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Probability Theory and Related Fields

A Useful Estimate in the Multidimensional Invariance Principle

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Summary. An estimate of the convergence speed in the multidimensional invariance principle is obtained. Using this estimate, we can prove strong invariance principles for partial sums of independent not necessarily identically distributed multidimensional random vectors.

1. Introduction and Main Results

We denote by $(\mathbb{R}^d, |\cdot|)$ the d-dimensional euclidean space. Let $C_d[0,1]$ be the space of all continuous $\mathbb{R}^d$-valued functions on $[0,1]$ endowed with the sup-norm $\|\cdot\|$. Log t stands for $\log(\max(t,e)$, $t \geqq 0$.

Let $\xi_1, \ldots, \xi_n$: $\Omega \to \mathbb{R}^d$ be independent random vectors with zero means, $\operatorname{cov}(\xi_k) = \sigma_k^2 I$, $1 \leqq k \leqq n$ and $\sum_1^n \sigma_k^2 = 1$, where I denotes the d-dimensional unit matrix.

Let $S_{(n)}$: $\Omega \to C_d[0,1]$ be the partial sum process which is defined by

$$S_{(n)}(t) := \sum_1^m \xi_k + \frac{t - t_m}{t_{m+1} - t_m} \xi_{m+1}, \qquad t_m \leqq t \leqq t_{m+1}, \tag{1.1}$$

$0 \leqq m < n$, where $t_m := \sum_1^m \sigma_k^2$, $0 \leqq m \leqq n$. $\left(\sum_1^0 \xi_k := \sum_1^0 \sigma_k^2 := 0\right)$.

Denote by $W_\Gamma | B(C_d[0,1])$ the Wiener-measure with covariance matrix Γ, i.e. the unique p-measure on the Borel-σ-algebra of $C_d[0,1]$ such that $W \circ \pi_t = N(0, t\Gamma)$, $0 \leqq t \leqq 1$ and $\{\pi_t\colon t \in [0,1]\}$ is a stochastic process with independent and stationary increments under W_Γ, where π_t; $0 \leqq t \leqq 1$ are the canonical projections (i.e. $\pi_t(f) := f(t)$, $f \in C_d[0,1]$, $0 \leqq t \leqq 1$). Let $W = W_I$.

If $Q_i | B(C_d[0,1])$, $i = 1, 2$, are p-measures, we put for each $\delta > 0$:

$$\lambda(Q_1, Q_2, \delta) := \sup\{Q_1(A) - Q_2(A^\delta)\colon A \subseteq C_d[0,1] \text{ closed}\}, \tag{1.2}$$

where $A^\delta := \{g \in C_d[0,1]\colon \exists f \in A,\ \|g - f\| < \delta\}$.

Call $\lambda(Q_1, Q_2, \delta)$ the "δ-distance" of Q_1 and Q_2.

These δ-distances have turned out to be extremely useful especially in treating the Prohorov-distance of p-measures (cf. [3, 4, 10, 11, 13]).

In this paper we prove an inequality for the δ-distances of the partial sum process $S_{(n)}$ from the Wiener-measure W (see Theorem 1). Our result concerns independent not necessarily identically distributed d-dimensional random vectors and does not require higher than second moments. Thus our inequality can be applied to rather different problems. Our inequality yields a multidimensional version of Prohorov's invariance principle for triangular arrays of rowwise independent r.v.'s fulfilling the Lindeberg-condition (see Corollary 4) as well as unimprovable estimates of the Prohorov-distance of the p-measures $P \circ S_{(n)} | B(C_d[0,1])$ and $W | B(C_d[0,1])$ (see Corollary 2); it can be used to obtain strong approximation results (see Theorems 2, 3, 5, 6) as well as functional (compact) laws of the iterated logarithm (see Corollaries 5/6).

Sahanenko (cf. [13]) obtained in the 1-dimensional case (i.e. $d=1$) the following estimate of the δ-distances of $P \circ S_{(n)}$ and W:

$$\lambda(P \circ S_{(n)}, W, \delta) \leqq c(s)\, \delta^{-s} K_{sn}, \qquad s > 2, \tag{1.3}$$

where $K_{sn} := \sum_1^n E[|\xi_k|^s]$, $c(s)$ is a positive constant depending on s only.

He used in his proof a refinement of the already rather complicated common probability space method of Komlós et al. (1976), which is restricted to the 1-dimensional case. The question remained open, whether (1.3) holds true in the multidimensional case (cf. [4], p. 64).

Using a recent result in connection with the multidimensional central limit theorem (cf. [8], Theorem 6), we now obtain by a comparatively simple construction method, which is related to the method of Csörgo and Révész (1975), Theorem 1 below enabling us to extend Sahanenko's result to the multidimensional case, if $2 < s < 4$. At the same time, we are able to prove (1.3) for $s \geqq 4$ in the multidimensional case, if δ is sufficiently large - in particular, if $\delta \geqq K_{sn}^{\gamma}$, where $\gamma < 1/(2s-4)$. This shows that (1.3) can be obtained for any given $s > 2$ by a much easier construction method than that in [13], if δ is large. On the other hand, it can be shown by similar arguments as in [2, Remark 3] that it is impossible to prove (1.3) by such a simple method if δ is small and $s > 4$.

Theorem 1. *Let* $\xi_1, \dots, \xi_n$: $\Omega \to \mathbb{R}^d$ *be independent random vectors with zero means,* $\operatorname{cov}(\xi_k) = \sigma_k^2 I$, $1 \leqq k \leqq n$, *and* $\sum_1^n \sigma_k^2 = 1$. *Let* $S_{(n)}$: $\Omega \to C_d[0,1]$ *be defined by* (1.1). *Let* $s > 2$, $0 < \gamma < 1/(2s-4)$. *Then we have for* $\delta \geqq (K_{sn}(\delta) + \delta^{s-2} L_n(\delta))^{\gamma}$:

$$\lambda(P \circ S_{(n)}, W, c_1 \delta) \leqq c_2(\delta^{-s} K_{sn}(\delta) + \delta^{-2} L_n(\delta)), \tag{1.4}$$

where $K_{sn}(\delta) := \sum_1^n E[|\xi_k|^s 1\{|\xi_k| \leqq \delta\}]$, $L_n(\delta) := \sum_1^n E[|\xi_k|^2 1\{|\xi_k| > \delta\}]$. $c_1 = c_1(\gamma, s, d) > 0$ *and* $c_2 = c_2(\gamma, s, d) > 0$ *are constants depending on* γ, s *and* d *only.*

Corollary 1. *Let $S_{(n)}$: $\Omega \to C_d[0,1]$ be as in Theorem 1. Let $2<s<4$. Then we have for $\delta>0$:*

$$\lambda(P\circ S_{(n)}, W, c_3\delta) \leqq c_4(\delta^{-s} K_{sn}(\delta) + \delta^{-2} L_n(\delta)), \tag{1.5}$$

where $c_3 = c_3(s,d)$, $c_4 = c_4(s,d)$ are positive constants depending on s and d only.

Since $\delta^{-s} K_{sn}(\delta) + \delta^{-2} L_n(\delta) \leqq \delta^{-s} K_{sn}$, $\delta>0$, we easily obtain from (1.5) a multidimensional version of (1.3), if $2<s<4$. Furthermore, Theorem 1 implies the following unimprovable estimate of the Prohorov distance of $P\circ S_{(n)}$ and W:

Corollary 2. *Let $S_{(n)}$: $\Omega \to C_d[0,1]$ be as in Theorem 1. Then we have for $2<s<5$:*

$$\rho(P\circ S_{(n)}, W) \leqq c_5 K_{sn}^{1/(s+1)}, \tag{1.6}$$

where $c_5 = c_5(s,d) > 0$.

Thus we have obtained as a byproduct of (1.4) an improvement of the main result of Borovkov and Sahanenko (1980) (cf. [3], Theorem 4).

Recall that the Prohorov-distance of two p-measures $Q_i|B(C_d[0,1])$, $i=1,2$, is defined by

$$\rho(Q_1, Q_2) = \inf\{\delta>0: \lambda(Q_1, Q_2, \delta) \leqq \delta\}.$$

Furthermore, it easily follows from Theorem 1:

Corollary 3. *Let $S_{(n)}$: $\Omega \to C_d[0,1]$ be as in Theorem 1, let $2<s<4$. Set $\varepsilon := \varepsilon(s) := (s+1)/(s-2)$. Suppose that $L_n(\delta^\varepsilon) \leqq \delta^3$. Then: $\rho(P\circ S_{(n)}, W) \leqq c_6\delta$, where $c_6 = c_6(s,d)$ is a positive constant depending on s and d only.*

An immediate consequence of Corollary 3 is the above announced invariance principle for triangular arrays of rowwise independent random vectors fulfilling the Lindeberg-condition.

