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A Note on Limit Theorems in Percolation

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Summary. Laws of large numbers and central limit theorems are proved for some cluster functions, e.g. the number of points in a large box which are $(+)$ connected to its boundary or the number of $(+)$ clusters in the box.

1. Introduction

We shall consider Bernoulli atom percolation in Z^2 and shall mainly adopt the notation of Russo [7], which is briefly as follows:

Nearest neighbours in Z^2 are called adjacent and points which are nearest neighbours or diagonal nearest neighbours are called * adjacent. A set $A \subset Z^2$ is connected [* connected] if for all $x, y \in A$ there is a chain of adjacent [* adjacent] points in A which has x and y as terminal points.

The configuration space is $\Omega = \{-1, 1\}^{Z^2}$ and ± 1 are sometimes called spins. A maximal connected [* connected] component of $\omega^{-1}(1)$ is called a $(+)$ cluster [$(+)^*$ cluster] of $\omega \in \Omega$.

The measure is

$P(p) = \prod_{x \in Z^2} \nu_p(x)$, where $0 \leqq p \leqq 1$ and ν_p assigns weights p and $1-p$ to 1 and -1.

For $x \in Z^2$, let $C(x)[C^*(x)]$ be those points which are $(+)$ connected [$(+)^*$ connected] to x. Let $N(x) = |C(x)|$. $N(0)$ is denoted simply as N and the variable $NI(N < \infty)$ is called N'.

Then some basic functions are:

The percolation function $P_\infty(p) = P(N = \infty)$.

The mean size of finite clusters (susceptibility) $S(p) = EN'$.

The number of clusters per site $K(p) = EN^{-1} I(0 < N)$.

The purpose of this note is to check some facts concerning the physical interpretation of these quantities. In Sect. 2 some ergodic properties are mentioned and Sect. 3 contains central limit theorems.

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We shall need some nice results concerning the moments of N', which were obtained independently by Russo [7], or Seymour and Welsh [8].

Let $p_c = \inf\{p\colon P_\infty > 0\}$, $\pi_c = \sup\{p\colon P_\infty = 0 \text{ and } S(p) < \infty\}$ and define p_c^* and π_c^* similarly.

Theorem 1.1 (Russo, Seymour, Welsh). a) $1 - p_c^* = \pi_c \leqq p_c = 1 - \pi_c^*$.

b) *For p off the interval $[\pi_c, p_c]$, $E(N')^r < \infty$ for any r.*

Especially b) will be repeatedly used in the sequel.

2. Ergodic Theorems

The following lemma is a well-known consequence of Birkhoff's ergodic theorem. Cf. e.g. Pitt [6], Theorem 5, p. 337.

Lemma 2.1. *Let $(\Omega, \mathscr{B}, P)$ be a probability space. Let T and U be ergodic transformations and suppose that $f \in L^r$, $r \geqq 1$. Then*

$$n^{-2} \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} f(T^i U^k \omega) \to Ef \quad \text{a.s. and in } L^r \text{ as } n \to \infty.$$

Let $T[U]$ be the translation of the spin configuration one step to the left [downward]. Then T and U are ergodic and the lemma may be applied to appropriate cluster functions to give alternative interpretations of the percolation functions.

Notation. Let K_n be the square $\{z \in Z^2\colon 0 \leqq z_1, z_2 \leqq n-1\}$ and let the (inner) boundary $\partial K_n = \{z \in K_n\colon z_1 \text{ or } z_2 = 0 \text{ or } n-1\}$.

Theorem 2.2. *Let N_n be the number of $(+)$ clusters in K_n which contain no boundary point. Then*

$$n^{-2} N_n \to K(p) \quad \text{a.s. and in any} \quad L^r, \text{ as } n \to \infty.$$

Remark. The convergence was shown by Grimmett [4], using a subadditive argument. The limit was identified as $K(p)$ by Smythe and Wierman, Theorem 3.7 in [9], where they show that $K(p)$ is differentiable a.e. We observe that the derivative exists and is continuous except possibly at p_c[1]. This follows from essentially the same arguments as Proposition 4 in [7]:

