_t = \sigma\{y_s, s \leqq t\}$.

We shall consider the filtering problem over a finite time interval $[0, T]$. Fix a function $h \in \mathscr{B}(S)$ and for $t \in [0, T]$ define

$$L_t = \exp\left(\int_0^t h(x_s)\, dy_s - \tfrac{1}{2}\int_0^t h^2(x_s)\, ds\right)$$

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[2] i.e. $\mathfrak{x}$ is a homogeneous strong Markov process with quasi-left-continuous paths. and there exists a measure ν on $(S, \mathscr{S})$ such that every λ-excessive function which is 0 a.e. $(d\nu)$ is identically zero, for any $\lambda > 0$

It is well-known that $E^0 L_t = 1$ for all t and that the formula

$$\frac{dP}{dP^0} = L_T$$

defines a measure P under which

(i) the distributions of $\mathfrak{x}$ are the same as under P^0

(ii) the process $\{w_t\}$ defined by (1) is a standard Brownian motion

(iii) $\{w_t\}$ and $\{x_t\}$ are independent.

Thus the filtering problem consists of calculating $E[f(x_t)|\mathscr{Y}_t]$ for a suitably large class of $f \in \mathscr{B}(S)$, where E denotes integration with respect to measure P. This can be expressed in terms of integration with respect to P^0 by the following standard formula

$$E[f(x)|\mathscr{Y}_t] = \frac{E^0[f(x_t)L_t|\mathscr{Y}_t]}{E^0[L_t|\mathscr{Y}_t]}$$

Denote by $\sigma_t(f)$ the numerator of this expression. It suffices to calculate $\sigma_t(f)$ for $f \in \mathscr{B}(S)$ because then

$$E[f(x_t)|\mathscr{Y}_t] = \sigma_t(f)/\sigma_t(1) \tag{5}$$

where $1 \in \mathscr{B}(S)$ is the constant function taking the value 1 for all $x \in S$. Since, under measure P^0, $\mathfrak{x}$ and $\mathfrak{y}$ are independent, the conditional expectation operator $E^0[\,.\,|\mathscr{Y}_t]$ amounts to "integrating out" the $\mathfrak{x}$-dependence and thus $\sigma_t(f)$ can be expressed in the following form:

$$\sigma_t(f) = \int_W f(x_t) \exp\left(\int_0^t h(x_s)\, dy_s - \tfrac{1}{2}\int_0^t h^2(x_s)\, ds\right) P_\pi(dx) \tag{6}$$

This heuristic reasoning was justified in [7], and (6) is known as the *Kallianpur-Striebel formula.* Now *suppose that $z_t = h(x_t)$ is a semimartingale.* Then the joint variation process $\langle y, z\rangle_t = 0$ in view of the independence of $\mathfrak{x}$ and $\mathfrak{y}$, and hence by the change of variables formula [6, (2.52)],

$$\int_0^t h(x_s)\, dy_s = y_t h(x_t) - \int_0^t y_s\, dh(x_s)$$

Thus the formula

$$\sigma_t(f, y) = \int_W f(x_t) \exp\left(y(t) h(x_t) - \int_0^t y(s)\, dh(x_s) - \tfrac{1}{2}\int_0^t h^2(x_s) ds\right) P_\pi(dx) \tag{7}$$

defines a functional $\sigma_t\colon \mathscr{B}(S) \times C[0, T] \to \mathbb{R}$ such that

(i) $\sigma_t(f, y)$ is well-defined for *all* $y \in C[0, T]$ (not just on a subset of Wiener measure 1)

(ii) $\sigma_t(f, y)/\sigma_t(1, y)$ is a version of the conditional expectation $E[f(x_t)|\mathscr{Y}_t]$.

It is shown by Clark [2] and Kushner [9] that $\sigma_t(f, y)$ is locally Lipschitz continuous in y and that this fact has important implications in terms of "robust estimation".

From now on $y=\{y(s), 0 \leqq s \leqq T\}$ will be an arbitrary, but fixed, continuous function. We shall have no further use for the factor $C(\mathbb{R}^+)$ of Ω and henceforth the "probability space" is $(W, \{P_x\})$, the canonical space for $\mathfrak{x}$.