Corollary 4. *Let $\{\xi_{nk}: 1\leqq k\leqq k_n\}$, $n\in\mathbb{N}$, be a triangular array of rowwise independent random vectors with zero means $\operatorname{cov}(\xi_{nk}) = \sigma_{nk}^2\Gamma$, $1\leqq k\leqq k_n$ and $\sum_{k=1}^{k_n}\sigma_{nk}^2 = 1$ ($n\in\mathbb{N}$). Let $S_{(n)}$: $\Omega\to C_d[0,1]$ be defined by*

$$S_{(n)}(t) := \sum_{k=1}^{m} \xi_{nk} + \frac{t-t_{nm}}{t_{nm+1}-t_{nm}}\xi_{nm+1}, \qquad t_{nm} \leqq k \leqq t_{nm+1},$$

$0\leqq m<k_n$, where $t_{nm} := \sum_{k=1}^{m}\sigma_{nk}^2$, $0\leqq m\leqq k_n$, $n\in\mathbb{N}$.

Assume that for all $\delta>0$ the following is true:

$$L_n(\delta) := \sum_{k=1}^{k_n} E[|\xi_{nk}|^2 1\{|\xi_{nk}|>\delta\}] \to 0 \qquad \text{as } n\to\infty.$$

Then $P\circ S_{(n)}|B(C_d[0,1])$ converges weakly to $W_\Gamma|B(C_d[0,1])$.

Theorem 1 is now used to obtain strong invariance principles for partial sums of independent not necessarily identically distributed d-dimensional random vectors.

We denote in the sequel by $\mathscr{H}$ the set of all continuous functions H: $[0,\infty)\to[0,\infty)$ such that $t^{-2}H(t)$ is non-decreasing and $t^{-4+r}H(t)$ is non-increasing for some $r>0$.

Theorem 2. *Let $\{\xi_k\}$ be a sequence of independent random vectors with zero means and* $\operatorname{cov}(\xi_k)=\sigma_k^2\Gamma$, $k\in\mathbb{N}$. *Assume that the following holds true for some* $s\in(2,4)$:

$$\sum_1^\infty a_k^{-s}E[|\xi_k|^s 1\{|\xi_k|\leqq a_k\}]<\infty \tag{1.7}$$

and

$$\sum_1^\infty a_k^{-2}E[|\xi_k|^2 1\{|\xi_k|>a_k\}]<\infty \quad \text{where } 0<a_k\uparrow\infty. \tag{1.8}$$

Then one can construct a p-space $(\Omega_0,\mathscr{A}_0,P_0)$ *and two sequences of independent random vectors* $\{X_k\}$, $\{Y_k\}$ *with* $P_0\circ X_k=P\circ\xi_k$, $P_0\circ Y_k=N(0,\sigma_k^2\Gamma)$, $k\in\mathbb{N}$, *such that the partial sums* $S_n:=\sum_1^n X_k$, $T_n:=\sum_1^n Y_k$ *fulfill:*

$$S_n-T_n=o(a_n) \quad a.s. \tag{1.9}$$

Remark. It is easy to see that the conditions (1.7) and (1.8) are fulfilled if

$$\sum_1^\infty H(a_k)^{-1}E[H(|\xi_k|)]<\infty \quad \text{for some } H\in\mathscr{H}. \tag{1.10}$$

From Theorem 2 we obtain

Theorem 3. *Let $\{\xi_k\}$ be a sequence of independent random vectors with zero means and* $\operatorname{cov}(\xi_k)=\sigma_k^2\Gamma$, $k\in\mathbb{N}$. *Let* f: $[0,\infty)\to(0,\infty)$ *be a non-decreasing function such that* $\sum_n \frac{1}{nf(n)}<\infty$. *Assume that* $\alpha_n:=\sum_1^n E[H(|\xi_k|)]\to\infty$, *where* $H\in\mathscr{H}$. *Then a construction is possible such that*

$$S_n-T_n=o(H^{-1}(\alpha_n f(\alpha_n))) \quad a.s. \tag{1.11}$$

Let $\{\xi_k\}$ be a sequence of independent random vectors such that $0<\varliminf_k E[H(|\xi_k|)]\leqq\varlimsup_k E[H(|\xi_k|)]<\infty$. According to (1.11), a construction is possible such that

$$S_n-T_n=o(H^{-1}(nf(n)) \quad \text{a.s., if } \sum_n \frac{1}{nf(n)}<\infty. \tag{1.12}$$

Since $f(n)/\log n\to\infty$ as $n\to\infty$ by Kronecker's Lemma, the convergence rate is considerably worse than that in the strong invariance principle for partial sums of i.i.d. random vectors. Recall that it was shown in [8], Theorem 2 for the i.i.d. case that if $H \in \mathscr{H}$ such that $t^{-2}(\operatorname{Log}\operatorname{Log} t)^{-1} H(t)$ is non-decreasing, a construction with error term $o(H^{-1}(n))$ is possible. If $H \in \mathscr{H}$ such that $t^{-2}(\operatorname{Log}\operatorname{Log} t)^{-1} H(t)$ is non-increasing it is still possible to obtain a construction with error term $o\left(H^{-1}(n)\sqrt{\frac{\operatorname{Log}\operatorname{Log} n}{h(n)}}\right)$, where $h(t) := t^{-2} H(t)$, $t \geqq 0$ (cf. [8], Theorem 3).

The following Theorem 4, however, shows that under the above assumptions no better convergence rate than (1.12) can be reached in general.

Theorem 4. *Let* $0 < f_n \uparrow \infty$ *be such that* $\sum_n \frac{1}{n f_n} = \infty$. *Let* $H \in \mathscr{H}$. *There exists a sequence of independent r.v.'s* $\xi_k \colon \Omega \to \mathbb{R}$, $k \in \mathbb{N}$ *with zero means,* $E[\xi_k^2] = 1$ *and* $E[H(|\xi_k|)] \leqq 1 + H(1)$, $k \in \mathbb{N}$, *such that for all sequences of independent r.v.'s* $\{X_k\}$ *with* $P_0 \circ X_k = P \circ \xi_k$, $k \in \mathbb{N}$, *and for all sequences of i.i.d. r.v.'s* $\{Y_k\}$ *with* $P_0 \circ Y_1 = N(0,1)$, *where* $(\Omega_0, \mathscr{A}_0, P_0)$ *is an arbitrary p-space, the following holds true:*

$$\overline{\lim_n} \frac{|S_n - T_n|}{H^{-1}(n f_n)} = \infty \quad a.s. \tag{1.13}$$

Our next result implies the above announced functional law of the iterated logarithm. To simplify our notations, we set $\operatorname{Log}_2 t := \operatorname{Log}(\operatorname{Log} t)$, $t \geqq 0$.

Theorem 5. *Let* $\{\xi_k\}$ *be a sequence of independent random vectors with zero means and* $\operatorname{cov}(\xi_k) = \sigma_k^2 \Gamma$, $k \in \mathbb{N}$. *Assume* $B_n := \sum_1^n \sigma_k^2 \to \infty$,

$$\sum_1^\infty (B_k \operatorname{Log}_2 B_k)^{-s/2} E[|\xi_k|^s 1\{|\xi_k| \leqq \sqrt{B_k \operatorname{Log}_2 B_k}\}] < \infty \quad \textit{for some } s > 2 \tag{1.14}$$

and

$$\sum_1^\infty (B_k \operatorname{Log}_2 B_k)^{-1} E[|\xi_k|^2 1\{|\xi_k| > \sqrt{B_k \operatorname{Log}_2 B_k}\}] < \infty. \tag{1.15}$$

Then a construction is possible such that

$$S_n - T_n = o(\sqrt{B_n \operatorname{Log}_2 B_n}) \quad a.s. \tag{1.16}$$

(1.14) and (1.15) are fulfilled if

$$\sum_1^\infty (B_k \operatorname{Log}_2 B_k)^{-s/2} E[|\xi_k|^s] < \infty \quad \text{for some } s > 2. \tag{1.17}$$

Let $\mathscr{F}$ be the set of all absolutely continuous functions $f\colon [0,1] \to \mathbb{R}^d$ such that $f(0) = 0$ and $\int_0^1 |f'(t)|^2 \, dt \leqq 1$. Denote by $\mathscr{F}_\Gamma$ the functions $\{t \to \Gamma^{1/2} f(t), f \in \mathscr{F}\}$.

