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# Recurrence and Transience for Random Walks with Stationary Increments

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**Summary.** Let $S_n = \xi_1 + \ldots + \xi_n$, $n \geqq 1$, be the partial sums of stationary, dependent random variables in $\mathbb{R}^m$. The probability space can be partitioned into $I_t \cup I_r$, where $I_t = \{\|S_n\| \to \infty\}$ and $I_r = \{$each $S_n$ is limit point of $(S_n)_{n \geqq 1}\}$. This result follows from the inclusion $\{\|S_n\| > \varepsilon$ for $n > 0\} \subset I_t$ a.s., which is obtained by using Kac's inequality.

## 1. Introduction and Results

Let $(S_n)_{n \geqq 0}$ be a process with values in the $m$-dimensional Euclidean space $\mathbb{R}^m$. We assume that $S_0 = 0$ and that the increments $\xi_n = S_n - S_{n-1}$, $n \geqq 1$, form a stationary sequence. The process $(S_n)_{n \geqq 0}$ will be called a *random walk with stationary increments.* We discuss a technique to investigate transience of $(S_n)_{n \geqq 0}$ and use it to prove a result that justifies the subdivision into transient and recurrent random walks, for the generalization of the class of random walks given above.

This class of processes was studied earlier in [5] and [7]. [1] discusses renewal theory for these random walks. Also in ergodic theory the process $(S_n)_{n \geqq 0}$ is studied. For example the limit behavior of $\frac{1}{n} S_n$ is described by the individual ergodic theorem. The results below were discussed earlier in [1] for random walks on the real line. The present approach is not only simpler but is also suitable for random walks on $\mathbb{R}^m$, $m > 1$.

The technique that enables us to investigate transience of a random walk, is presented in Sect. 3. Using this technique we obtain the following theorem. Let $\|.\|$ be the Euclidean norm on $\mathbb{R}^m$.

**Theorem 1.** *For any positive* $\varepsilon$

$$\{\|S_n\| > \varepsilon \textit{ for } n > 0\} \subset \{\lim_{n \to \infty} \|S_n\| = \infty\} \quad \textit{a.s.}$$

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Theorem 2 below is a consequence of this result. To formulate it, define the *sets of transience* $I_t$ and *of recurrence* $I_r$ by

$$I_t = \{\lim_{n\to\infty} \|S_n\| = \infty\},$$

$$I_t = \{\text{each } S_n,\ n \geqq 0, \text{ is a limit point of } (S_n)_{n\geqq 0}\}.$$

These definitions are justified by

**Theorem 2.** $P(I_t \cup I_r) = 1$.

Obviously this result implies the following recurrence criterion:

$$\{0 \text{ is a limit point of } (S_n)_{n\in 0}\} = I_r \quad \text{a.s.}$$

Note that in case the increments of the random walk are independent, $(\xi_n)_{n\geqq 1}$ is ergodic (compare [2]). So then we have, because $I_r$ is invariant for $(\xi_n)_{n\geqq 1}$, that $I_r$ has probability 0 or 1.

Suppose $P(I_r) = 1$. Define the random set

$$L = \{x \in \mathbb{R}^m\colon x \text{ is a limit point of } (S_n)_{n\geqq 0}\}.$$

If the increments of the random walk are independent, it is well-known that $L$ coincides a.s. with a (non random) lattice on $\mathbb{R}^m$. However without this independence assumption, the behavior of $L$ is much more irregular.

*Example.* Let $(X_n)_{n\geqq 0}$ consist of independent random variables, uniformly distributed on $(0, 1)$. Define $\xi_n = X_n - X_{n-1}$, $n \geqq 1$, and note that $S_n = X_n - X_0$, $n \geqq 0$. One easily proves that $L$ is a random set, given by $L = [-X_0, -X_0 + 1]$ a.s.

Any stationary sequence $(\xi_n)_{n\geqq 1}$ can be extended to a stationary sequence $(\xi_n)_{n\in\mathbb{Z}}$ (compare [2]). An important role in our discussions is played by the *extended* random walk $(S_n)_{n\in\mathbb{Z}}$, defined by requiring

$$S_0 = 0, \quad \xi_n = S_n - S_{n-1}, \qquad n \in \mathbb{Z}.$$

Note that $S_n = \sum_1^n \xi_i$, $n \geqq 1$, and $S_n = -\sum_{n+1}^0 \xi_i$, $n \leqq -1$.