For $0 \leqq s \leqq t \leqq T$ define

$$\alpha_t^s = \exp\left(-\int_s^t y(u)\, dh(x_u) - \tfrac{1}{2}\int_s^t h^2(x_u)\, du\right). \tag{8}$$

Then α_t^s is a multiplicative functional (m.f.) of $\mathfrak{x}$. (It is non-standard in that it is not homogeneous and does not satisfy $E_{x,s}[\alpha_t^s] \leqq 1$, the latter because $\sigma_t(f)$ is an *unnormalized* conditional expectation). The following formula thus defines a two-parameter semigroup of operators on $\mathscr{B}(S)$

$$T_{s,t}^y f(x) = E_{s,x}[f(x_t)\alpha_t^s]. \tag{9}$$

Combining (6)–(9) we can write

$$\sigma_t(f) = \sigma_t(f, y) = \langle T_{0,t}^y(e^{y(t)h} f), \pi \rangle. \tag{10}$$

This formula provides the starting point for the present paper. The idea behind our approach is that (10) leads to a recursive filter in a form in which no stochastic integration is involved. Write (10) in the form

$$\sigma_t(f) = \langle e^{y(t)h} f, \pi_t^y \rangle \tag{11}$$

where

$$\pi_t^y = U_{t,0}^y \pi$$

and $U_{t,s}^y$ is the semigroup adjoint to $T_{s,t}^y$ defined by

$$\langle f, U_{t,s}^y \mu \rangle = \langle T_{s,t}^y f, \mu \rangle.$$

Now formally π_t^y is the solution of the *forward equation*

$$\frac{d}{dt}\pi_t^y = (\bar{A}_t^y)^* \pi_t^y, \qquad \pi_0^y = \pi \tag{12}$$

where $\bar{A}_t^y$ is the generator of $T_{s,t}^y$, and this gives us a recursive filter in that (12) corresponds to the "updating equation" (2) while the "pointwise computation" (3) is given by (5) and (11).

Our interest is in substantiating this program and hence in investigating the infinitesimal characteristics of the semigroup $T_{s,t}^y$. In view of (4), the m.f. α_t^s of (8) is well-defined (i.e. $h(x_t)$ is a semimartingale) if $h \in \mathfrak{D}(A)$; our main result is Theorem (33) which states that under this condition and some other mild assumptions the extended generator of $T_{s,t}^y$ is

$$A_t^y f = e^{y(t)h}(A - \tfrac{1}{2}h^2)(e^{-y(t)h} f).$$

An explicit formula (35) for this is also given. In order that this result be useful we need to know that $T_{s,t}^y$ is actually determined by A_t^y, and we next investigate the case where $\mathfrak{x}$ is governed by a Lévy generator in the sense of Stroock [14].

We show using the results of [14] that the operator A_t^y then corresponds to a well-posed martingale problem and hence determines $T_{s,t}^y$. A special case of this has been investigated in greater detail by Pardoux [13]; see Remark (44) below.

If $\{x_t\}$ is a diffusion process with boundary conditions then the condition $h \in \mathfrak{D}(A)$ is unduly restrictive. In the concluding section we show that the m.f. transformation is nevertheless well defined and that its effect is to introduce y-dependent perturbations both into the operator coefficients *and into the boundary conditions.*

The Lévy System and the Local Characteristics of M^f

For $f \in \mathfrak{D}(A)$ the process M_t^f defined by (4) is both an additive functional and a locally square-integrable martingale. We can decompose M^f into a sum

(13)
$$M_t^f = M_t^{fc} + M_t^{fd}$$

where M^{fc} is a local martingale with continuous paths and M^{fd} is orthogonal to the stable subspace of continuous martingales.

In order to elucidate the structure of M^f more explicitly we need to consider the Lévy system of $\mathfrak{x}$. This was introduced by Watanabe [16] and further accounts can be found in Kunita and Watanabe [8], Meyer [12] and Jacod [6]. Under the conditions stated in the Introduction there exist a continuous additive functional ϕ_t^0 and a kernel $n(F, x)$ (i.e. $n(.,x)$ is a positive measure on $(S, \mathcal{S})$ for each $x \in S$ and $n(F,.)$ is a measurable function for each $F \in \mathcal{S}$) characterized by the fact that for every positive measurable function β: $S \times S \to \mathbb{R}^+$ such that $\beta(x,x) = 0$ for all $x \in S$,

$$E_x \sum_{\substack{s \leqq t \\ x_s \neq x_{s-}}} \beta(x_s, x_{s-}) = E_x \int_0^t \int_S \beta(z, x_{s-})\, n(dz, x_{s-})\, d\phi_s^0.$$

Let ρ be a metric on S, fix $\varepsilon > 0$ and for $F \in \mathcal{S}$ define

$$p_\varepsilon(t, F) = \sum_{s \leqq t} I_{(\rho(x_s, x_{s-}) > \varepsilon)} I_{(x_s \in F)}.$$

There exist sets F such that $E_x p_\varepsilon(t, F) < \infty$ and for such sets the compensator of $p_\varepsilon(t, F)$ is

$$\tilde{p}_\varepsilon(t, F) = \int_0^t \int_F I_{(\rho(z, x_{s-}) > \varepsilon)} n(dz, x_{s-})\, d\phi_s^0$$

i.e. $q_\varepsilon(t, F)$ is a martingale, where

$$q_\varepsilon(t, F) = p_\varepsilon(t, F) - \tilde{p}_\varepsilon(t, F).$$

Let $\mathfrak{F}_Q$ denote the set of measurable functions β: $S \times \mathbb{R}^+ \times W \to R$ such that $\beta(z, .,)$ is measurable with respect to the predictable σ-field on $\mathbb{R}^+ \times W$ for each $z \in S$ and

