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Distribution Functions Invariant Under Residual-Lifetime and Length-Biased Sampling

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Summary. The equation

$$F(qx)=\int_0^x (1-F(u))\,du, \qquad x \geqq 0 \tag{$*$}$$

where F is a distribution function (d.f.), arises when the limiting d.f. of the *residual-lifetime* in a renewal process is a scaled version of the *general-lifetime* d.f. F. The equation

$$G(qx)=\int_0^x u\,G(du), \qquad x \geqq 0 \tag{$**$}$$

on the other hand arises when the limiting d.f. of the *total-lifetime* in a renewal process is a scaled version of the *general-lifetime* d.f. G.

For $0<q<1$ the class, $\mathbf{F}_q$, of all d.f.'s satisfying ($*$) has been recently characterized and shown to include infinitely many d.f.'s. By explicitly exhibiting all the extreme points of $\mathbf{F}_q$, we recharacterize $\mathbf{F}_q$ as the convex hull of its extreme points and use this characterization to show that for q close to one the d.f. solution to ($*$) is "*nearly unique*." For example, if $q>0.8$ then all the infinitely many d.f.'s in $\mathbf{F}_q$ agree to more than 15 decimal places.

The class, $\mathbf{G}_q$, of all d.f. solutions to ($**$) is studied here, apparently for the first time, and shown to be in a one-to-one correspondence with $\mathbf{F}_q$; symbolically, $1-\mathbf{F}_q(x)$ is the Laplace transform of $\mathbf{G}_q(qx)$. For $0<q<1$, we characterize $\mathbf{G}_q$ as the convex hull of its extreme points and obtain results analogous to those for $\mathbf{F}_q$. For $q>1$ we give a simple argument to show that neither ($**$) nor ($*$) has a d.f. solution. We present a complete, self-contained, unified treatment of the two dual families, $\mathbf{G}_q$ and $\mathbf{F}_q$, and discuss previously known results.

A further application of the theory to graphical comparisons of two samples ($Q-Q$ plots) is described.

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1. Introduction and Results

Let $X_i \geqq 0$, independent with mean $\mu < \infty$ and common d.f. F, represent the lifetimes of a replaceable unit such as a lightbulb, and assume that lightbulbs are immediately replaced when they burn out. Denote by Y_t, Y_t' and $Z_t \equiv Y_t' + Y_t$ the *residual* lifetime, the *age*, and the *total* lifetime, respectively, of the lightbulb that is operating at time t. It is well known (see [2], for example) that as $t \uparrow \infty$ suitably, Y_t and Z_t (also Y_t' but this is not of importance here) converge stochastically to random variables Y and Z, respectively, the distributions of which are

$$P[Y \leqq x] = \mu^{-1} \int_0^x (1 - F(u))\, du, \qquad x \geqq 0, \tag{1.1}$$

$$P[Z \leqq x] = \mu^{-1} \int_0^x u F(du), \qquad x \geqq 0. \tag{1.2}$$

If in (1.1) $F(x) = 1 - e^{-x/\mu}$, so that X is exponential, then $P[Y \leqq x] = F(x)$ and Y has the same d.f. as X, $Y \sim X$. To generalize we ask *for which d.f.'s F is Y a multiple of X in law, $Y \sim X/q$*? If $Y \sim X/q$ then (1.1) becomes

$$F(qx) = \mu^{-1} \int_0^x (1 - F(u))\, du, \qquad x \geqq 0. \tag{1.3}$$

We note that without loss of generality we can take $\mu = 1$ in (1.3); indeed, replacing $F(x)$ by $F(\mu x)$ reduces (1.3) to

$$F(qx) = \int_0^x (1 - F(u))\, du, \qquad x \geqq 0 \tag{1.4}$$

where F is a d.f. which, as a consequence of (1.4) itself, has mean $\mu = 1$. If instead of $Y \sim X/q$, we ask for which d.f. F, $Z \sim X/q$, then (1.2) becomes

$$F(qx) = \mu^{-1} \int_0^x u F(du), \qquad x \geqq 0$$

which reduces to

$$G(qx) = \int_0^x u G(du), \qquad x \geqq 0 \tag{1.5}$$

upon substituting $G(x) = F(\mu x)$. Note that G is a d.f. with unit mean.

