Z. Wahrscheinlichkeitstheorie verw. Gebiete
56, 549–554 (1981)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete
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On General Versions of Erdős-Rényi Laws

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Summary. Rather general versions of the Erdős-Rényi [6] new law of large numbers have recently been given by S. Csörgő [5] for sequences of *rv*'s which have stationary and independent increments and satisfy a first order large deviation theorem. It is shown that Csörgő's results can be extended to cover also situations of stochastic processes where stationarity and independence of increments are not generally available, but for randomly chosen subsequences of the process. Examples demonstrate that the main result can be applied, for instance, to waiting-times in $G/G/1$ queuing models or cumulative processes in renewal theory, where Erdős-Rényi type laws cannot be derived from Csörgő's theorems.

1. Introduction and Results

In [6] Erdős and Rényi proved what they called a new law of large numbers dealing with an a.s. asymptotic behaviour of the maximum increments of partial sum sequences over blocks of certain lengths.

Theorem 1 (Erdős-Rényi). *Let $\{X_i\}_{i=1,2,\ldots}$ be an i.i.d. sequence on $(\Omega, \mathfrak{A}, P)$ with partial sums $S_0=0$, $S_n=X_1+\ldots+X_n$. Suppose (i) $\varphi(t)=E\exp(tX_1)<\infty$ for all $t\in(0,t_1)$. Then, for each $a\in A=\{\varphi'(t)/\varphi(t): t\in(0,t_1)\}$ and $C=C(a)$ such that $\exp(-1/C)=\inf_t \varphi(t)\exp(-ta)=\rho(a)$, we have*

$$\lim_{N\to\infty} \max_{0\leq n\leq N-[C\log N]} \frac{S_{n+[C\log N]}-S_n}{[C\log N]}=a \quad \text{w.p.1.} \tag{1}$$

It is a remarkable fact that the functional dependence $C=C(a)$, which can a.s. be obtained from (1) on the set A, determines the underlying distribution via its moment-generating function φ. This so-called Erdős-Rényi phenomenon seems to be the main essence of Theorem 1.

Various generalizations of the Erdős-Rényi law have been found, e.g. for weighted sums of i.i.d. *rv*'s or non i.i.d. *rv*'s with absolutely continuous distri-

0044-3719/81/0056/0549/$01.20

butions (Book [2, 3]), for continuous functionals of multivariate partial sum differences (Bártfai [1]) or, generally, for such functions of moving blocks of i.i.d. *rv*'s and those of empirical measures of these blocks, for which a first order large deviation theorem holds (S. Csörgő [5], who also gives a very complete list of further references). Moreover, the latter author provided a unified approach to Erdős-Rényi laws realizing that the essential facts for the proof are independence and stationarity of the increments of the considered process and an exponential large deviation behaviour. Here we state a reformulated version of S. Csörgő's theorems forgetting about the special dependence upon an i.i.d. sequence:

Theorem 2 (S. Csörgő). *Let* $\{T_{n,K}\}_{\substack{n=0,1,\ldots\\K=1,2,\ldots}}$ *be a double sequence of real-valued random variables on* $(\Omega, \mathfrak{A}, P)$ *satisfying*

(i) $$\lim_{K\to\infty} P(T_{0,K} \geqq Ka)^{1/K} = \rho(a)$$

exists for a-values in some interval (a_0, a_1), *where* ρ *is a strictly decreasing function,* $0<\rho(a)<1$;

(ii) *for each* K, $\{T_{n,K}\}_{n=0,1,\ldots}$ *is a stationary sequence;*

(iii) *for each* K, $\{T_{iK,K}\}_{i=0,1,\ldots}$ *is an i.i.d. sequence. Then, for* $a\in(a_0,a_1)$ *and* $C=C(a)$ *such that* $\exp(-1/C)=\rho(a)$, *we have*

$$\lim_{N\to\infty} \max_{0\leqq n\leqq N-[C\log N]} \frac{T_{n,[C\log N]}}{[C\log N]} = a \quad \text{w.p. 1.} \tag{2}$$

Setting $T_{n,K}=S_{n+K}-S_n$, Theorem 2 obviously implies Theorem 1. But assumptions (ii) and (iii) of Theorem 2 are essentially based upon an i.i.d. situation. On the other hand it has been shown (see e.g. [8, 9, 10] that Erdős-Rényi type laws are available even if the underlying processes have neither stationary nor independent increments. This prompted us to extend S. Csörgő's result in the following way:

Theorem 3. *Let* $\{T_{n,K}\}_{\substack{n=0,1,\ldots\\K=1,2,\ldots}}$ *be a double sequence of real-valued random variables on* $(\Omega, \mathfrak{A}, P)$ *satisfying*