Differentiating $K(p) = \sum_{0 \in \gamma} |\gamma|^{-1} p^{|\gamma|}(1-p)^{|\partial\gamma|}$ term by term one formally gets

$$\sum_{0 \in \gamma} p^{|\gamma|-1}(1-p)^{|\partial\gamma|} - \sum_{0 \in \gamma} \frac{|\partial\gamma|}{|\gamma|} p^{|\gamma|}(1-p)^{|\partial\gamma|-1}.$$

Here the summation index γ runs over all connected subsets of Z^2 containing the origin. Since $|\partial\gamma|/|\gamma| \leqq 4$ and

$$\sum_{\substack{0 \in \gamma \\ |\gamma| \geqq n}} p^{|\gamma|}(1-p)^{|\partial\gamma|} = P(n \leqq N < \infty)$$

1 A slight elaboration of the argument shows that $K'(p_c)$ exists if $P_\infty(p_c) = 0$

is an increasing function of p on $[0, p_0]$, where $P_\infty(p_0)=0$, the series above converges uniformly on the interval $[0, p_0]$. On an interval $[p_1, p_2]$, where $p_c < p_1 < p_2 < 1$, the uniform convergence follows from (4.4) of [7].

A similar argument using Theorem 1.1b shows that higher derivatives exist for $p < \pi_c$ or $p > p_c$.

Proof of Theorem 2.2. Let $X(x) = (N(x))^{-1} I(N(x) > 0)$. We then have the identity

$$\sum_{x \in K_n} X(x) = N_n + \sum_{x \in \partial K_n} Y_n(x) \tag{2.1}$$

where

$$Y_n(x) = \begin{cases} \dfrac{k_1}{k_2 k} & \text{if } x \text{ belongs to a } (+) \text{ cluster of size } k \\ & \text{with } k_1 \text{ points in } K_n \text{ and } k_2 \text{ points in } \partial K_n, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\sum_{x \in \partial K_n} Y_n(x) \leqq |\partial K_n| = o(n^2)$.

By Lemma 2.1 $n^{-2} \sum_{x \in K_n} X(x) \xrightarrow{\text{a.s.}} EX(0) = K(p)$ and the theorem follows.

Theorem 2.3. *Let the sizes of the* $(+)$ *clusters in* K_n *be* $d_1^{(n)}, \ldots, d_{N_n}^{(n)}$ *and let* $\tilde{S}_n = n^{-2} \sum_{i=1}^{N_n} (d_i^{(n)})^2$. *Then, if* $S(p) < \infty$, $\tilde{S}_n \to S(p)$ *a.s. and in any* L^r *as* $n \to \infty$, *if* $S(p) = \infty$, $\tilde{S}_n \to \infty$ *a.s.*

Remark. The result $E\tilde{S}_n \to S(p)$ has sometimes been used as a definition of $S(p)$. Cf. Essam [3], p. 221. A quantity much resembling $\tilde{S}_n$ has also been used in a Monte Carlo study of $S(p)$. See Dean [2].

Remark. The results of Russo show that $S(p)$ is infinitely differentiable for $p < \pi_c$ or $p > p_c$.

Proof. Suppose $S(p) < \infty$ and consider the identity

$$\sum_{x \in K_n} N'(x) = n^2 \tilde{S}_n + \sum_{x \in K_n} Y_n(x),$$

where $Y_n(x) = N'(x) I(C(x) \cap \partial K_n \neq \emptyset)$.

Since by Theorem 1.1 and Lemma 2.1 $n^{-2} \sum_{x \in K_n} N'(x) \to S(p)$ a.s. and in L^r, it suffices to check that $n^{-2} \sum_{x \in K_n} Y_n(x) \to 0$ a.s. and in L^r. Let $\varepsilon > 0$ and n_0 be given. Then, if n is large enough

$$\begin{aligned} n^{-2} \sum_{x \in K_n} Y_n(x) &= n^{-2} \sum_{x \in K_n \setminus K_{(1-\varepsilon)n}} Y_n(x) + n^{-2} \sum_{x \in K_{(1-\varepsilon)n}} Y_n(x) \\ &\leqq n^{-2} \sum_{x \in K_n \setminus K_{(1-\varepsilon)n}} N'(x) + n^{-2} \sum_{x \in K_{(1-\varepsilon)n}} N'(x) I(N'(x) \geqq n_0). \end{aligned}$$

It follows from Lemma 3.1 that both of these terms converge a.s. and in L^r, the first one towards $(1-(1-\varepsilon)^2)S(p)$ and the second one towards $(1-\varepsilon)^2 EN' I(N' \geqq n_0)$. As ε and n_0 are arbitrary this ends the proof.