The quantity $uG(du)$ is often called the probability density function corresponding to the *length-biased-sampling* of G [2, p. 65]. Its statistical interpretation is that we sample from a population in which the length of each individual is distributed according to $G(u)$ and the probability of selecting any individual in the population is proportional to its length, u. This type of sampling bias appears often in statistical applications and is independent of the

[1] Note that here, and throughout the paper, we do not distinguish in our notation between a d.f. and the corresponding measure, so that $F(du)$ and $dF(u)$ have the same meaning

context of renewal theory. It is therefore of interest to know *for which* d.f.'s G *the length-biased-sampling effect amounts to a change of scale in the original distribution.* More on the motivation of these problems after a short discussion of the literature and statement of results (known and new).

In the context of renewal theory, Eq. (1.4) was first considered by W. Harkness and R. Shantaram [5] and then by R. Shantaram and W. Harkness (SH) [7], and P. van Beek and J. Braat (vBB) [8]; the latter obtained the general d.f. solution of (1.4) for the case $0<q<1$ (Theorem 4.1 of vBB). Quite different than ours, the motivation of these authors in studying (1.4) was to *characterize all the possible limiting laws of (suitably normalized) sequences of iterated residual-lifetime distributions.* Nevertheless, from a probabilistic viewpoint Eq. (1.5) is of no less interest than (1.4). We note, in particular, that (1.5) *characterizes all the possible limiting laws of (suitably normalized) sequences of iterated length-biased distributions;* a result entirely parallel to that which motivated the work of these authors with (1.5) replaced by (1.4) and "*length-biased distributions*" replaced by "*residual-lifetime distributions*". (This statement is an easy consequence of vBB's result and Theorem 1 below.)

Outside of renewal theory, the interest in (1.4) arose much earlier, and of particular importance is the work of N.G. De Bruijn [3] who solved the functional equation

$$H'(x)=e^{\beta x+\delta}H(x-1)$$

which is equivalent to (1.4) upon substituting $H(y)=1-F(x)$, $x=q^y$, $\beta=\log q$, $e^\delta=-\beta e^{-\beta}$. More explicit results are obtained, however, when attention is restricted to d.f. solutions of (1.4).

Let $\mathbf{F}_q$ denote the set of all d.f.'s F satisfying (1.4) and $\mathbf{G}_q$ the set of all d.f.'s G satisfying (1.5), then we have

Theorem 1. *If* $F\in\mathbf{F}_q$ *then*

$$F(x)=1-\int_0^\infty e^{-xy}G(dq\,y) \tag{1.6}$$

for some $G\in\mathbf{G}_q$*; conversely, if* $G\in\mathbf{G}_q$ *then* F*, defined by* (1.6), *belongs to* $\mathbf{F}_q$.

Theorem 4.1 of vBB, combined with Theorem 1, characterizes $\mathbf{F}_q$ and $\mathbf{G}_q$, for $0<q\leqq 1$; the case $q>1$ is not treated in vBB and should be considered separately. For subsequent results, however, it is preferable to give the complete characterization of $\mathbf{F}_q$ and $\mathbf{G}_q$ separately, and because of (1.6) it is instructive to start with $\mathbf{G}_q$ and then deduce $\mathbf{F}_q$. This is done in Theorems 2 and 2′ respectively.

Theorem 2.

(i) *If* $q>1$, $\mathbf{G}_q$ *is empty.*

(ii) *If* $q=1$, *the unique member of* $\mathbf{G}_1$ *is* $G(x)=1$ *for* $x\geqq 1$ *and* 0 *otherwise.*

(iii) *If* $0<q<1$, *for any periodic measure* ν *(defined on the Borel sets of the whole real line) satisfying*

$$\nu(A)=\nu(A+1)\geqq 0 \quad \textit{for any set } A$$

$$\nu([0,1))=1, \tag{1.7}$$

the d.f. G_ν given by

$$G_\nu(du) = q^{v^2/2}\, \nu(dv) \Big/ \int_{-\infty}^{\infty} q^{s^2/2}\, \nu(ds), \qquad u \equiv q^{v+1/2}, \qquad -\infty < v < \infty \tag{1.8}$$

belongs to $\mathbf{G}_q$*; conversely, if* $G \in \mathbf{G}_q$ *it is of the form* (1.8) *for some periodic measure* ν *satisfying* (1.7).