Furthermore, we set:

$$S(t) := \sum_1^m \xi_k + \frac{t - B_m}{B_{m+1} - B_m} \xi_{m+1}, \qquad B_m \leqq t \leqq B_{m+1},$$

$0 \leqq m < \infty$, where $B_0 = \sum_1^0 \xi_k = 0$.

Corollary 5. *Let* $\{\xi_k\}$ *be as in Theorem* 5. *Let* $U_{(n)}: \Omega \to C_d[0,1]$ *be defined by* $U_{(n)}(t) := (2B_n \operatorname{Log}_2 B_n)^{-1/2} S(B_n t)$, $0 \leqq t \leqq 1$. *Suppose that* (1.14) *and* (1.15) *hold true*, $B_n \to \infty$ *and*

$$\overline{\lim_n} \frac{B_{n+1}}{B_n \exp(\sqrt{\operatorname{Log}_2 B_n})} < \infty. \tag{1.18}$$

With probability one, $\{U_{(n)}: n \in \mathbb{N}\}$ *is relatively compact in* $C_d[0,1]$ *and the set of its limit points coincides with* $\mathscr{F}_\Gamma$.

Corollary 5 implies immediately the following compact law of the iterated logarithm:

Corollary 6. *Under the assumptions of Corollary* 5 *we have with probability one*: $\left\{(2B_n \operatorname{Log}_2 B_n)^{-1/2} \sum_1^n \xi_k : n \in \mathbb{N}\right\}$ *is relatively compact in* $\mathbb{R}^d$ *and the set of its limit points coincides with* $\{\Gamma^{1/2} x : |x| \leqq 1\}$.

Thus we have obtained a strong extension of [15], Theorem 1.2, which was proved in [15] under the more restrictive assumptions "$\overline{\lim_n} (B_{n+1}/B_n) < \infty$" and (1.17), if $2 < s \leqq 4$. Furthermore, we have shown that [15] Theorem 1.2 remains valid for any $s > 4$ and that the condition "$E[X_n^3] = 0$ $(n \in \mathbb{N})$" can be omitted.

As a last application of Theorem 1 we present a sufficient condition which guarantees that in the strong invariance principle the convergence rate $o(\sqrt{B_n})$ can be reached.

Theorem 6. *Let* $\{\xi_k\}$ *be a sequence of independent random vectors with zero means and* $\operatorname{cov}(\xi_k) = \sigma_k^2 \Gamma$, $k \in \mathbb{N}$.

Assume that $B_n \to \infty$ *and*

$$\sum_{k=1}^{\infty} B_k^{-s/2} E[|\xi_k|^s] < \infty \quad \textit{for some } s > 2. \tag{1.19}$$

Then a construction is possible such that

$$S_n - T_n = o(\sqrt{B_n}) \quad a.s.$$

Using Kronecker's lemma, we obtain from (1.19):

$$\sum_1^n E[|\xi_k|^s] = o(B_n^{s/2}) \quad \text{for some } s > 2,$$

i.e. Lyapunov's condition – a well known sufficient condition for the invariance principle (in distribution).

The paper is now organized as follows: In Sect. 2 we prove Theorem 1 and its corollaries. Some more or less known lemmas needed here are formulated and proved in Sect. 6. In Sect. 3 we infer from Theorem 1 the above strong invariance principles for partial sums of independent random vectors (i.e. Theorems 2, 3, 5 and 6). Theorem 4 is proved in Sect. 4, whereas the proof of Corollaries 5 and 6 is given in Sect. 5.

2. Proof of Theorem 1 and its Corollaries

Main tool of the proof is the following Theorem 7, which is related to [8], Theorem 7.

Theorem 7. *Let* $\xi_1, \dots, \xi_n$: $\Omega \to \mathbb{R}^d$ *be independent random vectors with zero means. Assume* $\lambda_n/\Lambda_n \geqq 1/2$, *where* $\lambda_n(\Lambda_n)$ *is the smallest (largest) eigenvalue of* $\Gamma_n := \operatorname{cov}(\xi_1 + \dots + \xi_n)$. *Let* $3 \leqq \bar{s} < 4$, $s > \bar{s}$. *There exist positive constants* $c_7 = c_7(\bar{s}, s, d)$, $c_8 = c_8(\bar{s}, s, d)$ *such that, if* $|\xi_k| \leqq c_7 \sqrt{\lambda_n \operatorname{Log} 1/\rho_{sn}}$ *a.s.*, $1 \leqq k \leqq n$, *where* $\rho_{sn} := \lambda_n^{-s/2} \sum_1^n E[|\xi_k|^s]$, *the following holds true:*

One can construct a p-space $(\Omega_0, \mathscr{A}_0, P_0)$ *and random vectors* $S_n, T_n: \Omega_0 \to \mathbb{R}^d$ *such that* $P_0 \circ S_n = P \circ \sum_1^n \xi_k$, $P_0 \circ T_n = N(0, \Gamma_n)$, $E[|S_n - T_n|^s] \leqq c_8 \Lambda_n^{s/2} \rho_{sn}^{1+1/s}$ *and* $E[|S_n - T_n|^2] \leqq c_8 \Lambda_n \rho_{\bar{s}n}^{1+\eta}$ *for some* $\eta = \eta(\bar{s}, s) > 0$.

Proof (cf. [8], Theorem 7). We denote by $c_9, \dots, c_{14}$ constants depending on $\bar{s}, s$ and d only. Since by Lemma 1(b)

$$\rho_n := \rho_{3n} \leqq c_9 \cdot \rho_{\bar{s}n}^{1/(\bar{s}-2)} \leqq c_{10} \cdot \rho_{sn}^{1/(s-2)}, \tag{2.1}$$

we have

$$\operatorname{Log} 1/\rho_n \geqq c_{11} \cdot \operatorname{Log} 1/\rho_{sn}. \tag{2.2}$$

Setting $c_7 := \left(\frac{c_{11}}{8d}\right)^{1/2} \cdot (4 - \bar{s})(s+1)^{-2}$, we infer from (2.2):

$$|\xi_k| \leqq \frac{4 - \bar{s}}{\sqrt{8d}(s+1)^2} \sqrt{\lambda_n \operatorname{Log} 1/\rho_n} \quad \text{a.s.,} \quad 1 \leqq k \leqq n. \tag{2.3}$$

By [8], Theorem 6 (applied with $\varepsilon := (4 - \bar{s})/s$) we have:

$$\lambda \left(P \circ \Gamma_n^{-1/2} \sum_1^n \xi_k, N(0, I), c_{12} \rho_n^{1-\varepsilon}\right) \leqq c_{13} \rho_n^{s(s+1)}. \tag{2.4}$$

Thus we can obtain from the Strassen-Dudley theorem (cf. [7], Theorem 2) a p-space $(\Omega_0, \mathscr{A}_0, P_0)$ and random vectors S_n, T_n: $\Omega_0 \to \mathbb{R}^d$ with $P_0 \circ S_n = P \circ \sum_1^n \xi_k$, $P_0 \circ T_n = N(0, \Gamma_n)$, such that

$$P_0\{|S_n - T_n| > c_{12} \Lambda_n^{1/2} \rho_n^{1-\varepsilon}\} \leqq c_{13} \rho_n^{s(s+1)}. \tag{2.5}$$

Since

$$\begin{aligned}
&E[|S_n - T_n|^s] \\
&\quad \leqq x^s + E[|S_n - T_n|^{s+1}]^{s/(s+1)} P_0\{|S_n - T_n| > x\}^{1/(s+1)} \\
&\quad \leqq x^s + 2^{s-1}(E[|S_n|^{s+1}]^{s/(s+1)} + E[|T_n|^{s+1}]^{s/(s+1)}) P_0\{|S_n - T_n| > x\}^{1/(s+1)}
\end{aligned}$$

for $x>0$, we easily obtain from an obvious modification of [8], Lemma 3:

(2.6) $$E[|S_n - T_n|^s] \leqq c_{14} \Lambda_n^{s/2} \rho_n^{s-(4-\bar{s})}.$$

Using (2.1), we immediately obtain $E[|S_n - T_n|^s] \leqq c_8 \Lambda_n^{s/2} \rho_{sn}^{1+1/s}$. (Notice that $(s+\bar{s}-4)/(s-2) \geqq (s+1)/s$, since $\bar{s} \geqq 3$.)