The lemma in Sect. 3 is our main tool in the proof of the results above. To discuss it, we first have to introduce the concept of a return set.

## 2. Return Sets

In the theory of random walks with stationary, independent increments, regeneration epochs are frequently used. If only stationary is imposed, it is possible to make use of the so-called return times, that have weaker, but still useful properties. [6] shows that the class of return times is basic for the much larger class of stopping times. Below we introduce the narrowly related concept of a return set. By well-known arguments (see [2], Sect. 6.10), we prove a simple result for return sets.

Let $T$ be the *shift transformation* on $(\mathbb{R}^m)^{\mathbb{Z}}$, defined by $T(x_i)_{i\in\mathbb{Z}}=(x_{i+1})_{i\in\mathbb{Z}}$. Write $\xi$ for the stationary sequence $(\xi_n)_{n\in\mathbb{Z}}$. Let $\mathfrak{B}^m$ be the Borel $\sigma$-field on $\mathbb{R}^m$ and take $B\in\prod_{n\in\mathbb{Z}}\mathfrak{B}^m$. The *return set* $M$ of $B$ is defined as

$$M=\{n\in\mathbb{Z}\colon T^n\xi\in B\}.$$

An example of a return set is the set of ascending record times

$$\{n\in\mathbb{Z}\colon S_n>S_k \text{ for } k<n\},$$

with $B$ given by

$$B=\left\{(x_i)_{i\in\mathbb{Z}}\colon 0>-\sum_{k+1}^{0} x_i,\ k<0\right\}.$$

Let $M$ be a return set. Using Poincaré's recurrence principle (see [2] and [3]) one deduces easily that on $\{M\neq\emptyset\}$

$$\#M\cap\{\dots,-2,-1\}=\#M\cap\{1,2,\dots\}=\infty \quad \text{a.s.}$$

and hence the elements of $M$ can be written on this set as

$$\dots<\tau_{-1}<0\leqq\tau_0<\tau_1<\dots$$

Define on $\{M\neq\emptyset\}$ the process $\tilde{\xi}$ by $\tilde{\xi}=(\tilde{\xi}_n)_{n\in\mathbb{Z}}$, where

$$\tilde{\xi}_n=(\xi_{\tau_n+j})_{j=1}^{\tau_{n+1}-\tau_n}, \qquad n\in\mathbb{Z}.$$

These random vectors have their values in the space $\bigcup_{k\geqq 1}(\mathbb{R}^m)^k$.

**Proposition 3.** *Let $M$ be a return set that is non-empty with positive probability. Then $\{0\in M\}$ has positive probability and, given $\{0\in M\}$, the process $\tilde{\xi}$ is stationary.*

*Proof.* The event $\{0\in M\}$ has positive probability, for otherwise, by stationarity, $\{k\in M\}$ would be a $P$-null set for all $k$, thus contradicting $P(M\neq 0)>0$. Choose $A\in\prod_{n\in\mathbb{Z}}\mathfrak{B}^m$ arbitrary. Splitting up $\{\xi\in A,\ \tau_0=0\}$ by considering the occurrences of $\tau_{-1}=-n$, $n\geqq 1$, we get

$$\begin{aligned}
&P(\xi\in A,\ \tau_0=0)\\
&\quad=\sum_{n\geqq 1} P(\xi\in A,\ \xi\in B,\ T^{-1}\xi\notin B,\dots,T^{-n+1}\xi\notin B,\ T^{-n}\xi\in B)\\
&\quad=\sum_{n\geqq 1} P(T^n\xi\in A,\ T^n\xi\in B,\ T^{n-1}\xi\notin B,\dots,T\xi\notin B,\ \xi\in B)\\
&\quad=\sum_{n\geqq 1} P((\xi_{\tau_1+k})_{k\in\mathbb{Z}}\in A,\ \tau_1=n,\ \tau_0=0)\\
&\quad=P((\xi_{\tau_1+k})_{k\in\mathbb{Z}}\in A,\ \tau_0=0).
\end{aligned}$$