The case when $S(p)=\infty$ follows by truncation.

Theorem 2.4. *Let M_n be the number of points in K_n which are $(+)$ connected to ∂K_n. Then*

a) $n^{-2} M_n \to P_\infty$ *a.s. and in any L^r as $n\to\infty$.*

b) *For $p<\pi_c$, $n^{-1} M_n \to 4\mu$ in any L^r as $n\to\infty$,*

where $\mu = EY(0,0)$ and

$$Y(i,0)=\begin{cases} \dfrac{k_1}{k_2} \text{ if } (i,0) \text{ belongs to a } (+) \text{ cluster with } k_1 \text{ points} \\ \text{in the upper halfplane and } k_2 \text{ points on the } x_1\text{-axis,} \\ 0 \text{ otherwise.} \end{cases}$$

Remark on b) We shall prove b) by referring to the onedimensional ergodic theorem. This simple argument is insufficient to prove a.s. convergence. The reason for this is that the transformation $n\to n+1$ only adds one point to the lower side of K_n but changes all points in the upper side. Still, one may prove a.s. convergence by showing that the fourth central moment of M_n is $\mathcal{O}(n^2)$. This longer argument is omitted.

Proof of a) Write

$$M_n = \sum_{x\in K_n} I(N(x)=\infty) + \sum_{x\in K_n} I(N(x)<\infty,\, C(x)\cap\partial K_n \neq \emptyset)$$

and repeat the argument in the proof of Theorem 2.3.

Proof of b) Write $M_n = \sum_{x\in\partial K_n} Y_n(x)$, where

$$Y_n(x)=\begin{cases} \dfrac{k_1}{k_2} & \text{if } x \text{ belongs to a finite } (+) \text{ cluster with} \\ k_1 & \text{points in } K_n \text{ and } k_2 \text{ points in } \partial K_n, \\ 0 & \text{otherwise.} \end{cases}$$

By symmetry it suffices to check that $n^{-1}\sum_{i=0}^{n-1} Y_n(i,0) \to EY(i,0)$ in any L^r.

$$\begin{aligned} n^{-1}\sum_{i=0}^{n-1} Y_n(i,0) &= n^{-1}\sum_{i=0}^{n-1} Y(i,0) + n^{-1}\sum_{i=0}^{n_0-1} (Y_n(i,0)-Y(i,0)) \\ &\quad + n^{-1}\sum_{i=n_0}^{n-n_0-1} (Y_n(i,0)-Y(i,0)) + n^{-1}\sum_{i=n-n_0}^{n-1} (Y_n(i,0)-Y(i,0)). \end{aligned}$$

As $Y(i,0) \leqq N(i,0)$ it follows from Theorem 1.1 that the Y's have moments of all orders. Thus by the onedimensional ergodic theorem the first term above tends to $EY(0,0)$ in any L^r. In the third term

$$|Y(i,0)-Y_n(i,0)| \leqq 2N(i,0)\, I(N(i,0) \geqq n_0)$$

and by the ergodic theorem

$$\limsup_{n\to\infty} \|\text{third term}\|_r \leqq 2\,\|N(i,0)\, I(N(i,0) \geqq n_0)\|_r,$$

which is small for large n_0. Clearly, the norms of the second and fourth terms tend to zero.

3. Central Limit Theorems

3.1. *Some Lemmas.* Lemma 3.1 is a special case of Theorem 4.2, p. 25 in [1]. Lemma 3.2 is Lemma 20.3, p. 172 in [1], adapted to the case of a twodimensional index set. Its proof is immediate. Lemma 3.3 is a well-known result about m-dependent variables. Cf. e.g. [5], Theorem 19.2.1, p. 370, where it is stated for the case of a one-dimensional index set. For the sake of completeness a proof is given, using Lemma 3.1 and 3.2.