(iv) *The* d.f.'s G *belonging to* $\mathbf{G}_q$, $0<q<1$, *all have the same moments which are:*

$$\int_0^\infty x^n G(dx) = q^{-n(n-1)/2}, \qquad n = 0, \pm 1, \pm 2, \ldots \; \forall G \in \mathbf{G}_q.$$

Remark. It is interesting to note that the family $\mathbf{G}_q$ includes the family of C.C. Heyde [6] which demonstrated, for the first time, the indeterminacy of the lognormal distribution by its moments.

Corollary. *If* X *is a r.v. having a d.f.* $G_\nu \in \mathbf{G}_q$ *then* q/X *has a d.f.* $G_{\bar\nu} \in \mathbf{G}_q$*, where the measure* $\bar\nu$ *is given by* $\bar\nu(A) \equiv \nu(-A)$ *for any set* A*. In particular, if* ν *is symmetric about* 0, X *and* q/X *have the same d.f.*

With Theorem 1 *at hand, Theorem* 2 *is equivalent to*

Theorem 2′. (*Except* (i), *this is Theorem* 4.1 *of van Beek and Braat* [8].)

(i) *If* $q>1$, $\mathbf{F}_q$ *is empty.*

(ii) *If* $q=1$, *the unique member of* $\mathbf{F}_1$ *is* $F(x) = 1 - e^{-x}$, $x \geqq 0$.

(iii) *If* $0<q<1$, *for any periodic measure* ν *satisfying* (1.7), *the d.f.* F_ν *given by*

$$F_\nu(x) = 1 - \int_{-\infty}^{\infty} e^{-xq^{u-1/2}} q^{u^2/2}\, \nu(du) \Big/ \int_{-\infty}^{\infty} q^{u^2/2}\, \nu(du), \qquad x \geqq 0 \tag{1.9}$$

belongs to $\mathbf{F}_q$*; conversely, any d.f.* $F \in \mathbf{F}_q$ *is of the form* (1.9) *for some periodic measure* ν *satisfying* (1.7).

(iv) *The d.f.'s* F *belonging to* $\mathbf{F}_q$, $0<q<1$, *all have the same moments which are:*

$$\int_0^\infty x^n F(dx) = n!\, q^{-n(n-1)/2}, \qquad n = 0, 1, 2, \ldots \; \forall F \in \mathbf{F}_q.$$

The class of measures satisfying (1.7) is convex and compact and its extreme points are the measures ν_θ, $0 \leqq \theta < 1$, where ν_θ is a periodic train of unit point masses at $\{n+\theta; n=0, \pm 1, \ldots\}$. It follows from Theorem 2 (2′) that $\mathbf{G}_q(\mathbf{F}_q)$ *is convex and compact with extreme points* $G_\theta \equiv G_{\theta,q}$ ($F_\theta \equiv F_{\theta,q}$) *given by*

$$G_\theta(x) = \sum_{n \in A_{x,\theta}} q^{(n+\theta)^2/2} \Big/ \sum_{n=-\infty}^{\infty} q^{(n+\theta)^2/2}, \qquad x \geqq 0, \tag{1.10}$$

$$A_{x,\theta} = \left\{ n; n+\theta \geqq \frac{\log x}{\log q} - \frac{1}{2} \right\}$$

and

$$1-F_\theta(x)=\sum_{n=-\infty}^{\infty} e^{-xq^{u-1/2}} q^{u^2/2} \Big/ \sum_{n=-\infty}^{\infty} q^{u^2/2}, \qquad x \geqq 0,\ u=n+\theta. \tag{1.11}$$

The general d.f.'s in $\mathbf{G}_q$ and $\mathbf{F}_q$ are then given by

$$G(x)=\int_0^1 m(d\theta)\, G_\theta(x), \qquad F(x)=\int_0^1 m(d\theta)\, F_\theta(x) \tag{1.12}$$

where $m(d\theta)$ is a probability measure on $0 \leqq \theta < 1$.