(14)
$$E_x \int_0^t \int_S \beta^2(z, s, \mathfrak{x})\, n(dz, x_{s-})\, d\phi_s^0 < \infty$$

for all $x \in S$ and $t>0$. $\mathfrak{F}_Q^{\mathrm{loc}}$ then denotes those functions β such that $\beta I_{(s \leqq \tau_n)} \in \mathfrak{F}_Q$ for each n, for some sequence $\{\tau_n\}$ of stopping times such that $\tau_n \uparrow \infty$ a.s. With each $\beta \in \mathfrak{F}_Q^{\mathrm{loc}}$ it is possible to associate uniquely a locally square integrable martingale, denoted Q^β or

$$\int_0^t \int_S \beta(z,s,\mathfrak{x})\, q(dz,ds)$$

in such a way that

(i) $$Q_t^\beta = q_\varepsilon(t,F) \quad \text{if } \beta(z,t,\mathfrak{x}) = \chi_\varepsilon^F(z,x_{t-})$$

where

$$\chi_\varepsilon^F(z,x) = \begin{cases} 1 & \text{if } z \in F \text{ and } \rho(z,x) > \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

(ii) $$Q^{a\beta} + Q^{a'\beta'} = a Q^\beta + a' Q^{\beta'} \qquad \text{for } a,a' \in R,\ \beta,\beta' \in \mathfrak{F}_Q$$

(iii) $$\langle Q^\beta, Q^{\beta'} \rangle_t = \int_0^t \int_S \beta(z,s,\mathfrak{x})\, \beta'(z,s,\mathfrak{x})\, n(dz, x_{s-})\, d\phi_s^0.$$

We then have the following representation theorem for the space $\mathfrak{M}_{\mathrm{loc}}^d$ of locally square-integrable martingales orthogonal to all continuous martingales.

(15) **Theorem** [8, Proposition 5.2].

$$\mathfrak{M}_{\mathrm{loc}}^d = \{Q^\beta : \beta \in \mathfrak{F}_Q^{\mathrm{loc}}\}.$$

Watanabe [16, Theorem 3.1] shows that those elements of $\mathfrak{M}^d$ which are additive functionals correspond to $\beta \in \mathfrak{F}_Q$ of the form

(16) $$\beta(z,t,\mathfrak{x}) = \tilde{\beta}(z, x_{s-})$$

for some function $\tilde{\beta}$: $S \times S \to R$ such that $\tilde{\beta}(z,z)=0$ for all $z \in S$ and (14) is satisfied. Let $\tilde{\mathfrak{F}}_Q$ denote the set of such functions, with a corresponding definition for $\tilde{\mathfrak{F}}_Q^{\mathrm{loc}}$.

Let us now return to the local martingale M^{fd} introduced in (4), (13) above. It is also an additive functional and hence has a stochastic integral representation with integrand in $\tilde{\mathfrak{F}}_Q^{\mathrm{loc}}$. We can identify this integrand precisely.

(17) **Lemma.** *Let $f \in \mathfrak{D}(A)$. Then*

(18) $$M^{fd} = Q^{Bf}$$

where $Bf \in \tilde{\mathfrak{F}}_Q^{\mathrm{loc}}$ is given by

(19) $$Bf(x,z) = f(x) - f(z).$$

Proof. This fact is essentially established in the proof of Theorem 3.1 in [16]. There, the argument is given for f of the form $f = \mathfrak{G}_\lambda g$ ($g \in \mathscr{B}(S)$, $\lambda > 0$, $\mathfrak{G}_\lambda$ the resolvent operator) i.e. for f in the domain of the *strong* generator of $\mathfrak{x}$, but the same argument applies for $f \in \mathfrak{D}$(A). In outline, it is as follows. Let $\{\tau_n\}$ be a sequence of localizing times, fix n and define

$$U_\varepsilon^F(z,x)=(f(z)-f(z))\,\chi_\varepsilon^F(z,x)$$

for any set F such that $E_x p_\varepsilon(t\wedge\tau_n, F)<\infty$. Then $Q^{U_\varepsilon^F}$ is a martingale whose jumps of size $>\varepsilon$ into F coincide with those of $\{M^f_{t\wedge\tau_n}\}$, and one can show that $Q_t^{U_\varepsilon^F}\to M^f_{t\wedge\tau_n}$ as $F\uparrow S$, $\varepsilon\downarrow 0$. The result follows.

The next result is just a restatement of the fact that $Bf\in\mathfrak{F}_Q^{\mathrm{loc}}$.