Divide by the probability of $\{\tau_0=0\}=\{0\in M\}$. It follows that, given $\{0\in M\}$, the processes $(\xi_n)_{n\in\mathbb{Z}}$ and $(\xi_{n+\tau_1})_{n\in\mathbb{Z}}$ have the same distribution. This implies the assertion. To see this write $\xi'_n=\xi_{n+\tau_1}$, $n\in\mathbb{Z}$. If $\tau'_n$ is defined from $\xi'=(\xi'_n)_{n\in\mathbb{Z}}$ just as $\tau_n$ is from $\xi=(\xi_n)_{n\in\mathbb{Z}}$, then $\tau'_n=\tau_{n+1}-\tau_1$. Let $\tilde{\xi}'_n$ be defined from $\xi'$ just as $\tilde{\xi}_n$ is from $\xi$. Then

$$\tilde{\xi}'_n=(\xi'_{\tau'_n+j})_{j=1}^{\tau'_{n+1}-\tau'_n}=(\xi_{\tau_{n+1}+j})_{j=1}^{\tau_{n+2}-\tau_{n+1}}=\tilde{\xi}_{n+1}.$$

It was proved that $\xi$ and $\xi'$ are equally distributed, given $\{0\in M\}$. Hence this holds also for $\tilde{\xi}$ and $\tilde{\xi}'$ and so $(\tilde{\xi}_n)_{n\in\mathbb{Z}}$ is distributed as $(\tilde{\xi}_{n+1})_{n\in\mathbb{Z}}$, given $\{0\in M\}$. □

## 3. Inheritance of Transience

Let $s_K=(s_k)_{k\in K}$ be a sequence of elements of $\mathbb{R}^m$, indexed by $K\subset\mathbb{Z}$. We say that this sequence is *transient* if each bounded set $V\in\mathfrak{B}^m$ contains only finitely many elements of the sequence.

**Main Lemma.** *Let $M$ be a return set. If on $\{M\neq\emptyset\}$ the subsequence $(S_n)_{n\in M}$ is transient a.s., then also on $\{M\neq\emptyset\}$ the sequence $(S_n)_{n\in\mathbb{Z}}$ is transient a.s.*

*Proof.* We may suppose that $P(M\neq\emptyset)>0$. Define $\tau_n$ and $\tilde{\xi}_n$ as above. Each increment $S_{\tau_{n+1}}-S_{\tau_n}$ of the process $(S_{\tau_n})_{n\in\mathbb{Z}}$ equals the sum of the components of $\tilde{\xi}_n$. Hence by Proposition 3, given $\{0\in M\}$, the process $(S_{\tau_n})_{n\in\mathbb{Z}}$ is a random walk with stationary increments. By the assumption of the lemma, this random walk is transient a.s.

We consider the random walk $(S_n)_{n\in\mathbb{Z}}$ on $\{M\neq\emptyset\}$. We associate to each point $S_n$ a point of $(S_n)_{n\in M}$, or better, we associate to each $n\in\mathbb{Z}$ an element of $M$, given by

$$\sigma(n)=\inf\{\tau_k\colon \|S_n-S_{\tau_k}\|=\inf_i\|S_n-S_{\tau_i}\|,\ k\in\mathbb{Z}\}.$$

So $\sigma(n)$ is the smallest element in $M$ for which the distance of $S_n$ to $S_M$ is minimized. Because $S_M$ is transient a.s., $\sigma(n)$ is properly defined on $\{M\neq\emptyset\}$. We consider for any $k$ the number $\#\sigma^{-1}\{\tau_k\}$ of elements $n\in\mathbb{Z}$ associated to $\tau_k\in M$. Using Proposition 3 one easily observes that $(\#\sigma^{-1}\{\tau_k\})_{k\in\mathbb{Z}}$ is stationary, given $\{0\in M\}$. We calculate the expected value of the random variables of this sequence. Write $P_0$ and $E_0$ for probability and expectation, given $\{0\in M\}$. Using Proposition 3 in the third equality below, we get

$$\begin{aligned}
E_0\#\sigma^{-1}\{0\}&=\sum_{k\in\mathbb{Z}}E_0\#\{n\colon \sigma(n)=0,\ \tau_k<n\leqq\tau_{k+1}\}\\
&=\sum_{k\in\mathbb{Z}}E_0\#\{j\colon \sigma(j+\tau_k)=0,\ 0<j\leqq\tau_{k+1}-\tau_k\}\\
&=\sum_{k\in\mathbb{Z}}E_0\#\{j\colon \sigma(j)=\tau_{-k},\ 0<j\leqq\tau_1\}\\
&=E_0\sum_{j=1}^{\tau_1}\sum_{k\in\mathbb{Z}}I_{\{\sigma(j)=\tau_{-k}\}}\\
&=E_0\tau_1=1/P(0\in M\mid M\neq\emptyset).
\end{aligned}$$

The last equality is due to Kac (compare [2]). Hence $\#\sigma^{-1}\{0\}<\infty$ $P_0$-a.s. and by stationarity $\#\sigma^{-1}\{\tau_k\}<\infty$ $P_0$-a.s. for all integers $k$.