Since F_θ are the extreme points of $\mathbf{F}_q$ we see that the maximum difference, at x, between any two d.f.'s in $\mathbf{F}_q$,

$$\varepsilon_F(\mathrm{x},\mathrm{q})=\sup_{F,\bar{F}\in\mathbf{F}_q} |F(x)-\bar{F}(x)| \tag{1.13}$$

is given by the supremum of $F_\theta - F_{\bar\theta}$ over $0 \leqq \theta,\ \bar\theta < 1$,

$$\varepsilon_F(x,q)=\sup_{\theta,\bar\theta} |F_\theta(x)-F_{\bar\theta}(x)|. \tag{1.14}$$

A bound on $\varepsilon_F(x,q)$ is given in

Theorem 3. *For* $q > e^{-2\pi} \cong 0.001867$,

$$\varepsilon_F(x,q) \leqq 4 \frac{\sqrt{\frac{2\pi}{\alpha}}\, e^{\frac{\pi^2}{8\alpha}}}{\left(\sqrt{\frac{2\pi}{\alpha}}-1\right)\sinh(\pi^2/\alpha)} \sim 8\,\mathrm{e}^{-\frac{7\pi^2}{8\alpha}}, \qquad \alpha \equiv \log q^{-1}. \tag{1.15}$$

In particular, it follows that for q *close to* 1 *the family* $\mathbf{F}_q$ *is very "tight" and, from a numerical view point, it can be thought of as consisting of only one d.f.;* this is evident upon observing that if, for example, $q > 0.8$ then (1.15) implies $\varepsilon_F(x,q) < 10^{-15}$ for all x (we have numerical evidence that the bound in (1.15) is conservative and the number 10^{-15} can be safely replaced by 10^{-20}). In Figs. 1-3 we give graphs of $F_\theta(x)=F_{\theta,q}(x)$ for various values of q and θ; it is apparent from the Figures that for q near 1 all the d.f.'s F satisfying (1.4) have their graphs lying in a narrow "cloud" and thus in a numerical sense F is "*nearly unique*".

Though the family $\mathbf{G}_q$ is not as "tight" as $\mathbf{F}_q$, its behavior as $q \uparrow 1$ is similar in principle and for

$$\varepsilon_G(x,q)=\sup_{G,\tilde{G}\in\mathbf{G}_q} |G(x)-\tilde{G}(x)| \tag{1.16}$$

we have

Theorem 4. *Let* $\bar{G}(x)=\sup_{G\in\mathbf{G}_q} G(x)$, $\underline{G}(x)=\inf_{G\in\mathbf{G}_q} G(x)$, *then*

$$\varepsilon_G(x,q)=\bar{G}(x)-\underline{G}(x)=\left(\sum_{n=-\infty}^{\infty} q^{n^2/2+nv}\right)^{-1}, \qquad x=q^{v+1/2} \tag{1.17}$$

$$\sup_x \varepsilon_G(x,q)=\varepsilon_G(\sqrt{q},q)=\left(\sum_{n=-\infty}^{\infty} q^{n^2/2}\right)^{-1} \sim \sqrt{\frac{\alpha}{2\pi}}, \qquad \alpha \equiv \log q^{-1}. \tag{1.18}$$

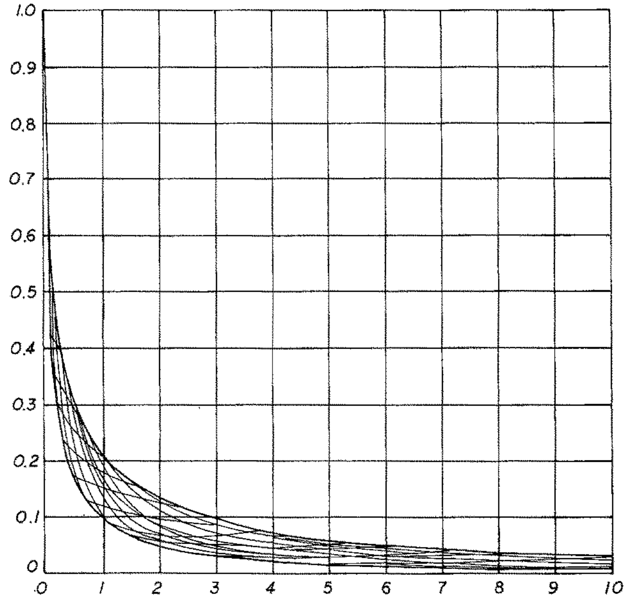

Fig. 1. The "cloud" of d.f.'s belonging to $\mathbf{F}_{0.03125}$, $1-F_{\theta,0.03125}(x)$ for $0 \leqq \theta < 1$ superimposed