Lemma 3.1. *Let* $\{Y_n\}_1^\infty$ *be r.v. such that for any integer u there is a partition* $Y_n = X_{un} + \delta_{un}$, *such that*

(i) $X_{un} \xrightarrow{d} X_u$, *as* $n\to\infty$ *for u fixed.*
(ii) $X_u \xrightarrow{d} X$, *as* $u\to\infty$.
(iii) $\lim_{u\to\infty}\limsup_{n\to\infty} E\,\delta_{un}^2 = 0$.

Then $Y_n \xrightarrow{d} X$, *as* $n\to\infty$.

Lemma 3.2. *Let* $\{X(x)\}_{x\in Z^2}$ *be a stationary process in* L^2. *Suppose that* $EX(0)=0$ *and* $\sum_{x\in Z^2} |E(X(0)\,X(x))| = \bar{\sigma}^2 < \infty$. *For a finite subset A of* Z^2, *let* $S(A) = \sum_{x\in A} X(x)$. *Then*

a) $|A|^{-1}\, E(S(A))^2 \leqq \bar{\sigma}^2$,
b) $n^{-2}\, E(S(K_n))^2 \to \sigma^2 = \sum_{x\in Z^2} E(X(0)\,X(x))$, *as* $n\to\infty$.

Notation. For $x, y\in Z^2$, let $\|x\| = |x_1| + |x_2|$ and $d(x,y) = \|x-y\|$. Let $\Lambda_n(x) = \{y: d(x,y) = n\}$.

A process $\{X(x)\}_{x\in Z^2}$ is called m-dependent if for all finite subsets A and B of Z^2 such that $d(A,B) > m$, the families $\{X(x)\}_{x\in A}$ and $\{X(x)\}_{x\in B}$ are independent.

Lemma 3.3. *Let* $\{X(x)\}_{x\in Z^2}$ *be a stationary, m-dependent process and assume that* $EX(0)=0$, $E(X(0))^2 < \infty$. *Then*

$$n^{-1} \sum_{x\in K_n} X(x) \xrightarrow{d} N(0,\sigma^2), \quad \textit{as } n\to\infty, \quad \textit{where}$$

$$\sigma^2 = \sum_x E(X(0)\,X(x)).$$

Remark. $\sigma^2 < \infty$ since the sum of covariances contains finitely many terms. In general, however, it may happen that $\sigma^2 = 0$. In this case the assertion of the lemma could be sharpened. For example one may check that in this case $\lim_{n\to\infty} E(S(K_n))^2/n$ exists. In the applications of the lemma which are to follow, unfortunately, I have been unable to prove that this pathological case does not happen.

Proof. Divide K_n into smaller squares (side u) separated by channels of width m. Write for u fixed $n=k(u+m)+s$, $0\leqq s<u+m$, and let the union of the k^2 smaller squares be $A_n=B_n\times B_n$, where

$$B_n=\{z\colon i(u+m)\leqq z<i(u+m)+u, i=0,1,\dots,k-1\}.$$

Consider the partition $n^{-1}S(K_n)=n^{-1}\sum_{x\in A_n} X(x)+\delta_{un}=X_{un}+\delta_{un}$. It is easy to verify conditions (i)–(iii) in Lemma 3.1. By m-dependence $\sum_{x\in A_n} X(x)$ is a sum of k^2 independent sums, each distributed as $S(K_u)$. It follows that

$$k^{-1}\sum_{x\in A_n} X(x)\xrightarrow{d} N(0, E(S(K_u))^2)$$

as $n\to\infty$. Thus $X_{un}\xrightarrow{d} N(0,\sigma_u^2)$, as $n\to\infty$, where $\sigma_u^2=ES(K_u)^2/(u+m)^2$. This verifies (i).

Secondly, it follows from Lemma 3.2b, that $\lim_{n\to\infty}\sigma_u^2=\sigma^2$, which verifies (ii).