(20) **Corollary.** *If $f\in\mathfrak{D}(A)$ and $\{\tau_n\}$ is a sequence of localizing times, then*

$$E_x\int_0^{t\wedge\tau_n}(f(z)-f(x_{s-}))^2\,n(dz,x_{s-})\,d\phi_s^0<\infty$$

for all n, $x\in S$, $t>0$.

Equations (18), (19) above give a representation for M^{fd} in the form of a stochastic integral. As regards the continuous part, we need the quadratic variation $\langle M^{fc}, M^{gc}\rangle_t$ and this can be calculated explicitly for those functions $f, g\in\mathfrak{D}(A)$ such that the product $fg\in\mathfrak{D}(A)$ $(fg(x)=f(x)\,g(x))$. First, some notation.

(21) *Notation.* Suppose G is any operator acting on a domain $\mathfrak{D}(G)$ of real-valued functions and that $f, g, fg\in\mathfrak{D}(G)$. Then we denote

$$\Delta_G^{fg}=G(fg)-fGg-gGf.$$

A simple calculation given by Jacod [6, Proposition 13.42] shows that if f, g, $fg\in\mathfrak{D}(A)$ then

$$\langle M^f, M^g\rangle_t=\int_0^t\Delta_A^{fg}(x_s)\,ds.$$

A similar line of reasoning gives the following result.

(22) **Lemma.** *Suppose f, g, $fg\in\mathfrak{D}(A)$. Then*

$$\langle M^{fc}, M^{gc}\rangle_t=\int_0^t\Delta_A^{fg}(x_s)\,ds-\int_0^t\int_S\Delta_B^{fg}(z,x_{s-})\,n(dz,x_{s-})\,d\phi_s^0.$$

Proof. From (4) and Lemma (17) we have

(23) $$df(x_s)=Af(x_s)\,ds+dM_s^{fc}+dQ_s^{Bf}$$

and

(24) $$dfg(x_s)=A(fg)(x_s)\,ds+dM_s^{fg}.$$

Now calculate the product $f(x_t)\,g(x_t)$ using (23), the similar expression for $dg(x_s)$ and the differential formula of stochastic calculus. It is convenient, here and below, to use this in the explicit form given by Kunita and Watanabe [8, Theorem 5.1] (which will be valid, since, in the notation of [8], each $f^{(i)}$ will always be bounded) rather than in the general semimartingale form [6, Theorem 2.52]. We obtain:

$$\text{(25)}\qquad fg(x_t)-fg(x_0)=\int_0^t g(x_s)\,dM_s^{fc}+\int_0^t f(x_s)\,dM_s^{gc}$$
$$+Q^{B(fg)}+\int_0^t\int_S \Delta_B^{fg}(z,x_{s-})\,n(dz,x_{s-})\,d\phi_s^0$$
$$+\int_0^t (gAf+fAg)\,(x_s)\,ds+\langle M^{fc},M^{gc}\rangle_t.$$

The result follows from (24), (25), using the uniqueness of the special semimartingale decomposition of $fg(x_t)$.

(26) **Corollary.** *Suppose in addition that* $\phi_t^0=t$. *Then*

$$\langle M^{fc},M^{gc}\rangle_t=\int_0^t Dfg(x_s)\,ds$$

where

$$\text{(27)}\qquad Dfg(x)=\Delta_A^{fg}(x)-\int_S \Delta_B^{fg}(z,x)\,n(dz,x).$$

The Case $h\in\mathfrak{D}(A)$

If $h\in\mathfrak{D}(A)$ then the multiplicative functional α_t^s of (8) is well-defined. It can be factored as follows:

(28) **Theorem.** $\alpha_t^s=\gamma_t^s\delta_t^s$

where γ, δ *are multiplicative functionals with the following properties:*

(i) γ_t^s *is a local martingale and satisfies the equation* (*we write* γ_t *for* γ_t^0)

$$\text{(29)}\qquad \gamma_t=1-\int_0^t \gamma_{s-}y(s)\,dM_s^{hc}+\int_0^t\int_S \gamma_{s-}(e^{-yBh}-1)\,q(dz,ds).$$

(ii) δ_t^s *is a continuous process of bounded variation and is given explicitly by*

$$\text{(30)}\qquad \ln\delta_t=\int_0^t \tfrac{1}{2}y^2(s)\,d\langle M^{hc}\rangle_s+\int_0^t\int_S (e^{-yBh}-1+yBh)\,n(dz,x_{s-})\,d\phi_s^0$$
$$-\int_0^t y(s)\,Ah(x_s)\,ds-\tfrac{1}{2}\int_0^t h^2(x_s)\,ds.$$