(i) *for a-values in some interval* (a_0, a_1) *there exists a strictly decreasing function* $\rho_1(a)$, $0<\rho_1(a)<1$, *such that for each* $C>0$

$$\limsup_{N\to\infty} P(T_{n,[C\log N]} \geqq [C\log N]\,a)^{1/[C\log N]} \leqq \rho_1(a)$$

holds uniformly with respect to (w.r.t.) n;

(ii) *There exists a sequence* $\nu_{0,N}<\nu_{1,N}<\ldots$ *of random indices such that, for each* $N=1,2,\ldots$ *and* $C>0$:

(ii.1) $\{T_{\nu_{i,N};[C\log N]}\}_{i=0,1,\ldots}$ *is a sequence of independent rv's*

(ii.2) $\liminf_{N\to\infty} P(T_{\nu_{i,N};[C\log N]} \geqq [C\log N]\,a)^{1/[C\log N]} \geqq \rho_2(a)$

holds uniformly w.r.t. i, where $\rho_2(a)$, $0<\rho_2(a)<1$, *is another strictly decreasing function on* (a_0, a_1);

(ii.3) $\sum_{N=1}^{\infty} P(L_N \leqq (1-\delta)[N/C\log N]) < \infty$

for some $0<\delta<1$, *where* $L_N = \#\{\nu_{i,N}\colon \nu_{i,N} \leqq N-[C\log N]\}$.

Then, for $a \in (a_0, a_1)$ *and* $C_1 = C_1(a)$ *such that* $\exp(-1/C_1) = \rho_1(a)$, *we have*

$$\limsup_{N\to\infty} \max_{0\leqq n\leqq N-[C_1 \log N]} \frac{T_{n,[C_1 \log N]}}{[C_1 \log N]} \leqq a \quad \text{w.p. } 1, \tag{3}$$

and, for $a \in (a_0, a_1)$ *and* $C_2 = C_2(a)$ *such that* $\exp(-1/C_2) = \rho_2(a)$, *we have*

$$\liminf_{N\to\infty} \max_{0\leqq \nu_{i,N}\leqq N-[C_2 \log N]} \frac{T_{\nu_{i,N};[C_2 \log N]}}{[C_2 \log N]} \geqq a \quad \text{w.p. } 1. \tag{4}$$

Remark. Using the deterministic sequence $\nu_{i,N} = i[C \log N]$ $(i = 0, 1, \ldots)$ of indices and $\rho_1(a) = \rho_2(a)$ we see that Theorem 2 is an immediate consequence of Theorem 3. However, the latter is mainly constructed for other than i.i.d. situations, as will be indicated by a couple of examples.

2. Proof of Theorem 3

The proof is similar to the i.i.d. case. Therefore we only outline some necessary modifications. For details we refer to [11].

Proof. a) From assumption (i) it can be shown that, for $a < a' \in (a_0, a_1)$ and N sufficiently large,

$$P\Big(\max_{0\leqq n\leqq N-[C_1 \log N]} T_{n,[C_1 \log N]} \geqq [C_1 \log N]\, a'\Big) \leqq N^{-\beta}, \tag{5}$$

for some $\beta > 0$. Defining N_j = largest integer N such that $[C_1 \log N] = j$, and using the Borel-Cantelli lemma, we obtain from (5) the relation

$$\limsup_{j\to\infty} \max_{0\leqq n\leqq N_j-[C_1 \log N_j]} \frac{T_{n,[C_1 \log N_j]}}{[C_1 \log N_j]} \leqq a' \quad \text{w.p. } 1.$$

This turns out to hold for the whole sequence, too, since $\{N_j\}$ has been chosen in an appropriate way. Hence, letting a' tend to a, assertion (3) is proved.

b) By assumptions (ii.1) and (ii.2) one can estimate, for $a > a'' \in (a_0, a_1)$ and N sufficiently large,

$$P\Big(\max_{i=0,1,\ldots,l_N-1} T_{\nu_{i,N};[C_2 \log N]} < [C_2 \log N]\, a''\Big) \leqq \exp(-N^{\gamma}) \tag{6}$$

for some $\gamma > 0$, where $l_N = [(1-\delta)[N/C_2 \log N]]$.

Combining (6) with (ii.3), we immediately obtain

$$\sum_{N=1}^{\infty} P\Big(\max_{0\leqq \nu_{i,N}\leqq N-[C_2 \log N]} T_{\nu_{i,N};[C_2 \log N]} < [C_2 \log N]\, a''\Big) < \infty.$$

By using the Borel-Cantelli lemma again and letting a'' tend to a, we get (4) which renders the proof complete.

3. Some Examples

A series of examples has been given by S. Csörgő in [5] illustrating the use of Theorem 2. It would be difficult to extend this list essentially, giving other Erdős-Rényi laws depending upon an i.i.d. situation. Therefore we concentrate on some situations which are not covered by Theorem 2 and thus indicate the need for an extended version, as given in Theorem 3.