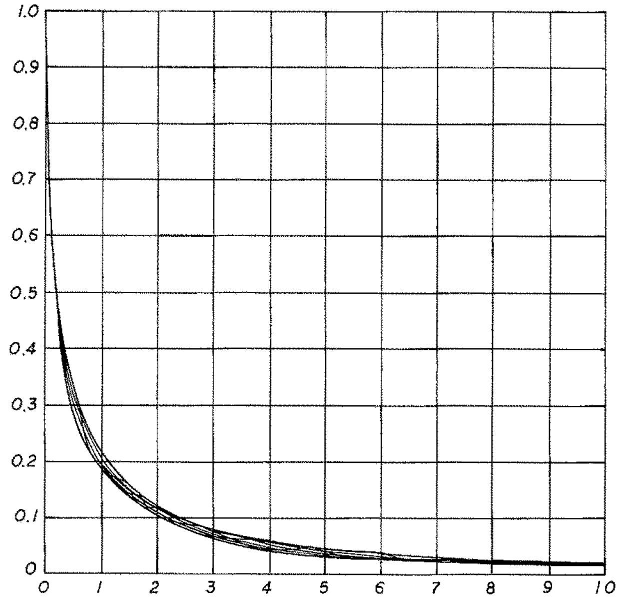

Fig. 2. The "cloud" of d.f.'s belonging to $\mathbf{F}_{0.125}$, $1-F_{\theta,0.125}(x)$ for $0 \leqq \theta < 1$ superimposed

In fact, $\bar{G}(x) = H(v)$ and $\underline{G}(x) = H(v+1)$ where $x = q^{v+1/2}$ and

$$H(y) = \sum_{n=0}^{\infty} q^{(y+n)^2/2} \Big/ \sum_{k=-\infty}^{\infty} q^{(y+k)^2/2}. \tag{1.19}$$

Relations to other families of d.f.'s are given in the following additional

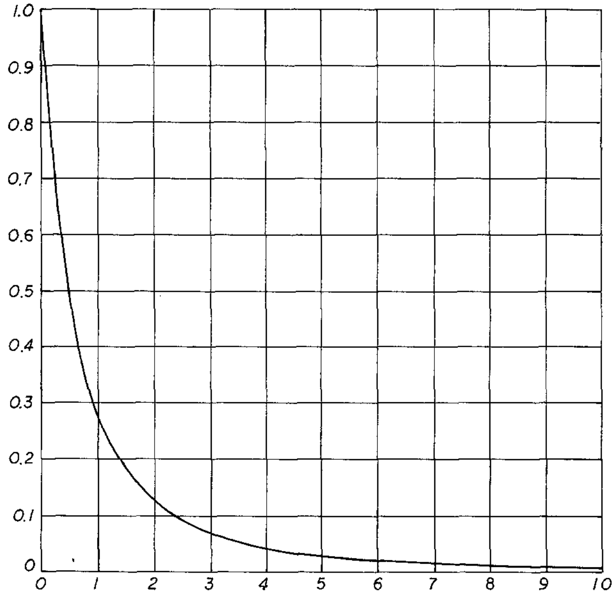

Fig. 3. The "cloud" of d.f.'s belonging to $\mathbf{F}_{0.5}$, $1-F_{\theta,0.5}(x)$ for $0 \leqq \theta < 1$ superimposed

Properties.
P1. *If* $F \in \mathbf{F}_q$ *then* F *has a decreasing failure rate, and*

$$r(x) \equiv -\frac{\partial}{\partial x} \log(1-F(x)) = q^{-1}(1-F(q^{-1}x))/(1-F(x)), \qquad x \geqq 0.$$

P2. *Denoting by* $N(\zeta, \sigma^2)$ *a normal r.v. with mean* ζ *and variance* σ^2, *the d.f. of* $Z_q \equiv \exp\{N(2^{-1}\log q, \log q)\}$, *which is lognormal with unit mean, belongs to* $\mathbf{G}_q$.
P3. (*Shantaram and Harkness* [7]) *Denoting by* V *an exponential r.v. with unit mean, independent of* Z_q *given above, the d.f. of* $Z_q V$ *belongs to* $\mathbf{F}_q$.