Finally

$$E[|S_n - T_n|^2] \leqq E[|S_n - T_n|^s]^{2/s} \leqq c_{14}^{2/s} \Lambda_n \cdot \rho_n^{2-2(4-\bar{s})/s} \underset{(2.1)}{\leqq} c_8 \Lambda_n \rho_{\bar{s}n}^{1+\eta},$$

where $\eta := \dfrac{(s-2)(4-\bar{s})}{s(\bar{s}-2)}$.

We now proceed to the proof of Theorem 1. Since $\delta^{-s} K_{sn}(\delta) \leqq \delta^{-s'} K_{s'n}(\delta)$, $s \geqq s'$, it suffices to prove Theorem 1 for $s \geqq 3$.

To simplify our notations, we set for $1 \leqq m \leqq n$:

$$\alpha_m := \sum_{k=1}^{m} (E[|\xi_k|^s 1\{|\xi_k| \leqq \delta\}] + \delta^{s-2} E[|\xi_k|^2 1\{|\xi_k| > \delta\}]).$$

Furthermore, we define:

$$\bar{\xi}_k := \xi_k 1\{|\xi_k| \leqq \delta\}, \qquad \tilde{\xi}_k := \bar{\xi}_k - E[\bar{\xi}_k], \qquad 1 \leqq k \leqq n.$$

We denote by $c_{15}, \ldots, c_{33}$ positive constants depending on γ, s and d only.

(i) The purpose of this part of the proof is to show that there exist positive constants $\bar{s} = \bar{s}(\gamma, s) \in [3, 4)$, $\varepsilon = \varepsilon(\gamma, s) \in (0, 1]$ and c_{15} such that

(2.7) $$\delta^{-\bar{s}} K_{\bar{s}n}(\delta) \leqq c_{15} (\delta^{-s} K_{sn}(\delta))^{\varepsilon}.$$

Since $\sum_1^n E[|\tilde{\xi}_k|^2] \leqq d$, we infer from Lemma 1(a):

(2.8) $$K_{\bar{s}n}(\delta) \leqq c_{15} K_{sn}(\delta)^{(\bar{s}-2)/(s-2)} \quad \text{if } \bar{s} \leqq s.$$

We choose ε and $\bar{s}$ in a way such that

(2.9) $$\bar{s} - s\varepsilon \geqq 0$$

and

(2.10) $$(\bar{s} - s\varepsilon)\gamma = (\bar{s}-2)/(s-2) - \varepsilon.$$

Since $\gamma = \frac{1}{2}(1-\hat{\gamma})/(s-2)$, where $\hat{\gamma} > 0$, (2.10) can be rewritten:

(2.11) $$\bar{s} = \frac{4}{\hat{\gamma}+1} + \left(s - \frac{4}{\hat{\gamma}+1}\right)\varepsilon.$$

It is now easy to see that a possible choice is:

$$\bar{s}=\frac{4+3\hat{\gamma}}{1+\hat{\gamma}}, \qquad \varepsilon=\frac{3\hat{\gamma}}{(s-4)+s\hat{\gamma}} \quad (s\geqq 4)$$

and (trivially) $\bar{s}=s$, $\varepsilon=1$, if $s<4$.

Since $\delta\geqq(K_{sn}(\delta))^{\gamma}$, we easily obtain (2.7) from (2.8), (2.9) and (2.10).

Let now $(\varepsilon,\bar{s})$ be fixed. We set for $1\leqq m\leqq n$:

$$\beta_m := \sum_{k=1}^{m} (E[|\xi_k|^{\bar{s}}\,1\{|\xi_k|\leqq\delta\}]+\delta^{\bar{s}-2}E[|\xi_k|^2\,1\{|\xi_k|>\delta\}]).$$

Using (2.7), we obtain

$$\delta^{-\bar{s}}\beta_n\leqq c_{15}(\delta^{-s}K_{sn}(\delta))^{\varepsilon}+\delta^{-2}L_n(\delta)\leqq(c_{15}+1)\,2^{1-\varepsilon}(\delta^{-s}\alpha_n)^{\varepsilon},$$

if we w.l.o.g. assume

$$\alpha_n\leqq e^{-2}\delta^{s}. \tag{2.12}$$

(In particular: $\delta^{-2}L_n(\delta)\leqq e^{-2}$.)

Thus we have

$$\delta^{-\bar{s}}\beta_n\leqq c_{16}(\delta^{-s}\alpha_n)^{\varepsilon}. \tag{2.13}$$

(ii) Let $\tilde{S}_{(n)}: \Omega\to C_d[0,1]$ be defined by

$$\tilde{S}_{(n)}(t):=\sum_{1}^{m}\tilde{\xi}_k+\frac{t-t_m}{t_{m+1}-t_m}\tilde{\xi}_{m+1}, \quad t_m\leqq t\leqq t_{m+1}, \quad 0\leqq m<n. \tag{2.14}$$

An application of Lemma 2 yields

$$\lambda(P\circ S_{(n)}, P\circ\tilde{S}_{(n)}, 2\delta)\leqq\delta^{-s}\alpha_n. \tag{2.15}$$

(iii) Let $c_{17}>0$ be a sufficiently large chosen constant, which is determined by (2.25).

We now define a finite sequence of non-negative integers $0=m_0<\ldots<m_r=n$ by the following recursion: $m_0:=0$. Let m_{j-1} be defined for a $j\geqq 1$. Then we set

$$m_j:=\min\left\{k>m_{j-1}: c_{17}\,\delta^2\left(\log\frac{\delta^s}{\alpha_k-\alpha_{m_{j-1}}}\right)^{-1}\leqq t_k-t_{m_{j-1}}\right\}\wedge n, \tag{2.16}$$

where $\min\emptyset:=n+1$.

Put $r:=\min\{j\in\mathbb{N}: m_j=n\}$.

Since

$$\begin{aligned} d\sigma_k^2 &= E[|\xi_k|^2\,1\{|\xi_k|\leqq\delta\}]+E[|\xi_k|^2\,1\{|\xi_k|>\delta\}] \\ &\leqq(\alpha_k-\alpha_{k-1})^{2/s}+\delta^{2-s}(\alpha_k-\alpha_{k-1}), \end{aligned}$$

we obtain from (2.12):

$$d\sigma_k^2\leqq 2(\alpha_k-\alpha_{k-1})^{2/s}\leqq s\,\delta^2\left(\log\frac{\delta^s}{\alpha_k-\alpha_{k-1}}\right)^{-1}, \quad 1\leqq k\leqq n. \tag{2.17}$$

From the definition of the m_j's and (2.17) it follows immediately

$$t_{m_j}-t_{m_{j-1}}\leqq c_{17}\,\delta^2\left(\log\frac{\delta^s}{\alpha_{m_j}-\alpha_{m_{j-1}}}\right)^{-1}+\sigma^2_{m_j} \tag{2.18}$$
$$\leqq (c_{17}+s)\,\delta^2\left(\log\frac{\delta^s}{\alpha_{m_j}-\alpha_{m_{j-1}}}\right)^{-1},\qquad 1\leqq j\leqq r.$$

Furthermore, we infer from the definition of the m_j's

$$c_{17}\,\delta^2\left(\log\frac{\delta^s}{\alpha_{m_j}-\alpha_{m_{j-1}}}\right)^{-1}\leqq t_{m_j}-t_{m_{j-1}},\qquad 1\leqq j\leqq r-1. \tag{2.19}$$

(iv) Let $\lambda_j(\Lambda_j)$ be the smallest (largest) eigenvalue of

$$\Gamma_j:=\operatorname{cov}(\tilde{\xi}_{m_{j-1}+1}+\ldots+\tilde{\xi}_{m_j}),\qquad \rho_{\bar{s}j}:=\lambda_j^{-\bar{s}/2}\sum_{m_{j-1}+1}^{m_j}E[|\tilde{\xi}_k|^{\bar{s}}],$$
$$\rho_{sj}:=\lambda_j^{-s/2}\sum_{m_{j-1}+1}^{m_j}E[|\tilde{\xi}_k|^s],\qquad 1\leqq j\leqq r.$$