Proof. This is a similar decomposition to that given by Kunita and Watanabe [8, Theorem 6.1]. In their formulation δ was monotone decreasing; this occurs if $E_{x,s}\alpha_t^s\leqq 1$ which, as remarked earlier, is not generally the case here. From (4), (8), (18) we have

$$\ln\alpha_t=-\int_0^t y(s)\,dh(x_s)-\tfrac{1}{2}\int_0^t h^2(x_s)\,ds$$
$$=-\int_0^t y(s)\,Ah(x_s)\,ds-\int_0^t y(s)\,dM^{hc}-Q^{yBh}-\tfrac{1}{2}\int_0^t h^2(x_s)ds$$

where

$$yBh(t,z,\mathfrak{x}) = y(t)\,Bh(z,x_{t-}).$$

Now define

$$\ln\gamma_t = -\int_0^t y(s)\,dM_s^{hc} - \tfrac{1}{2}\int_0^t y^2(s)\,d\langle M^{hc}\rangle_s$$
$$-Q^{yBh} - \int_0^t\int_S (e^{-yBh} - 1 + yBh)\,n(dz,x_{s-})\,d\phi_s^0.$$

Note that

$$\int_0^{t\wedge\tau_n}\int_S |e^{-yBh} - 1 + yBh|\,n(dz,x_{s-})\,d\phi_s^0 < \infty,$$

from Corollary (20), since by the mean value theorem

$$e^{-yBh} - 1 + yBh = \tfrac{1}{2}e^{\theta_s}y^2(s)(f(z) - f(x_{s-}))^2$$

for some $\theta_s \in [0, -yBh(s)]$, and Bh is bounded. Applying the differential formula, we find that γ_t satisfies (29) and hence is a local martingale. Defining $\delta_t^s = \alpha_t^s(\gamma_t^s)^{-1}$ gives (30). This completes the proof.

We now wish to calculate the "generator" of $T_{s,t}^y$. Since, however, this is not a Markovian semigroup, we cannot define its extended generator in precisely the way stated in the Introduction. The appropriate definition is as follows.

(31) *Definition.* Let η_t^s be a multiplicative functional of $\mathfrak{x}$ (not necessarily normalized) and $V_{s,t}$ be the corresponding semigroup: $V_{s,t}f(x) = E_{x,s}[f(x_t)\eta_t^s]$. Let $\mathscr{B}^1(\mathbb{R}^+ \times S)$ denote those functions in $\mathscr{B}(\mathbb{R}^+ \times S)$ which are C^1 in the first variable for each $x \in S$. Then $(J, \mathfrak{D}(J))$ is the *extended generator* of $V_{s,t}$ if for each $f \in \mathfrak{D}(J) \subset \mathscr{B}^1(\mathbb{R}^+ \times S)$ there exists $Jf \in \mathscr{B}(\mathbb{R}^+ \times S)$ such that $\{N_{s,t}^f,\ t \geqq s\}$ is a local martingale, where

$$N_{s,t}^f = \eta_t^s f(t,x_t) - f(s,x_s) - \int_s^t \eta_u^s\left(\frac{\partial}{\partial u} + J\right) f(u,x_u)\,du.$$

This definition clearly coincides with that given in the Introduction if V is Markovian and attention is restricted to time-invariant functions f (i.e. $f \in \mathscr{B}(S)$). In fact attention can, and will be, restricted to $f \in \mathscr{B}(S)$ below except in the final section dealing with boundary conditions, where the time variation must be brought in.

The factorization (28) splits the multiplicative functional transformation (9) into two stages. The second of these, corresponding to δ, just adds a "potential" term to the generator, so it remains to consider the effect of γ. Let

$$T_{s,t}^{\gamma} f(x) = E_{x.s}[f(x_t)\gamma_t^s],$$

Now use the Kunita-Watanabe differential formula to compute the product $f(x_t)\gamma_t^s$ from (4) and (29), for $f \in \mathfrak{D}(A)$. This gives

$$(32)\quad f(x_t)\gamma_t^s = f(x_s) + \int_s^t \gamma_{u-}^s Af(x_u)\,du - \int_s^t \gamma_{u-}^s y(u)\,d\langle M^{fc}, M^{hc}\rangle_u$$
$$+\int_s^t \gamma_{u-}^s \int_S Bf(z,x_{u-})(e^{-y(u)Bh(z,x_{u-})} - 1)\,n(dz,x_{u-})\,d\phi_u^0 + N_t$$

where N_t is a local martingale. We can now formulate the main result concerning the extended generator A_t^y of the semigroup $T_{s,t}^y$

(33) **Theorem.** *Suppose $\mathfrak{x}$ is a Hunt process as defined in the Introduction, and*

(i) *There is a kernel $n(.,.)$ such that $(n(.,.),t)$ is a Lévy system for $\mathfrak{x}$ (i.e. it is possible to take $\phi_t^0 = t$)*