Thirdly, by Lemma 3.2a),

$$E\,\delta_{un}^2=n^{-2}E(\sum_{x\in K_n\setminus A_n} X(x))^2\leqq\frac{|K_n\setminus A_n|}{|K_n|}\bar{\sigma}^2.$$

Thus $\limsup_{n\to\infty} E\,\delta_{un}^2\leqq\left(1-\left(\frac{u}{u+m}\right)^2\right)\bar{\sigma}^2$, which tends to zero, as $u\to\infty$. This verifies (iii) and by Lemma 3.1 $n^{-1}S(K_n)\xrightarrow{d} N(0,\sigma^2)$ as $n\to\infty$.

3.2. *Bounded Clusters.* Lemma 3.3 leads immediately to limit theorems for cluster functions, which depend only on the spins in a bounded part of the plane. As an example one has

Theorem 3.4. *Let $N_n(k)$ be the number of $(+)$ clusters in K_n of size k which contain no point in ∂K_n. Then*

$$n^{-1}(N_n(k)-n^2 p_k/k)\xrightarrow{d} N(-\mu_k,\sigma_k^2),$$

as $n\to\infty$, where

$$p_k=P(N=k),\qquad \sigma_k^2=k^{-2}\sum_{x\in Z^2}(P(N(0)=N(x)=k)-p_k^2)$$

and the edge effect $\mu_k=4EX$, where

$$X=\begin{cases}\dfrac{k_1}{k_2 k} & \text{if 0 belongs to a } (+) \text{ cluster of size } k \text{ with } k_1 \text{ points}\\ & \text{in the upper halfplane and } k_2 \text{ points on the } x_1\text{-axis,}\\ 0 & \text{otherwise.}\end{cases}$$

Remark. The edge effect μ_k may be eliminated by assuming toroidal boundary conditions. This remark applies also in the sequel.

Proof. Letting $X(x)=k^{-1} I$ (x belongs to a $(+)$ cluster of size k),

$$\sum_{x\in K_n} X(x)=N_n(x)+\sum_{x\in\partial K_n} Y_n(x),$$

where

$$Y_n(x)=\begin{cases}\frac{k_1}{k\cdot k_2} \text{ if } x \text{ belongs to a } (+) \text{ cluster of size } k\\ \text{with } k_1 \text{ points in } K_n \text{ and } k_2 \text{ points in } \partial K_n,\\ 0 \text{ otherwise.}\end{cases}$$

By Lemma 3.3 the left hand side converges (after norming) towards $N(0,\sigma_k^2)$. By symmetry it then suffices to show

$$n^{-1}\sum_{i=0}^{n-1} Y_n(i,0)\xrightarrow{p} EX, \qquad \text{as } n\to\infty.$$

This is clearly true since $\{Y_n(i,0)\}_{i=k}^{n-k}$ are $2k$-dependent r.v. distributed as X.

Example. For $k=3$,

$$p_3/3=2p^3q^7(2+q),$$
$$\mu_3=12p^3q^7(2+q),$$
$$\sigma_3^2=2p^3q^7(2+q)+4p^6q^{11}(1+27q+57q^2-85q^3-123q^4-35q^5).$$

It is of course a difficult combinatorial problem to compute these parameters for large values of k.

3.3. *Unbounded Cluster Functions.* In order to prove central limit theorems for the quantities treated in Sect. 2 one needs to combine Theorem 1.1 and Lemma 3.3 using some truncation argument.

Theorem 3.5. *Let N_n be as in Theorem 2.2 and assume that $p<\pi_c$ or $p>p_c$. Then*

$$n^{-1}(N_n-n^2K(p))\xrightarrow{d} N(-\mu,\sigma^2),$$

as $n\to\infty$, where

$$\sigma^2=\sum_x C(N^{-1}I(N>0),(N(x))^{-1}I(N(x)>0))$$

and $\mu=4EX$,

$$X=\begin{cases}\frac{k_1}{k\cdot k_2} \text{ if } 0 \text{ belongs to a } (+) \text{ cluster of size } k \text{ with } k_1 \text{ points}\\ \text{in the upper halfplane and } k_2 \text{ points on the } x_1\text{-axis,}\\ 0 \text{ otherwise.}\end{cases}$$

Remark. The condition on p looks unnatural in this context.