Since

$$\lambda_j\geqq t_{m_j}-t_{m_{j-1}}-2\sum_{m_{j-1}+1}^{m_j}E[|\xi_k|^2\,1\{|\xi_k|>\delta\}],$$

we obtain

$$t_{m_j}-t_{m_{j-1}}-\lambda_j\leqq 2\delta^{2-s}(\alpha_{m_j}-\alpha_{m_{j-1}})\leqq 2\delta^2\left(\log\frac{\delta^s}{\alpha_{m_j}-\alpha_{m_{j-1}}}\right)^{-1},\qquad 1\leqq j\leqq r-1. \tag{2.20}$$

Thus we have by (2.19)

$$\lambda_j\geqq (c_{17}-2)\,\delta^2\left(\log\frac{\delta^s}{\alpha_{m_j}-\alpha_{m_{j-1}}}\right)^{-1},\qquad 1\leqq j\leqq r-1. \tag{2.21}$$

Since $\Lambda_j\leqq t_{m_j}-t_{m_{j-1}}$, we infer from (2.18):

$$\Lambda_j\leqq (c_{17}+s)\,\delta^2\left(\log\frac{\delta^s}{\alpha_{m_j}-\alpha_{m_{j-1}}}\right)^{-1},\qquad 1\leqq j\leqq r. \tag{2.22}$$

Using the Hölder-inequality, we easily obtain:

$$E[|\tilde{\xi}_k|^s]\leqq 2^s E[|\bar{\xi}_k|^s],\qquad 1\leqq k\leqq n.$$

Together with (2.21) this implies:

$$\rho_{sj}\leqq\frac{2^s}{(c_{17}-2)^{s/2}}\,\frac{\alpha_{m_j}-\alpha_{m_{j-1}}}{\delta^s}\left(\log\frac{\delta^s}{\alpha_{m_j}-\alpha_{m_{j-1}}}\right)^{s/2},\qquad 1\leqq j\leqq r-1. \tag{2.23}$$

From (2.23) it easily follows if we choose c_{17} in a way such that $c_{17}\geqq c_{18}:=4s+2$

$$\operatorname{Log} 1/\rho_{sj}\geqq\tfrac{1}{2}\log\frac{\delta^s}{\alpha_{m_j}-\alpha_{m_{j-1}}},\qquad 1\leqq j\leqq r-1. \tag{2.24}$$

We put $c_{19}:=c_7(\bar{s},s,d)$, where $\bar{s}=\bar{s}(\gamma,s)$ is defined by (2.7). Recalling relations (2.21) and (2.24), it is straightforward to check that $2\delta\leqq c_{19}\sqrt{\lambda_j\operatorname{Log}1/\rho_{sj}}$, $1\leqq j\leqq r-1$, when setting

(2.25) $$c_{17}:=\max(c_{18},10c_{19}^{-2}).$$

Hence:

(2.26) $$|\xi_k|\leqq c_{19}\sqrt{\lambda_j\operatorname{Log}1/\rho_{sj}},\quad m_{j-1}<k\leqq m_j,\quad 1\leqq j\leqq r-1.$$

(v) We apply Theorem 7 and obtain a p-space $(\Omega_1,\mathscr{A}_1,P_1)$ with independent random vectors $(\tilde{U}_j,\tilde{V}_j)$, $1\leqq j\leqq r-1$ such that

(2.27) $$P_1\circ\tilde{U}_j=P\circ\sum_{m_{j-1}+1}^{m_j}\tilde{\xi}_k,\quad P_1\circ\tilde{V}_j=N(0,\Gamma_j),$$

(2.28) $$E[|\tilde{U}_j-\tilde{V}_j|^s]\leqq c_{20}(\alpha_{m_j}-\alpha_{m_{j-1}})$$

and

(2.29) $$E[|\tilde{U}_j-\tilde{V}_j|^2]\leqq c_{21}\cdot\Lambda_j\cdot\rho_{\bar{s}j}^{1+\eta},\quad 1\leqq j\leqq r-1.$$

(Notice that $\lambda_j\geqq\frac{1}{2}\Lambda_j$, $1=j\leqq r-1$, since $c_{17}\geqq c_{18}$. Use (2.21) and (2.22).)
A similar argument as in (2.23) shows

$$\rho_{\bar{s}j}\leqq\frac{2^{\bar{s}}}{(c_{17}-2)^{\bar{s}/2}}\cdot\frac{\beta_{m_j}-\beta_{m_{j-1}}}{\delta^{\bar{s}}}\left(\log\frac{\delta^s}{\alpha_{m_j}-\alpha_{m_{j-1}}}\right)^{\bar{s}/2},\quad 1\leqq j\leqq r-1.$$

Since $\operatorname{Log}1/\rho_{\bar{s}j}\geqq c_{22}\operatorname{Log}1/\rho_{sj}$ by Lemma 1, we infer from (2.22) and (2.24):

$$\begin{aligned}\Lambda_j\cdot\rho_{\bar{s}j}^{1+\eta}&\leqq c_{23}\cdot\frac{\beta_{m_j}-\beta_{m_{j-1}}}{\delta^{\bar{s}-2}}\rho_{\bar{s}j}^{\eta}(\operatorname{Log}1/\rho_{\bar{s}j})^{\bar{s}/2}\\&\leqq c_{24}\cdot\frac{\beta_{m_j}-\beta_{m_{j-1}}}{\delta^{\bar{s}-2}},\quad 1\leqq j\leqq r-1.\end{aligned}$$

(Note that $\rho_{\bar{s}j}$ remains bounded, $1\leqq j\leqq r-1$. Use (2.12) and (2.13).)
Therefore

(2.30) $$E[|\tilde{U}_j-\tilde{V}_j|^2]\leqq c_{25}\cdot\delta^{2-\bar{s}}(\beta_{m_j}-\beta_{m_{j-1}}),\quad 1\leqq j\leqq r-1.$$

We set $V_j:=(t_{m_j}-t_{m_{j-1}})^{1/2}\Gamma_j^{-1/2}\tilde{V}_j$, $1\leqq j\leqq r-1$.
Then $P_1\circ V_j=N(0,(t_{m_j}-t_{m_{j-1}})I)$ and

(2.31) $$P_1\circ(V_j-\tilde{V}_j)=N(0,((t_{m_j}-t_{m_{j-1}})^{1/2}I-\Gamma_j^{1/2})^2),\quad 1\leqq j\leqq r-1.$$

Denoting the largest eigenvalue of $((t_{m_j}-t_{m_{j-1}})^{1/2}I-\Gamma_j^{1/2})^2$ by μ_j, we obtain from the obvious inequality $\mu_j\leqq(t_{m_j}-t_{m_{j-1}}-\lambda_j)^2/(t_{m_j}-t_{m_{j-1}})$, $1\leqq j\leqq r-1$, (2.19) and (2.20):

$$\mu_j\leqq\frac{4}{c_{17}}\frac{(\alpha_{m_j}-\alpha_{m_{j-1}})^2}{\delta^{2s-2}}\log\frac{\delta^s}{\alpha_{m_j}-\alpha_{m_{j-1}}}\leqq c_{26}\frac{\alpha_{m_j}-\alpha_{m_{j-1}}}{\delta^{s-2}},\quad 1\leqq j\leqq r-1.$$

Since $E[|V_j-\tilde{V}_j|^2]\leqq d\mu_j$, $\delta^{2-s}(\alpha_{m_j}-\alpha_{m_{j-1}})\leqq\delta^{2-\bar{s}}(\beta_{m_j}-\beta_{m_{j-1}})$, $1\leqq j\leqq r-1$, we infer from the above inequality and (2.30)

(2.32) $$E[|\tilde{U}_j-V_j|^2]\leqq c_{27}\,\delta^{2-\bar{s}}(\beta_{m_j}-\beta_{m_{j-1}}),\quad 1\leqq j\leqq r-1.$$