Example 1 (Weighted sums). Let $S_n = \sum_{i=1}^{n} \alpha_i X_i$ denote the weighted sums of a standardized i.i.d. sequence $\{X_i\}_{i=1,2,\ldots}$ under weights $\{\alpha_i\}_{i=1,2,\ldots}$. Put $A_n = \sum_{i=1}^{n} \alpha_i$ and $D(N,K) = \max_{0 \leq n \leq N-K} (S_{n+K} - S_n)/(A_{n+K} - A_n)$. Then Book's [2, 3] 'Erdős-Rényi law of large numbers for weighted sums', which is

$$\limsup_{N\to\infty} D(N, [C_1(a) \log N]) \leq a \qquad \text{w.p. } 1,$$

$$\liminf_{N\to\infty} D(N, [C_2(a) \log N]) \geq a \qquad \text{w.p. } 1$$

for certain functions $C_1(a)$, $C_2(a)$, is covered by Theorem 3. Just set

$$T_{n,K} = K \frac{S_{n+K} - S_n}{A_{n+K} - A_n},$$

$$\rho_i(a) = \exp(-1/C_i(a)) \qquad (i = 1, 2),$$

and use a deterministic sequence $\{v_{i,N}\}_{i=1,2,\ldots}$, i.e.

$$v_{i,N} = i[C \log N], \qquad i = 1, 2, \ldots.$$

Then, under Book's conditions, the assumptions of Theorem 3 are fulfilled.

Example 2 (Waiting-times). In [8] the following Erdős-Rényi law has been proved for waiting-times $\{W_n\}_{n=0,1,\ldots}$ in a queuing model $G/G/1$:

$$\lim_{N\to\infty} \max_{0 \leq n \leq N - [C \log N]} \frac{W_{n+[C \log N]} - W_n}{[C \log N]} = a \qquad \text{w.p. } 1$$

for a's in some interval $(0, a_1)$ and C such that $\exp(-1/C) = \rho(a)$, where $\rho(a)$ denotes the first order large deviation rate of the sequence $\{W_n\}_{n=0,1,\ldots}$. Using a random sequence of indices, defined by

$$v_{0,N} = 0,$$

$$v_{i,N} = \text{smallest integer} > v_{i-1,N} + [C \log N] \text{ such that } W_{v_{i,N}} = 0$$

$(i = 1, 2, \ldots)$ and $\rho_1(a) = \rho_2(a) = \rho(a)$, this result can also be derived from Theorem 3.

Theorem 3 may even be helpful when treating stochastic processes with continuous parameters. Consider, for instance, the following

Example 3 (Renewal processes). Let $N(t)$ denote the number of renewals occuring up to time $t(\geqq 0)$ under an i.i.d. sequence $\{X_i\}_{i=1,2,\ldots}$ of nonnegative failure times. In [9, 10] Erdős-Rényi type laws were stated for processes associated with such a renewal counting process $\{N(t)\}_{t\geqq 0}$, the simplest version appearing as follows:

$$\lim_{T\to\infty} \sup_{0\leqq t\leqq T-C\log T} \frac{N(t+C\log T)-N(t)}{C\log T}=a \qquad \text{w.p. } 1,$$

where the functional dependence $C=C(a)$ is determined by the exponential large deviation rate $\rho(a)$ of $\{N(t)\}_{t\geqq 0}$. Using some monotonicity properties of the process, it can easily be seen that the limit assertions need only be proved for T being integer-valued. So, the methods of Theorem 3 can be used again to derive the desired result. For the lim inf-part take a sequence of random times defined by

$$\tau_{0,T}=0,$$
$$\tau_{i,T}=\text{first renewal time after } \tau_{i-1,T}+C\log T \quad (i=1,2,\ldots),$$

which provides i.i.d. increments $N(\tau_{i,T}+C\log T)-N(\tau_{i,T})$, $i=0,1,\ldots$.

4. Concluding Remarks

In certain situations, improved versions of Erdős-Rényi type laws are available which can also be viewed as convergence rate statements in the given limit theorems (see e.g. Révész [7], M. Csörgő and Steinebach [4]). Such improvements can be obtained whenever a better than a first order large deviation rate holds.

The results in [4] are mainly based on improved large deviation rates for partial sums, which, by identity $P(N(t)\geqq n)=P(S_n\leqq t)$, can at least be carried over to the renewal counting process and thus lead to improved Erdős-Rényi laws in that case. But, from the arguments in [4] and the proof of Theorem 3, it is obvious that improved general versions of Erdős-Rényi laws are also available, if assumptions (i) and (ii.2) can be replaced by large deviation results including second order terms.

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Received August 17, 1980; in revised form January 25, 1981