(ii) *$\mathfrak{D} \subset \mathfrak{D}(A)$ is a set such that $h \in \mathfrak{D}$ and $hf \in \mathfrak{D}(A)$ for all $f \in \mathfrak{D}$.*

Then, for any $y \in C[0,T]$ and $t \in [0,T]$, $\mathfrak{D}(A_t^y) \supset \mathfrak{D}$ and

$$A_t^y f(x) = e^{y(t)h(x)} A(e^{-y(t)h} f)(x) - \tfrac{1}{2} h^2(x) f(x) \tag{34}$$

for $f \in \mathfrak{D}$. A_t^y is given explicitly by

$$\begin{aligned} A_t^y f(x) = {} & Af(x) - y(t) Dhf(x) + \int_S Bf(z,x)(e^{-y(t)Bh(z,x)} - 1)\, n(dz,x) \\ & + \big[\tfrac{1}{2} y^2(t) Dhh(x) - y(t) Ah(x) - \tfrac{1}{2} h^2(x) \\ & + \int_S (e^{-y(t)Bh(z,x)} - 1 + y(t) Bh(z,x))\, n(dz,x)\big] f(x) \end{aligned} \tag{35}$$

where D is defined by (21), (27) *and B by* (19).

Proof. Under conditions (i) and (ii), Corollary (26) applies and (32) becomes

$$\begin{aligned} f(x_t)\gamma_t^s = f(x_s) + \int_s^t \gamma_t^s [& Af(x_u) - y(u) Dhf(x_u) \\ & + \int_S Bf(x_u)(e^{-y(u)Bh(z,x_u)} - 1)\, n(dz,x_u)]\, du + N_t. \end{aligned} \tag{36}$$

This identifies the extended generator A^γ of T^γ.

Similarly, under the stated conditions the expression (30) for δ becomes

$$\begin{aligned} \delta_t^s = \exp\Big[& \int_s^t (\tfrac{1}{2} y^2(u) Dhh(x_{u-}) - y(u) Ah(x_{u-}) - \tfrac{1}{2} h^2(x_{u-}) \\ & + \int_S (e^{-y(u)Bh} - 1 + Bh)\, n(dz, x_{u-}))\, du\Big]. \end{aligned} \tag{37}$$

This is a Feynman-Kac type transformation the effect of which is, by standard calculations, to add a potential term to A^γ. Thus combining (36) and (37) gives the expression (35) for A_t^y. Next, notice that throughout this calculation y is an arbitrary function from $C[0,T]$ but the result depends only on $y(t)$. Hence $A_t^y = A_t^{\bar{y}}$ where $\bar{y}(s) = y(t)$ for all $s \in [0,T]$. But

$$\alpha_t^s(\bar{y}) = \exp\Big[-y(t)(h(x_t) - h(x_s)) - \tfrac{1}{2} \int_s^t h^2(x_u)\, du \Big] \tag{38}$$

and

$$\begin{aligned} & E_{x,s}(f(x_t) \exp[-y(t)(h(x_t) - h(x_s))]) \\ & \quad = e^{y(t)h(x)} T_{t-s}(e^{-y(t)h} f). \end{aligned} \tag{39}$$

The expression (34) follows from (38) and (39).

Lévy Generators

In order to use the preceding results to construct recursive filtering algorithms along the lines suggested in the Introduction, we need to be assured that the set $\mathfrak{D}$ of functions on which the extended generator A_s^y is defined is sufficiently large to determine the semi-group $T_{s,t}^y$. This is the case, in particular, when x is an $\mathbb{R}^d$-valued process governed by a Lévy generator [14]. Denote $\mathfrak{D}=C^2(\mathbb{R}^d)$ and for $f\in\mathfrak{D}$ define an operator $\tilde{A}$ by

$$\tilde{A}f(x)=\tfrac{1}{2}\sum_{i,j}a_{ij}(x)\frac{\partial^2 f}{\partial x_i\partial x_j}+\sum_i m_i(x)\frac{\partial f}{\partial x_i} + \int_{R^d}\left(f(x+z)-f(x)-\frac{z'\nabla f(x)}{1+|z|^2}\right)M(dz;x). \tag{40}$$

Suppose the following conditions are satisfied:

(41) (i) $a_{ij}(.)$, $m_i(.)$ are bounded and continuous for all i,j

(ii) $[a_{ij}(x)]\geqq\eta I$ for some $\eta>0$ not depending on x

(iii) $M(dz;x)$ is a σ-finite measure on $\mathbb{R}^d-\{0\}$ such that

$$\int_B\frac{|z|^2}{1+|z|^2}M(dz;x)$$

is bounded and continuous for all Borel sets B of $\mathbb{R}^d\setminus\{0\}$.