Since $E[|V_j-\tilde{V}_j|^s]\leqq c_{28}\cdot\mu_j^{s/2}$ by (2.31),

$$\delta^{2-s}(\alpha_{m_j}-\alpha_{m_{j-1}})\leqq e^{-2(s-2)/s}(\alpha_{m_j}-\alpha_{m_{j-1}})^{2/s}$$

by (2.12), we finally infer from (2.29) and the above estimate of μ_j

(2.33) $$E[|\tilde{U}_j-V_j|^s]\leqq c_{29}(\alpha_{m_j}-\alpha_{m_{j-1}}),\quad 1\leqq j\leqq r-1.$$

Using Lemma A.1., [1] we obtain a p-space $(\Omega_2,\mathscr{A}_2,P_2)$ and two sequences of independent random vectors $\tilde{X}_k$, $1\leqq k\leqq n$ and Y_k, $1\leqq k\leqq n$ such that

(2.34) $$P_2\circ\tilde{X}_k=P\circ\tilde{\xi}_k,\quad P_2\circ Y_k=N(0,\sigma_k^2 I),\quad 1\leqq k\leqq n$$

and

(2.35) $$P_2\circ\left(\sum_{m_{j-1}+1}^{m_j}\tilde{X}_k,\sum_{m_{j-1}+1}^{m_j}Y_k\right)=P_1\circ(\tilde{U}_j,V_j),\quad 1\leqq j\leqq r-1.$$

We set $\tilde{S}_m:=\sum_1^m\tilde{X}_k$, $T_m:=\sum_1^m Y_k$, $1\leqq m\leqq n$.

(vi) From Lemma 3, (2.35), (2.32) and (2.33) we obtain

$$P_2\{\max_{1\leqq j\leqq r-1}|\tilde{S}_{m_j}-T_{m_j}|\geqq\delta\}=P_1\left\{\max_{1\leqq j\leqq r-1}\left|\sum_{k=1}^{j}(\tilde{U}_k-V_k)\right|\geqq\delta\right\}$$
$$\leqq\bar{c}_1\left(\exp\left(-\frac{\bar{c}_2}{c_{27}}\frac{\delta^{\bar{s}}}{\beta_n}\right)+c_{29}\,\delta^{-s}\alpha_n\right).$$

Therefore by (2.13):

(2.36) $$P_2\{\max_{1\leqq j\leqq r-1}|\tilde{S}_{m_j}-T_{m_j}|\geqq\delta\}\leqq c_{30}\,\delta^{-s}\alpha_n.$$

(vii) If the constant c_{31} is chosen large enough, we infer from Lemma 3, and (2.22) for $1\leqq j\leqq r$

$$P_2\{\max_{m_{j-1}\leqq k\leqq m_j}|\tilde{S}_k-\tilde{S}_{m_{j-1}}|\geqq c_{31}\cdot\delta\}\leqq c_{32}\,\delta^{-s}(\alpha_{m_j}-\alpha_{m_{j-1}}).$$

(Notice that $E[|\tilde{\xi}_k|^s]\leqq 2^s E[|\bar{\xi}_k|^s]$, $1\leqq k\leqq n$.)

Hence

(2.37) $$P_2\{\max_{1\leqq j\leqq r}\max_{m_{j-1}\leqq k\leqq m_j}|\tilde{S}_k-\tilde{S}_{m_{j-1}}|\geqq c_{31}\,\delta\}\leqq c_{32}\,\delta^{-s}\alpha_n.$$

Using (2.18) instead of (2.22), we obtain similarly:

(2.38) $$P_2\{\max_{1\leqq j\leqq r}\max_{m_{j-1}\leqq k\leqq m_j}|T_k-T_{m_{j-1}}|\geqq c_{31}\,\delta\}\leqq c_{32}\,\delta^{-s}\alpha_n.$$

(viii) (2.36), (2.37) and (2.38) imply immediately:

$$P_2\{\max_{1\leqq k\leqq n} |\tilde{S}_k - T_k| \geqq (2c_{31}+1)\delta\} \leqq (c_{30}+2c_{32})\delta^{-s}\alpha_n. \tag{2.39}$$

Therefore

$$\lambda(P\circ\tilde{S}_{(n)}, P_2\circ T_{(n)}, (2c_{31}+1)\delta) \leqq (c_{30}+2c_{32})\delta^{-s}\alpha_n, \tag{2.39'}$$

where $T_{(n)}$: $\Omega_2 \to C_d[0,1]$ is defined by

$$T_{(n)}(t) := T_m + \frac{t-t_m}{t_{m+1}-t_m} Y_{m+1} \quad \text{for } t_m \leqq t \leqq t_{m+1},$$

$0 \leqq m < n$.

To finish the proof is suffices to show

$$\lambda(P_2\circ T_n, W, \delta) \leqq c_{33}\delta^{-s}\alpha_n. \tag{2.40}$$

Let w: $\Omega \to C_d[0,1]$ be a Brownian motion, i.e. $P\circ w = W$.

Let $\hat{w}$: $\Omega \to C_d[0,1]$ be defined by

$$\hat{w}(t) := w(t_m) + \frac{t-t_m}{t_{m+1}-t_m}(w(t_{m+1}) - w(t_m)) \quad \text{for } t_m \leqq t \leqq t_{m+1}, \tag{2.41}$$

$0 \leqq m < n$.

Since $w = (w_1, \ldots, w_d)$, where w_i: $\Omega \to C[0,1]$ is a 1-dimensional Brownian motion ($i = 1, \ldots, d$), we obtain from the definition of $\hat{w}$, ([6], Theorem 1.5.1) and (2.17)

$$\begin{aligned} &P\{\sup_{0\leqq t\leqq 1} |\hat{w}(t) - w(t)| \geqq \delta\} \\ &\leqq P\{\max_{1\leqq k\leqq n} \sup_{t_{k-1}\leqq t\leqq t_k} |w(t) - w(t_{k-1})| \geqq \delta/2\} \\ &\leqq 2d \sum_{k=1}^{n} P\{|w_1(\sigma_k^2)| \geqq \delta/(2\sqrt{d})\} \leqq 4d \sum_{k=1}^{n} \exp\left(-\frac{\delta^2}{8\sigma_k^2 d}\right) \leqq c_{33}\delta^{-s}\alpha_n. \end{aligned}$$

This proves (2.40), since $P\circ\hat{w} = P_2\circ T_{(n)}$.

Corollaries 1 and 2 are immediate consequences of Theorem 1 applied for $\gamma = 1/s$ and $\gamma = 1/(s+1)$, respectively. (Notice that Corollary 1 is trivial for $\delta < (K_{sn}(\delta) + \delta^{s-2} L_n(\delta))^{1/s}$.) Corollary 3 follows from Corollary 1, using the obvious inequalities

$$K_{sn}(\delta) \leqq d\delta^{(s-2)\varepsilon} + \delta^{s-2} L_n(\delta^\varepsilon)$$

and

$$L_n(\delta) \leqq L_n(\delta^\varepsilon), \qquad \delta \leqq 1.$$

It suffices to prove Corollary 4 for $\Gamma = I$. Since the Prohorov distance metrizes the weak convergence, Corollary 4 is proved if we show that $\rho(P\circ S_{(n)}, W) \to 0$ as $n \to \infty$.

Since $\delta^{-1} L_n(\delta) \to 0$ for all $\delta > 0$, there exists a sequence $\delta_n \downarrow 0$ such that $\delta_n^{-3} L_n(\delta_n^3) \to 0$. (Use Lemma 6, [12], Chap. X.) Hence, $L_n(\delta_n^3) \leqq \delta_n^3$ for sufficiently large n. From Corollary 3, applied with $s = 7/2$, we obtain: $\rho(P \circ S_{(n)}, W) \leqq \delta_n \downarrow 0$.

3. Proof of Theorems 2, 3, 5 and 6

Using the well known Strassen-Dudley theorem (cf. [7], Theorem 2), we obtain from the proof of Theorem 1 the following proposition which improves ([11], Theorem 2).