Remark. Here and in the following theorems it will be clear from the proofs that $\sigma^2 < \infty$.

Theorem 3.6. *Let $\tilde{S}_n$ be as in Theorem 2.3 and assume that $p < \pi_c$ or $p > p_c$. Then*

$$n(\tilde{S}_n - S(p)) \xrightarrow{d} N(-\mu, \sigma^2),$$

where

$$\sigma^2 = \sum_x C(N'(0), N'(x))$$

and $\mu = 4EX$

$$X = \begin{cases} \dfrac{k_1 k}{k_2} & \text{if 0 belongs to a } (+) \text{ cluster of size } k \text{ with } k_1 \text{ points} \\ & \text{in the upper halfplane and } k_2 \text{ points on the } x_1\text{-axis}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark. For $p < \pi_c$, one may check rigorously that $\sigma^2 > 0$. In this case one may replace $N'(x)$ by $N(x)$ which is an increasing function of the $(+)$ spins. Thus by the F.K.G. inequalities (cf. [7] Lemma 1, p. 42) each covariance in the sum is nonnegative and at least one term is positive.

Theorem 3.7. *Let M_n be as in Theorem 2.4 and let Y be the process defined there. Then* a) *For $p > p_c$*

$$n^{-1}(M_n - n^2 P_\infty) \xrightarrow{d} N(4\mu, \sigma^2),$$

as $n \to \infty$, where $\mu = EY(0,0)$ and $\sigma^2 = \sum_x (P(0, x$ belong to the infinite cluster$) - P_\infty^2)$.

b) *For $p < \pi_c$*

$$n^{-1/2}(M_n - 4n\mu) \xrightarrow{d} N(0, 4\gamma^2),$$

where $\gamma^2 = \sum_i C(Y(0,0), Y(i,0))$.

Remark. In a) one may check that $\sigma^2 > 0$ by noting that $I(N(x) = \infty)$ is an increasing function of the $(+)$ spins.

In the proofs of Theorems 3.5 and 3.6 we need the following:

Lemma 3.8. *Suppose $E(N')^7 < \infty$. Then for any $\varepsilon > 0$ there exists n_0 such that*

$$\sum_{\|x\| \geqq n_0} |C(g(C(0)), g(C(x)))| < \varepsilon$$

for all functions $g(C(x))$ such that $0 \leqq g(C(x)) \leqq N'(x)$.

Proof. Applying the elementary inequality

$$|C(U_1+V_1, U_2+V_2)| \leqq |C(U_1, U_2)| + \sqrt{EU_1^2 EV_2^2} + \sqrt{EU_2^2 EV_1^2} + \sqrt{EV_1^2 EV_2^2}$$

to

$$U_1 = g(C(0))\, I(C(0) \cap \Lambda_{\left[\frac{\|x\|}{2}\right]}(0) = \emptyset),$$
$$U_2 = g(C(x))\, I(C(x) \cap \Lambda_{\left[\frac{\|x\|}{2}\right]}(x) = \emptyset),$$
$$V_1 = g(C(0)) - U_1,$$
$$V_2 = g(C(x)) - U_2,$$

using that

a) U_1 and U_2 are independent.
b) $EU_1^2 = EU_2^2 \leqq E(N')^2 < \infty$ and
c) $EV_1^2 = EV_2^2 \leqq E\left[(N')^2 I\left(N' \geqq \left[\frac{\|x\|}{2}\right]\right)\right] \leqq O(\|x\|^{-5})$

one gets

$$\sum_{\|x\| \geqq n_0} C(g(C(0)), g(C(x))) \leqq \sum_{k=n_0}^{\infty} 4k(0 + \text{const} \cdot k^{-5/2} + \text{const} \cdot k^{-5})$$

which is less than ε if n_0 is large.

Proof of Theorem 3.5. Consider (2.1). As in the proof of Theorem 2.4b) it is easy to see that $n^{-1} \sum_{x \in \partial K_n} Y_n(x) \xrightarrow{p} \mu$, as $n \to \infty$.