Our interest in this problem arose in studying data comprised of two samples, $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_m)$ say, for which there was a strong reason to believe that, due to the sampling method, the Y_i's follow a d.f. $H(y) = \mu^{-1} \int_0^y (1-F(u))\,du$ where F denotes the common d.f. of the X_i's. Comparing the two samples by means of a $Q-Q$ plot [9], we asked ourselves *which are the possible candidates for* F, *if the* $Q-Q$ *plot is approximately a straight line (say, with slope* q)? Since the graph of a $Q-Q$ plot of F vs. H is $Q(x) = F^{-1}H(x)$, it follows that F must satisfy (1.3), so that *the only candidates for* F *are d.f.'s that after rescaling (to have* $\mu = 1$) *belong to* $\mathbf{F}_q$. Other statistical questions related to the above two-sample problem will be discussed in a separate paper.

2. Proofs

Proof of Theorem 1. The implication in the first direction follows from the argument given in Eqs. (2)–(6) of Shantaram and Harkness [7]. It is short and elegant, and for the sake of completeness we repeat it here. If F satisfies (1.4), it is easy to prove by induction that F is infinitely differentiable and the n-th derivative of $1-F(x)$ is given by

$$(-1)^n \frac{\partial^n}{\partial x^n}(1-F(x)) = q^{-n(n+1)/2}(1-F(q^{-n}x)) \geqq 0, \quad x \geqq 0, \; n=0,1,\ldots.$$

Since $1-F(0)=1$, it follows from Bernstein's Theorem [3, p. 439] that $1-F(x)$ is the Laplace transform of a probability measure, say M. Using this and (1.4), we get

$$1-F(qx) = \int_0^\infty e^{-qxy} M(dy) = \int_x^\infty (1-F(u))\,du = \int_x^\infty \int_0^\infty e^{-uy} M(dy)\,du, \quad x \geqq 0$$

so that

$$\int_0^\infty e^{-xy} M(dq^{-1}y) = \int_0^\infty e^{-xy} y^{-1} M(dy), \quad x \geqq 0$$

and by the uniqueness of the Laplace transform, we have

$$M(dq^{-1}y) = y^{-1} M(dy), \quad y \geqq 0.$$

Setting $G(y) = M(q^{-1}y)$ above, it follows that G satisfies (1.5) and thus belongs to $\mathbf{G}_q$. To prove the converse we note that if F is of the form (1.6) and G satisfies (1.5), then

$$\int_0^x (1-F(u))\,du = \int_0^x du \int_0^\infty e^{-uy} G(dqy) = \int_0^\infty y^{-1}(1-e^{-xy})\,G(dqy)$$
$$= \int_0^\infty (1-e^{-xy})\,G(dy) = 1 - \int_0^\infty e^{-qxy} G(dqy) = F(qx)$$

so that F satisfies (1.4) and thus belongs to $\mathbf{F}_q$.

Proof of Theorem 2. (i), (iv). If G satisfies (1.5) then

$$\mu_{G,n} \equiv \int_0^\infty (qx)^n G(dqx) = q^n \int_0^\infty x^{n+1} G(dx) = q^n \mu_{G,n+1}$$

so that

$$\mu_{G,n} = q^{-n(n-1)/2}, \quad n = 0, \pm 1, \pm 2, \ldots$$

For $q>1$, however, $q^{-n(n-1)/2}$ cannot be a moment sequence since $(\mu_{G,n})^{1/n}$ must increase [4, p. 155] while $q^{-(n-1)/2}$ decreases.

(ii) For $q=1$, (1.5) implies $G(dx) = xG(dx)$, $x \geqq 0$, and the only probability measure satisfying it is a unit mass at 1.

(iii) (cf. proof of Theorem 4.1 in [8]) Denoting the denominator of (1.8) by C^{-1} we have

$$\begin{aligned}\int_0^x u\,G_\nu(du) &= C\int_{\frac{\log x}{\log q}-1/2}^{\infty} q^{v+1/2}\,q^{v^2/2}\,\nu(dv) = C\int_{\frac{\log x}{\log q}-1/2}^{\infty} q^{(v+1)^2/2}\,\nu(dv)\\ &= C\int_{\frac{\log x}{\log q}+1/2}^{\infty} q^{v^2/2}\,\nu(dv-1) = C\int_{\frac{\log x}{\log q}+1/2}^{\infty} q^{v^2/2}\,\nu(dv) = G_\nu(qx),\end{aligned}$$