Proposition 1. *Let $\xi_1, \ldots, \xi_n$: $\Omega \to \mathbb{R}^d$ be independent random vectors, defined on a p-space $(\Omega, \mathscr{A}, P)$, with zero means and $\operatorname{cov}(\xi_k) = \sigma_k^2 I$, $1 \leqq k \leqq n$. Let $s > 2$, $0 < \gamma < 1/(2s-4)$. Let $\delta/\sqrt{B_n} \geqq [B_n^{-s/2}(K_{sn}(\delta) + \delta^{s-2} L_n(\delta))]^\gamma$, where $B_n := \sum_1^n \sigma_k^2$, $n \in \mathbb{N}$. One can construct a p-space $(\Omega_0, \mathscr{A}_0, P_0)$ and two finite sequences of independent random vectors $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ with $P_0 \circ X_k = P \circ \xi_k$, $P_0 \circ Y_k = N(0, \operatorname{cov}(\xi_k))$, $1 \leqq k \leqq n$ such that the partial sums $S_m := \sum_1^m X_k$, $T_m := \sum_1^m Y_k$, $1 \leqq m \leqq n$, fulfill:*

$$P_0\{\max_{1 \leqq k \leqq n} |S_k - T_k| > c_{34} \delta\} \leqq c_{35}(\delta^{-s} K_{sn}(\delta) + \delta^{-2} L_n(\delta)),$$

where $c_{34}, c_{35} > 0$ are constants depending on γ, s and d only.

Proof. Using relations (2.15) and (2.39′) of Theorem 1 (applied with $\xi_1/\sqrt{B_n}, \ldots, \xi_n/\sqrt{B_n}$ and $\delta/\sqrt{B_n}$), we obtain

$$\lambda\left(P \circ S_{(n)}, P_2 \circ T_{(n)}, c_{34} \frac{\delta}{\sqrt{B_n}}\right) \leqq c_{35}(\delta^{-s} K_{sn}(\delta) + \delta^{-2} L_n(\delta)), \tag{3.1}$$

where $S_{(n)}$: $\Omega \to C_d[0,1]$ is the partial sum process of the random vectors $\xi_k/\sqrt{B_n}$, $1 \leqq k \leqq n$ and $T_{(n)}$: $\Omega_2 \to C_d[0,1]$ is the partial sum process of a finite sequence of independent random vectors $\eta_1, \ldots, \eta_n$: $\Omega_2 \to \mathbb{R}^d$ such that $P_2 \circ \eta_k = N\left(0, \frac{\sigma_k^2}{B_n} I\right)$, $1 \leqq k \leqq n$. The assertion follows by a straightforward application of [7], Theorem 2.

Remarks (a) By obvious modifications of the proof of Theorem 1 one can easily show that the above proposition remains valid, if the covariance matrices $\Sigma_k := \operatorname{cov}(\xi_k)$, $1 \leqq k \leqq n$, fulfill the condition

$$\sigma_k^2 \mu_1 \leqq \langle t, \Sigma_k t \rangle \leqq \sigma_k^2 \mu_2 \quad \text{for } |t| = 1 \quad (1 \leqq k \leqq n) \tag{3.2}$$

where μ_1, μ_2 are positive constants. We only have to replace the constants c_{34}, c_{35} by constants depending on μ_1, μ_2, γ, s and d.

(b) Let $2<s<4$. Then it is easy to see that for any given $\delta>0$ a construction is possible such that

$$(3.3)\qquad P_0\{\max_{1\leqq k\leqq n}|S_k-T_k|>c_{36}\,\delta\}\leqq c_{37}(\delta^{-s}K_{sn}(\delta)+\delta^{-2}L_n(\delta)),$$

where c_{36}, c_{37} are positive constants depending on s and d only.

(c) Since $K_{sn}(\delta)+\delta^{s-2}L_n(\delta)\leqq d\delta^{s-2}B_n$, the condition

$$\text{“}\delta\geqq\sqrt{B_n}\,[B_n^{-s/2}(K_{sn}(\delta)+\delta^{s-2}L_n(\delta))^{\gamma}\quad\text{for some } \gamma<1/(2s-4)\text{”}$$

is always fulfilled, if $\delta\geqq d^{1/(s-2)}\sqrt{B_n}$.

Now we start to prove Theorem 2. W.l.o.g. we assume that $\Gamma=I$.

Let $0<\tilde a_k\uparrow\infty$ such that

$$(3.4)\qquad \tilde a_k/a_k\to 0\quad\text{as } k\to\infty$$

and

$$(3.5)\qquad \sum_{k=1}^{\infty}\tilde a_k^{-s}(E[|\xi_k|^s 1\{|\xi_k|\leqq\tilde a_k\}]+\tilde a_k^{s-2}E[|\xi_k|^2 1\{|\xi_k|>\tilde a_k\}])<\infty.$$

(The existence of such a sequence follows easily from (1.7) and (1.8), since $E[|\xi_k|^s 1\{|\xi_k|\leqq\delta\}]+\delta^{s-2}E[|\xi_k|^2 1\{|\xi_k|>\delta\}]$, $\delta>0$, is non-decreasing.)

We define $\{m_n\colon n\in\mathbb{N}\}$ by the following recursion:

$$m_0:=1,\qquad m_n:=\min\{k\colon \tilde a_k\geqq 2\tilde a_{m_{n-1}}\},\qquad n\geqq 1.$$

By definition we have

$$(3.6)\qquad \tilde a_{m_n-1}\leqq 2\tilde a_{m_{n-1}},\qquad n\geqq 1.$$

Applying the above remark (b) to $\{\xi_k\colon m_{n-1}\leqq k<m_n\}$ $(n\in\mathbb{N})$ we obtain a p-space $(\Omega_0,\mathscr{A}_0,P_0)$ and two sequences of independent random vectors $\{X_n\}$, $\{Y_n\}$ such that the following holds true:

$$\begin{aligned}(3.7)\quad P_0\{&\max_{m_{n-1}\leqq k<m_n}|(S_k-S_{m_{n-1}-1})-(T_k-T_{m_{n-1}-1}))|>c_{36}\,\tilde a_{m_{n-1}}\}\\ &\leqq c_{37}\,\tilde a_{m_{n-1}}^{-s}\sum_{m_{n-1}}^{m_n-1}(E[|\xi_k|^s 1\{|\xi_k|\leqq\tilde a_{m_{n-1}}\}]+\tilde a_{m_{n-1}}^{s-2}E[|\xi_k|^2 1\{|\xi_k|>\tilde a_{m_{n-1}}\}])\\ &\leqq 2^s c_{37}\sum_{m_{n-1}}^{m_n-1}\tilde a_k^{-s}(E[|\xi_k|^s 1\{|\xi_k|\leqq\tilde a_k\}]+\tilde a_k^{s-2}E[|\xi_k|^2 1\{|\xi_k|\geqq\tilde a_k\}]),\end{aligned}$$

where $S_0:=T_0:=0$.

Thus we obtain from the Borel-Cantelli lemma, (3.5) and the definition of the m_n's that almost surely

$$\begin{aligned}|S_k-T_k|&\leqq K(\omega)+\sum_{j=1}^{n}c_{36}\,\tilde a_{m_{j-1}}\leqq K(\omega)+2c_{36}\,\tilde a_{m_{n-1}}\\ &\leqq K(\omega)+2c_{36}\,\tilde a_k\quad\text{if } m_{n-1}\leqq k<m_n.\end{aligned}$$

This proves the assertion (recall (3.4)).

Theorem 3 is an immediate consequence of the remark after Theorem 2 and Lemma 15, ([12], Chap. IX).

Theorems 5 and 6 are special cases of Theorem 2, if $2<s<4$. Thus we have to prove these results only for $s\geqq 4$. We will show that the above proof of Theorem 2 works in these cases, too. But we have to be a little more careful, since we can apply our proposition only for sufficiently large δ, if $s\geqq 4$.

The main problem is (3.7). Let $a_n=\sqrt{B_n \operatorname{Log}_2 B_n}$ (resp. $a_n=\sqrt{B_n}$). It suffices to show that there exists a sequence $\tilde{a}_n\uparrow\infty$ such that (3.4) and (3.5) as well as

$$\tilde{a}_n\geqq\sqrt{B_n}(B_n^{-s/2}(K_{sn}(\tilde{a}_n)+\tilde{a}_n^{s-2}L_n(\tilde{a}_n)))^\gamma \tag{3.8}$$

for some $\gamma<1/(2s-4)$ hold true.

It is easy to see that such a sequence can be found if

$$[B_n^{-s/2}(K_{sn}(a_n)+a_n^{s-2}L_n(a_n))]^\gamma=o(a_n/\sqrt{B_n}) \tag{3.9}$$

for some $\gamma<1/(2s-4)$.

Using the same argument as in the remark (c) after the proposition, we immediately obtain (3.9) for $a_n=\sqrt{B_n \operatorname{Log}_2 B_n}$, hence Theorem 5.