Under these conditions, the martingale problem associated with $\tilde{A}$ acting on functions in $C_0^\infty(\mathbb{R}^d)$ is well posed and the corresponding Markov process x is strongly Feller; see [14, Theorem 4.3]. Thus if A denotes the extended generator of x then $\mathfrak{D}\subset\mathfrak{D}(A)$ and $A=\tilde{A}$ on $\mathfrak{D}$. From Theorem 4.3 of [16] (or by a similar argument given by Meyer [11, pp. 159–160] it is easy to see that the Lévy system of x is $\phi_t^0=t$, $n(F,x)=M(F-x;x)$. Thus the conditions of Theorem (33) are satisfied if $h\in C^2(\mathbb{R}^d)$. From (27) we find, using an obvious notation, that

$$Dhf=\sum_{i,j}a_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_j}-(BhBf)\circ n$$

and hence that the extended generator of $T_{s,t}^y$ is given for $f\in\mathfrak{D}$ by

$$A_t^y f=Af-y(t)\sum_{i,j}a_{ij}\frac{\partial h}{\partial x_i}\frac{\partial f}{\partial x_j}+Bf(e^{y(t)Bh}-1-y(t)Bh)\circ n + \psi(t,x)f \tag{42}$$

where

$$\psi(t,x)=\tfrac{1}{2}y^2(t)\left(\sum_{i,j}a_{ij}\frac{\partial h}{\partial x_i}\frac{\partial h}{\partial x_j}-(Bh)^2\circ n\right) - y(t)Ah+(e^{-y(t)Bh}-1+y(t)Bh)\circ n-\tfrac{1}{2}h^2.$$

Thus $\psi(t,x)$ is bounded and continuous.

(43) **Theorem.** *The martingale problem associated with* (42) *is well posed.*

Proof. This is an application of Stroock's results [14] and the details will be found in [3]. One shows that $\tilde{A}_t^y = A_t^y - \psi$ is a Lévy generator satisfying conditions similar to (41) and hence that the martingale problem associated with $\tilde{A}_t^y$ is well-posed. One now constructs the measure corresponding to A_t^y by Feynman-Kac transformation as before, and it is easy to see that the uniqueness is not destroyed by this transformation.

(44) *Remark.* Consider the diffusion case ($M=0$). Then, from (42), A_t^y is a diffusion-type operator whose second-order part is the same as that of A. Thus essentially the same conditions required for existence of a solution to the forward equation for A also assure existence of a solution to the forward equation for A_t^y. This has been studied by Pardoux [13] who states conditions under which the equation

$$\frac{dq}{dt} = (A_t^y)^* q(t), \qquad q(0) = p_0 \in L_2(\mathbb{R}^d) \tag{45}$$

has a unique solution in $L_2([0, T], H^1) \cap C([0, T], L_2(\mathbb{R}^d))$. Here p_0 is the density of the initial distribution π. It then follows from the preceding results that

$$\sigma_t(f) = \int_{\mathbb{R}^d} f(x) e^{y(t)h(x)} q(t, x)\, dx$$

and hence that the conditional density of x_t given y_t is

$$p_t^y(x) = \Big[\int_{R^d} e^{y(t)h(z)} q(t, z)\, dz\Big]^{-1} e^{y(t)h(x)} q(t, x). \tag{46}$$

Now (45), (46) constitute a recursive filter in the sense of (1), (2) and thus the programme for recursive filtering outlined in the Introduction is competely substantiated in this case.

Reflecting Diffusions

The results outlined above in Remark (44) extend to diffusions with boundary conditions, with the interesting feature that the observation sample path y now appears in the boundary conditions specifying $\mathfrak{D}(A_t^y)$ as well as in the coefficients of A_t^y. Here we consider a class of reflecting diffusions, using results of Stroock and Varadhan [16] and Friedman [5]. Of course the results of preceding sections apply directly if $h \in \mathfrak{D}(A)$, but it is highly artificial to assume that h satisfies the relevant boundary conditions, nor is this necessary for our basic stipulation that $h(x_t)$ be a semimartingale.

Let G be a bounded domain in $\mathbb{R}^d$ defined by $G = \{x : \phi(x) > 0\}$ for some C^2 function ϕ, with boundary $\partial G = \{x : \phi(x) = 0\}$. We denote $\bar{G} = G \cup \partial G$. $\tilde{v}(x)$ is the inward normal at $x \in \partial G$. The operator $\tilde{A}$ is given by (40) where $M = 0$ and $a_{ij}(.)$, $m(.)$ are defined on $\bar{G}$ and satisfy (41) (i), (ii). The conormal vector field v is now defined by

$$\nu_i(x)=\sum_j a_{ij}(x)\,\tilde{\nu}_j(x), \qquad x\in\partial G.$$

According to [15], under these conditions there is, for each $x\in\bar{G}$, one and only one probability measure P_x on $C([0,T];\bar{G})$ such that $P_x[x_0=x]=1$ and such that