so that G_ν satisfies (1.5) and hence belongs to $\mathbf{G}_q$. Conversely, if G is a d.f. satisfying (1.5) and $u \equiv q^{v+1/2}$, define a measure ν (on the Borel sets of the whole real line) by

$$G(du) = c\,q^{v^2/2}\,\nu(dv) = c\,q^{\left(\frac{\log u}{\log q}-1/2\right)^2/2}\,\nu\left(d\,\frac{\log u}{\log q}-1/2\right)$$

for $c>0$, a constant. Then

$$c\,q^{(v+1)^2/2}\,\nu(dv+1) = G(dqu) = u\,G(du) = c\,q^{v+1/2+v^2/2}\,\nu(dv)$$

so that

$$\nu(dv+1) = \nu(dv) \geqq 0.$$

Choosing the constant c to satisfy $\nu([0,1))=1$, the results follows.

Proof of Theorem 3. Let

$$s(\lambda,\theta) = \sum_{n=-\infty}^{\infty} e^{-\lambda q^{n+\theta}}\,q^{(n+\theta)^2/2}, \qquad \lambda \geqq 0,\ 0 \leqq \theta < 1,\ 0<q<1$$

$$s(\theta) \equiv s(0,\theta)$$

Lemma 1. *For* $q > e^{-2\pi}$, $|F_\theta(\lambda) - F_{\bar\theta}(\lambda)| \leqq 2\left(\sqrt{\frac{2\pi}{\alpha}}-1\right)^{-1} \max_{\lambda\geqq 0} |s(\lambda,\theta) - s(\lambda,\bar\theta)|$,

Proof.

$$\begin{aligned}|F_\theta(q^{1/2}\lambda) - F_{\bar\theta}(q^{1/2}\lambda)| &= \left|\frac{s(\lambda,\theta)}{s(\theta)} - \frac{s(\lambda,\bar\theta)}{s(\bar\theta)}\right|\\ &= \frac{1}{s(\theta)\,s(\bar\theta)}\,|s(\bar\theta)(s(\lambda,\theta)-s(\lambda,\bar\theta)) + s(\lambda,\bar\theta)(s(\bar\theta)-s(\theta))|\\ &\leqq \frac{1}{s(\theta)}\,|s(\lambda,\theta)-s(\lambda,\bar\theta)| + \frac{s(\lambda,\bar\theta)}{s(\bar\theta)\,s(\theta)}\,|s(\bar\theta)-s(\theta)|\\ &\leqq \frac{1}{s(\theta)}\,|s(\lambda,\theta)-s(\lambda,\bar\theta)| + \frac{1}{s(\theta)}\,|s(\bar\theta)-s(\theta)|\\ &\leqq \frac{2}{s(\theta)}\max_{\lambda\geqq 0}|s(\lambda,\theta)-s(\lambda,\bar\theta)|.\end{aligned}$$

The proof is completed by observing that

$$s(\theta)=\sum_{n=-\infty}^{\infty} e^{-\alpha(n+\theta)^2/2}+1-1\geqq \int_{-\infty}^{\infty} e^{-\alpha(x+\theta)^2/2}\,dx-1=\sqrt{\frac{2\pi}{\alpha}}-1>0.$$

Lemma 2. $|s(\lambda,\theta)-s(\lambda,\bar{\theta})|\leqq \dfrac{2\sqrt{\frac{2\pi}{\alpha}}\,e^{\frac{\pi^2}{8\alpha}}}{\sinh(\pi^2/\alpha)}.$

Proof. Let $\phi(z)=\exp\{-\lambda e^{-\alpha z}-\alpha z^2/2\}$ and let C be a counter clockwise circuit defined by

$$C=\begin{Bmatrix} x-iy, & -\infty<x<\infty \\ x+iy, & -\infty<x<\infty \end{Bmatrix}, \qquad |y|\leqq \pi/2\alpha$$

We then have

$$s(\lambda,\theta)=\frac{i}{2}\int_C \phi(z)\tan\pi(z-\theta-1/2)\,dz,$$

$$\begin{aligned}|s(\lambda,\theta)-s(\lambda,\bar{\theta})|&\leqq \tfrac{1}{2}\int_C |\phi(z)|\,|\tan\pi(z-\theta-1/2)-\tan\pi(z-\bar{\theta}-1/2)|\,|dz|\\ &=I,\end{aligned}$$

say.