It remains to show

$$n^{-1} \sum_{x \in K_n} (X(x) - K(p)) \xrightarrow{d} N(0, \sigma^2), \quad \text{as } n \to \infty. \tag{3.1}$$

Write $X(x) - EX(x) = N(x)^{-1} I(C(x) \neq \emptyset) - K(p)$ as $Y_u'(x) + Y_u''(x)$, where

$$Y_u'(x) = (N(x))^{-1} I(C(x) \neq \emptyset,\ C(x) \cap \Lambda_u(x) = \emptyset)$$
$$- EN(x)^{-1} I(C(x) \neq \emptyset,\ C(x) \cap \Lambda_u(x) = \emptyset).$$

The rest of the proof is to apply Lemma 3.1 to the partition

$$n^{-1} \sum_{x \in K_n} (X(x) - K(p)) = X_{un} + \delta_{un},$$

where

$$X_{un} = n^{-1} \sum_{x \in K_n} Y_u'(x).$$

Since $\{Y_u'(x)\}$ are $2u$-dependent it follows from Lemma 3.3 that $X_{un} \xrightarrow{d} N(0, \sigma_u^2)$, as $n \to \infty$, where

$$\sigma_u^2 = \sum_x C(Y_u'(0), Y_u'(x)). \tag{3.2}$$

This verifies (i).

Secondly, $\lim_{u \to \infty} \sigma_u^2 = \sigma^2$, since we have termwise convergence in (3.2) and the sum (3.2) converges uniformly in u by Lemma 3.8. This verifies (ii).

To verify (iii) it suffices by Lemma 3.2a) to show that

$$\lim_{u\to\infty} \sum_x |C(Y_u''(0), Y_u''(x))| = 0.$$

Here again termwise convergence is immediate and the sum converges uniformly in u by Lemma 3.8.

Hence Lemma 3.1 applies and (3.1) is proved.

The proof of Theorem 3.6 is omitted since it is almost the same as that of Theorem 3.5.

In the proof of Theorem 3.7 one needs to replace Lemma 3.8 by the following

Lemma 3.9. *Let $I(x) = I(N(x) = \infty)$, $I_u(x) = I(C(x) \cap \Lambda_u(x) \neq \emptyset)$. Then, for $p > p_c$,*

$$0 \leqq \begin{cases} C(I(0), I(x)) \\ C(I(0), I_u(x)) \leqq 2\,E(I_{\left[\frac{\|x\|}{2}\right]}(0) - I(0)) \leqq O(\|x\|^{-r}) \\ C(I_u(0), I_u(x)) \end{cases}$$

uniformly in u for any r.

Proof. The right relation follows from Theorem 1.1. as

$$E(I_u(0) - I(0)) \leqq P(N' \geqq u).$$

The left inequalities follow from the F.K.G. inequality since $I(x)$ and $I_u(x)$ are increasing functions.

Concerning the middle inequalities, suppose first that $u \geqq \left[\frac{\|x\|}{2}\right]$. Then

$$E(I_u(0)\, I_u(x)) \leqq E(I_{\left[\frac{\|x\|}{2}\right]}(0)\, I_{\left[\frac{\|x\|}{2}\right]}(x)) = (E\, I_{\left[\frac{\|x\|}{2}\right]}(0))^2$$

by independence and

$$C(I_u(0), I_u(x)) \leqq (E\, I_{\left[\frac{\|x\|}{2}\right]}(0))^2 - (E\, I_u(0))^2 \leqq 2\, E(I_{\left[\frac{\|x\|}{2}\right]}(0) - I(0)).$$

The same relation holds for $C(I(0), I_u(x))$ and $C(I(0), I(x))$. If $u < \left[\frac{\|x\|}{2}\right]$ the lower covariance is 0 while

$$E(I_u(0)\, I(x)) \leqq E(I_u(0) \cdot I_{\left[\frac{\|x\|}{2}\right]}(x)) = E\, I_u(0)\, E\, I_{\left[\frac{\|x\|}{2}\right]}(x)$$

and

$$C(I_u(0), I(x)) \leqq E\, I_u(0) \cdot E\, I_{\left[\frac{\|x\|}{2}\right]}(0) - E\, I_u(0)\, E I(0) \leqq E(I_{\left[\frac{\|x\|}{2}\right]}(0) - I(0)).$$

This shows the middle inequality.