$$f(x_t)-f(x_0)-\int_0^t \tilde{A}f(x_s)\,ds \tag{47}$$

is a P_x-submartingale for all smooth functions f satisfying[3]

$$\nu(x)\,\nabla f(x)\geqq 0 \qquad x\in\partial G.$$

The corresponding process $\mathfrak{x}=\{x_t\}$ is a strong Markov process. Further, there exists a local time, i.e. an increasing continuous additive functional $\{\mathfrak{k}_t\}$, increasing only when $x_t\in\partial G$, such that for any function $f\in C^2(\bar{G})$

$$M_t^f=f(x_t)-f(x_0)-\int_0^t \tilde{A}f(x_s)\,ds-\int_0^t \nu\nabla f(x_s)\,d\mathfrak{k}_s \tag{48}$$

is a continuous martingale. Calculations as in the proof of Lemma (22) show that

$$\langle M^f,M^g\rangle_t=\int_0^t Dfg(x_s)\,ds \tag{49}$$

where

$$Dfg(x)=\sum_{i,j}a_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}.$$

Now suppose

$$h\in C^2(\bar{G}). \tag{50}$$

Then $h(x_t)$ is (from (48)) a semimartingale, so that the m.f. α_t^s of (8) is well-defined. Using (48) and (49), we find that $\alpha_t=\alpha_t^0$ satisfies

$$d\alpha_t^s=\alpha_t^s(\tfrac{1}{2}y^2(t)Dhh-y(t)\tilde{A}h-\tfrac{1}{2}h^2)dt-\alpha_t^s y(t)\nu\nabla h\;d\mathfrak{k}_t-\alpha_t^s y(t)\,dM_t^h. \tag{51}$$

For $y\in C[0,T]$, let $\mathfrak{D}^y$ be the set of real-valued continuous functions f on $\Xi=[0,T]\times\bar{G}$ such that f is C^1 in t and C^2 in x in the interior of Ξ and

$$\nu\nabla f(t,x)=[\nu\nabla h(x)\,y(t)]\,f(t,x) \tag{52}$$

for $(t,x)\in]0,T[\times\partial G$. Now calculate $f(t,x_t)\alpha_t^s$ for $f\in\mathfrak{D}^y$, using (48) and (51). This gives

$$\begin{aligned} f(t,x_t)\alpha_t^s&=f(s,x_s)+\int_s^t \alpha_u^s\left(\frac{\partial f}{\partial u}+A_u^y f\right)(u,x_u)\,du\\ &\quad+\int_s^t \alpha_u^s(dM_u^f-f(u,x_s)\,y(u)\,dM_u^h)\end{aligned} \tag{53}$$

---

[3] Juxtaposition denotes inner product, so that $\nu\nabla f=\sum_i \nu_i(x)\frac{\partial f}{\partial x_i}(x)$

where A_t^y is given by (42) with $M=0$. According to Definition (31) this says that A_t^y is the extended generator of $T_{s,y}^y$, acting on $\mathfrak{D}^y$, and is enough to establish the connection between the filtering problem and the following parabolic equation with mixed boundary conditions:

$$
\begin{aligned}
&\frac{\partial}{\partial s}u(s,x)+A_s^y u(s,x)=0, && (s,x)\in[0,t[\times G\\
(54)\qquad &u(t,x)=g(x), && x\in\bar{G}\\
&v(x)\,u_x(s,x)+b(s,x)\,u(s,x)=0, && (s,x)\in[0,t[\times\partial G.
\end{aligned}
$$

Here, $b(s,x)=-y(s)\,v(x)\,\nabla h(x)$.

(55) **Theorem.** *Suppose the coefficients of A satisfy conditions* (41) (i), (ii) *and in addition a_{ij} is continuously differentiable, for each i, j. Then* (54) *has a unique continuously differentiable solution u for any $g\in C^2(\bar{G})$, and*

$$u(0,x)=T_{0,t}^y g(x) \tag{56}$$

where $T_{s,t}^y$ is defined by (9) *above.*

Proof. Equation (54) is, in the terminology of Friedman [5], a second initial-boundary problem and the existence and uniqueness follow, under the stated conditions, from Theorem 5.2 (and corollaries) of [5]. Now, $u(s,x)$ satisfies (52) and hence from formula (53) we see that

$$u(0,x)=E_x[\alpha_t^0 g(x_t)].$$

But this is equivalent to (56).

(57) *Remark.* Suppose the initial distribution π has a density function p_0. Then from (10) and (56)

$$\sigma_t(f)=\int_G u(0,x)\,p_0(x)\,dx$$

where u is the solution of (54) with $g(x)=e^{y(t)h(x)}f(x)$. Under additional smoothness conditions on the coefficients of A we can derive the corresponding forward equation, as in Remark (44) above.

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