Now for x, $\bar{x}$ and y real, it is not hard to show

$$|\tan(x+iy)-\tan(\bar{x}+iy)|\leqq \frac{2}{|\sinh 2y|},$$

$$|\phi(z)|\leqq e^{-\alpha(x^2-y^2)/2} \qquad \text{if } |y|\leqq \pi/2\alpha,$$

so that by setting $y=\pi/2\alpha$, we obtain

$$I\leqq \int_C \frac{e^{-\alpha(x^2-y^2)/2}}{|\sinh 2\pi y|}\,dx=\frac{2\sqrt{\frac{2\pi}{\alpha}}\,e^{\frac{\pi^2}{8\alpha}}}{\sinh(\pi^2/\alpha)},$$

as asserted.

Lemmas 1 and 2 combine to prove Theorem 3.

Proof of Theorem 4. Since G_θ, $0\leqq\theta\leqq 1$, are the extreme points of $\mathbf{G}_q$ we have $\bar{G}(x)=\sup_{0\leqq\theta\leqq 1} G_\theta(x)$ and $\underline{G}(x)=\inf_{0\leqq\theta\leqq 1} G_\theta(x)$. From (1.10) we have $\bar{G}(q^{v+1/2})=\sup_{v\leqq y\leqq v+1} H(y)$ and $\underline{G}(q^{v+1/2})=\inf_{v\leqq y\leqq v+1} H(y)$, where H is defined in (1.19).

Claim: $H'(y)<0$, $-\infty<y<\infty$.

Proof.

$$H'(y)\left(\sum_{k=-\infty}^{\infty} q^{(y+k)^2/2}\right)^2=(\log q)\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}(n-k)\,q^{(y+n)^2/2+(y+k)^2/2}$$

$$=(\log q)\sum_{n=0}^{\infty}\sum_{j=1}^{\infty} j\,q^{(y+n)^2/2}\left(q^{(y+n-j)^2/2}-q^{(y+n+j)^2/2}\right)$$

$$=(\log q)\sum_{n=0}^{\infty}\sum_{j=1}^{\infty} j\,q^{(y+n)^2+j^2/2}\left(q^{-(y+n)j}-q^{(y+n)j}\right).$$

It follows that for $y\geqq 0$, $H'(y)<0$. But from (1.19), $H(1-y)+H(y)\equiv 1$ so that $H'(1-y)\equiv H'(y)$, and $H'(y)>0$ for all y which proves the claim.

It follows that $H(y)$ is decreasing, so that $\bar{G}(q^{v+1/2})=H(v)$, $\underline{G}(q^{v+1/2})=H(v+1)$ and the results of Theorem 4 follow immediately.

Proof of "Properties"

P1. From Theorem 1, if $F\in\mathbf{F}_q$ then F is a mixture of exponential d.f.'s and as such has a decreasing failure rate [1, p. 103] (A.M. Odlyzko proved, however, that the failure rate functions of the extreme d.f.'s F_θ, $0\leqq\theta<1$, are not convex).
P2. By choosing, in Theorem 2, the measure v to be Lebesgue measure the result follows.
P3. From Theorem 1, the distribution of qV/Z_q belongs to $\mathbf{F}_q$ whenever the distribution of Z_q belongs to $\mathbf{G}_q$. The result now follows from P2 upon observing that q/Z_q has the same d.f. as Z_q itself.

As a comment we note that properties P2 and P3 may be useful in designing a simulation study of processes with the discussed invariance properties and also that the extreme d.f.'s G_θ and F_θ, $0\leqq\theta<1$ are very easy to compute due to the fact that only a few terms dominate the value of the infinite sums involved.

Final Remark. Our results raise some additional interesting problems of more abstract nature: The set $\mathbf{M}_q$, of all d.f.'s with moments as in Theorem 2′ (iv), can be shown to *strictly* include $\mathbf{F}_q$; a complete description of $\mathbf{M}_q$ (which is convex and compact) would be of interest. In particular, we hope to show that as q approaches 1, $\mathbf{M}_q$ also becomes "tight" in the sense of Theorem 3 (i.e., $\sup_x \sup_{F,\bar{F}\in\mathbf{M}_q} |F(x)-\bar{F}(x)|\to 0$ as $q\to 1$). If this is true, then the (nonunique) moment problem of Theorem 2′ (iv) would be "nearly unique" in a numerical sense for q close to 1.

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