Let $V\in\mathfrak{B}^m$ be a bounded set. Consider, given $\{0\in M\}$, the random variable

$$N(V)=\#\{n\in\mathbb{Z}\colon S_n\in V\}\leqq\sum_{k\in K}\#\sigma^{-1}\{\tau_k\}$$

where

$$K=\{k\in\mathbb{Z}\colon \sigma(n)=\tau_k,\ S_n\in V \text{ for some integer } n\}.$$

If $k\in K$ we have for some $S_n\in V$ that $\sigma(n)=\tau_k$. So then for any $j$

$$d(S_{\tau_k}, V)\leqq\|S_{\tau_k}-S_n\|\leqq\|S_{\tau_j}-S_n\|,$$

where $d(x, V)$ denotes the distance of $x\in\mathbb{R}^m$ to $V$. Because $S_n\in V$

$$d(S_{\tau_k}, V)\leqq\operatorname{diam} V+d(S_{\tau_j}, V)$$

for any $j$, if $k\in K$. Hence

$$K\subset\{k\in\mathbb{Z}\colon d(S_{\tau_k}, V)\leqq\operatorname{diam} V+\inf_j d(S_{\tau_j}, V)\}.$$

Because $S_M$ is transient a.s. on $\{0\in M\}$, it follows that $K$ is finite a.s. on $\{0\in M\}$. Hence $N(V)$ is finite a.s. on $\{0\in M\}$, for any bounded $V\in\mathfrak{B}^m$. It follows that $(S_n)_{n\in\mathbb{Z}}$ is transient a.s. on $\{0\in M\}$, and, by stationarity, on $\{k\in M\}$ for any $k$. This proves the assertion. □

*Note.* Part of the argument above can be found in a proof of a result in [4] on transience of random walks.

## 4. Proofs of the Theorems

*Proof* of Theorem 1. Define a return set $M$ by

$$M=\{k\in\mathbb{Z}\colon \|S_n-S_k\|>\varepsilon \text{ for } n>k\}.$$

All points $S_n$, $n\in M$, are separated by distances at least $\varepsilon$, so $(S_n)_{n\in M}$ is transient. By the main lemma also the full sequence $(S_n)_{n\in\mathbb{Z}}$ is transient a.s. on $\{0\in M\}$, so we have

$$\{\|S_n\|>\varepsilon \text{ for } n>0\}\subset\{\lim_{n\to\pm\infty}\|S_n\|=\infty\} \text{ a.s. } \square \tag{1}$$

*Proof* of Theorem 2. Let $\varepsilon\downarrow 0$ in (1). We obtain

$$\{0 \text{ is not a limit point of } (S_n)_{n\geqq 0}\}\subset\{\lim_{n\to\pm\infty}\|S_n\|=\infty\} \text{ a.s.} \tag{2}$$

and so, by stationarity for any $k$

$$A_k=\{S_k \text{ is not a limit point of } (S_n)_{n\geqq 0}\}\subset\{\lim_{n\to\pm\infty}\|S_n\|=\infty\} \text{ a.s.}$$

The assertion is obtained by using that

$$(\bigcup_k A_k)^c = \{\text{each } S_k \text{ is a limit point of } (S_n)_{n \geqq 0}\}. \quad \square$$

As a side result we obtain

**Proposition 4.** $\{\lim_{n\to\infty} \|S_n\| = \infty\} = \{\lim_{n\to -\infty} \|S_n\| = \infty\}$ *a.s.*

*Proof.* By (2) we have

$$\{\lim_{n\to\infty} \|S_n\| = \infty\} \subset \{\lim_{n\to \pm\infty} \|S_n\| = \infty\} \text{ a.s.}$$

Obviously equality holds. Together with the corresponding result for $(S_{-n})$ instead of $(S_n)$, this yields the assertion. $\square$

Note. The connections between $(S_n)_{n \geqq 0}$ and $(S_{-n})_{n \geqq 0}$ are not always obvious or simple to derive. For instance in [1] a random walk is constructed with stationary increments on the real line for which

$$P(\lim_{n\to\infty} S_n = \infty) = 1, \qquad P(\lim_{n\to\infty} S_{-n} \text{ exists}) = 0.$$

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