Proof of Theorem 3.7a). Start from the partition

$$M_n = \sum_{x \in K_n} I(x) + \sum_{x \in \partial K_n} Y_n(x),$$

where $I(x)=I(N(x)=\infty)$ and $Y_n(x)$ was defined in the proof of Theorem 2.4. Since it is easy to check that $n^{-1}\sum_{x\in\partial K_n}\cdot Y_n(x)\xrightarrow{p}4\mu$, it remains to show that

$$n^{-1}\sum_{x\in K_n}(I(x)-P_\infty)\xrightarrow{d}N(0,\sigma^2).$$

Write $I(x)-P_\infty=Y_u'(x)+Y_u''(x)$, where $Y_u'(x)=I_u(x)-E\,I_u(x)$ and $I_u(x)$ was defined in Lemma 3.9.

From here on, the arguments in the proof of Theorem 3.5 may be repeated almost literally, referring to Lemma 3.9 concerning the uniform convergence.

Proof of Theorem 3.7b). In this case we are to show that

$$n^{-\frac{1}{2}}\sum_{x\in\partial K_n}(Y_n(x)-\mu)\xrightarrow{d}N(0,4\gamma^2),\qquad \text{as } n\to\infty. \tag{3.3}$$

Extend the definition of $Y(x)$ in Theorem 2.4 in the natural way to all x in ∂K_n. Then, one may drop the indices of the Y's in (3.3), as

$$\begin{aligned}E\,|n^{-\frac{1}{2}}\sum_{x\in\partial K_n}(Y_n(x)-Y(x))|&\leqq 8\,n^{-\frac{1}{2}}\sum_{i=0}^{[\frac{n}{2}]}E\,|Y_n(i,0)-Y(i,0)|\\&\leqq 16n^{-\frac{1}{2}}\sum_{i=0}^{\infty}EN(i,0)\,I(N(i,0)\geqq i),\end{aligned}$$

which tends to zero since $EN^2<\infty$.

Introduce $Y_u'(x)=Y(x)\,I(C(x)\cap A_u(x)=\emptyset)$ and $Y''(x)$ by

$$Y(x)=Y_u'(x)+Y_u''(x).$$

Let further $J_{n,u}$ be those points in ∂K_n which are at a distance no less than $2u$ from any corner of K_n and consider the partition

$$\begin{aligned}n^{-\frac{1}{2}}\sum_{x\in\partial K_n}(Y(x)-\mu)&=n^{-\frac{1}{2}}\sum_{x\in J_{n,u}}(Y_u'(x)-E\,Y_u'(x))+n^{-\frac{1}{2}}\sum_{x\in\partial K_n\setminus J_{n,u}}(Y(x)-\mu)\\&\quad+n^{-\frac{1}{2}}\sum_{x\in J_{n,u}}(Y_u''(x)-E\,Y_u''(x))=X_{un}+\delta_{un}=X_{un}+\delta_{un}^{(1)}+\delta_{un}^{(2)}.\end{aligned}$$

We shall apply Lemma 3.1 to this partition.

X_{un} can be split into four independent terms and the one-dimensional analogue of Lemma 3.3 may be applied to each part. Thus

$$X_{un}\xrightarrow{d}N(0,4\gamma_u^2),$$

where

$$\gamma_u^2=\sum_i C(Y_u'(0,0),\,Y_u'(i,0)).$$

This verifies (i). The sum further converges uniformly in u by Lemma 3.8 and it follows that

$$\lim_{u\to\infty}\gamma_u^2=\sum_i C(Y(0,0),Y(i,0))=\gamma^2,$$

which verifies (ii). To verify (iii) it remains to check that

$$\lim_{u\to\infty} \limsup_{n\to\infty} E\,\delta_{un}^{(v)2} = 0 \qquad \text{for } v=1,2.$$

For $v=1$ this is immediate. For $v=2$ it suffices by Lemma 3.2a) to show that

$$\lim_{u\to\infty} \overline{\lim_{n\to\infty}} \sum_i |C(Y_u''(0,0), Y_u''(i,0))| = 0$$

and this follows as before since the sum converges uniformly in u by Lemma 3.8.

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