Since $B_n^{-s/2}(K_{sn}(a_n)+a_n^{s-2}L_n(a_n))\leqq B_n^{-s/2}\sum_1^n E[|\xi_k|^s]$, we have by (1.19):

$$B_n^{-s/2}(K_{sn}(a_n)+a_n^{s-2}L_n(a_n))=o(n^0),$$

therefore (3.9) for $a_n=\sqrt{B_n}$, hence Theorem 6.

4. Proof of Theorem 4

To simplify our notations, we set: $G(x):=H^{-1}(x)$, $x>0$. Let $1+2/f_1<b_k\uparrow\infty$ such that

$$\sum_{k=1}^{\infty}\frac{1}{k\,b_k f_k}=\infty \tag{4.1}$$

and

$$0\leqq\frac{b_k^{1/2}G(kf_k)^2-1}{k\,b_k f_k}\leqq\frac{1}{2},\qquad k\in\mathbb{N}. \tag{4.2}$$

Let $\{\xi_k\}$ be a sequence of independent random variables with symmetric distributions such that

$$\begin{aligned}
&P\{|\xi_k|=b_k^{1/4}G(kf_k)\}=\frac{1}{k\,b_k f_k},\\
&P\{|\xi_k|=1\}=1-\frac{G(kf_k)^2}{k\,b_k^{1/2}f_k}\quad\text{and}\\
&P\{\xi_k=0\}=\frac{b_k^{1/2}G(kf_k)^2-1}{k\,b_k f_k},\qquad k\in\mathbb{N}.
\end{aligned}\tag{4.3}$$

From (4.3) it follows immediately that $E[\xi_k]=0$, $E[\xi_k^2]=1$, $k\in\mathbb{N}$. Furthermore, we have

$$E[H(|\xi_k|)] \leqq \frac{H(b_k^{1/4}\,G(kf_k))}{k\,b_k f_k} + H(1) \leqq 1 + H(1), \qquad k\in\mathbb{N},$$

since $t^{-4}H(t)$ is non-increasing and $H(0)=0$.

Using the obvious inequality: $|X_n| \leqq |S_n - T_n| + |S_{n-1} - T_{n-1}| + |Y_n|$, we obtain:

$$P_0\left\{\overline{\lim_n} \frac{|S_n - T_n|}{G(nf_n)} < \infty\right\} \leqq P_0\left\{\overline{\lim_n} \frac{|X_n|}{G(nf_n)} < \infty\right\}. \tag{4.4}$$

(Notice that $P_0\left\{\overline{\lim_n} \frac{|Y_n|}{G(nf_n)} < \infty\right\} = 1$, since $\{Y_n\}$ is a sequence of independent $N(0,1)$-distributed r.v.'s.)

From the Borel-Cantelli lemma, (4.1) and (4.3) we infer

$$P_0\left\{\overline{\lim_n} \frac{|X_n|}{G(nf_n)} < \infty\right\} = 0, \qquad \text{hence the assertion by (4.4).}$$

5. Proof of Corollaries 5 and 6

We use the following proposition which is related to Strassen's functional law of the iterated logarithm for the Brownian motion (cf. [14] Theorem 1).

Proposition 2. *Let $\{\zeta(t)\colon t\geqq 0\}$ be the Brownian motion in $\mathbb{R}^d$. Let $B_n \uparrow \infty$ such that*

$$\overline{\lim_n} \left(\frac{B_{n+1}}{B_n \exp(\sqrt{\operatorname{Log}_2 B_n})}\right) < \infty. \tag{5.1}$$

Set $\zeta_n(t) := (2B_n \operatorname{Log}_2 B_n)^{-1/2}\,\zeta(B_n t)$, $0\leqq t\leqq 1$.

With probability one, $\{\zeta_n\colon n\in\mathbb{N}\}$ is relatively compact in $C_d[0,1]$ and the set of its limit points coincides with $\mathscr{F}$.

Proof (cf. [14], Theorem 1). Since the first part of the assertion follows easily from [14], Corollary 1, we have only to show the second part.

We use similar arguments as in [14], Theorem 1 (cf. p. 214/215). It suffices to show that for given $x\in\mathscr{F}$ and $\varepsilon>0$ the following holds true:

$$P(\overline{\lim_n}\, \{\|\zeta_n - x\| < \varepsilon\}) = 1. \tag{5.2}$$

Let m be a positive integer, let $\delta>0$.

We denote by A_n the event

$$\left\{\left|\zeta_n^\kappa\left(\frac{i}{m}\right) - \zeta_n^\kappa\left(\frac{i-1}{m}\right) - \left(x^\kappa\left(\frac{i}{m}\right) - x^\kappa\left(\frac{i-1}{m}\right)\right)\right| < \delta \quad \text{for } 2\leqq i\leqq m, \quad 1\leqq\kappa\leqq d\right\},$$

where $\zeta_n^\kappa(t)\,(x^\kappa(t))$ denotes the κ-th coordinate of $\zeta_n(t)$ $(x(t))$, $1\leqq\kappa\leqq d$ $(t\in[0,1])$.

Using the same arguments as in [14], we obtain for sufficiently large n

$$P(A_n) \geqq \frac{c(\delta)}{\operatorname{Log} B_n \sqrt{m \operatorname{Log}\operatorname{Log} B_n}}, \tag{5.3}$$

where $c(\delta) > 0$ depends only on δ.

We define the subsequence $\{n_j : j \geqq 1\} \subseteq \mathbb{N}$ by the following recursion:

$$n_1 := 1, \quad n_j := \min\{k > n_{j-1} : B_k \geqq m \cdot B_{n_{j-1}}\}, \quad j \geqq 2.$$

By definition we have:

$$B_{n_j} \geqq m \cdot B_{n_{j-1}}, \quad j \geqq 2. \tag{5.4}$$

Furthermore, we obtain from (5.1) and the definition of the n_j's:

$$B_{n_j} \leqq K \cdot m \cdot B_{n_{j-1}} \exp(\sqrt{\operatorname{Log}_2 B_{n_j-1}}), \quad j \geqq 2, \tag{5.5}$$

where K is a positive constant such that $B_n \leqq K \cdot B_{n-1} \exp(\sqrt{\operatorname{Log}_2 B_{n-1}})$, $n \geqq 2$ and $\sigma_1^2 = B_1 \leqq K$.

This implies immediately:

$$B_{n_j} \leqq (K \cdot m)^j \exp(j \sqrt{\operatorname{Log}_2 B_{n_j}}). \tag{5.6}$$

From (5.6) we easily obtain: $B_{n_j} \leqq \exp(\bar{K} j \sqrt{\operatorname{Log} j})$, $j \geqq 1$, where $\bar{K} > 0$ is an appropriate constant.

Therefore by (5.3)

$$\sum_{j=1}^{\infty} P(A_{n_j}) = \infty. \tag{5.7}$$

Since the events A_{n_j}, $j \geqq 1$ are independent by (5.4), we infer from the Borel-Cantelli lemma

$$1 = P(\overline{\lim_j}\, A_{n_j}) = P(\overline{\lim_n}\, A_n).$$

We can now prove (5.2) by the same arguments as in [14], p. 215.

To prove Corollary 5, we use standard arguments. First, we assume w.l.o.g. that the sequence T_n, $n \geqq 1$ in Theorem 5 fulfills

$$T_n = \Gamma^{1/2} \zeta(B_n), \quad n \in \mathbb{N}, \tag{5.8}$$

where $\{\zeta(t) : t \geqq 0\}$ is a d-dimensional Brownian motion on $(\Omega_0, \mathscr{A}_0, P_0)$. This assumption can be justified by [6], Proposition 1.4.1 or [1], Lemma A.1.

Using the same arguments as in the proof of (2.40), we obtain from Theorem 5 and (5.8)

$$\sup_{0 \leqq t \leqq 1} |\hat{U}_{(n)}(t) - \Gamma^{1/2} \zeta_n(t)| \to 0 \quad \text{a.s.}, \tag{5.9}$$

where $\{\zeta_n(t) : 0 \leqq t \leqq 1\}$ is defined as in Proposition 2 and $\{\hat{U}_{(n)} : n \in \mathbb{N}\}$ is defined as $\{U_{(n)} : n \in \mathbb{N}\}$ with ξ_k replaced by X_k, $k \in \mathbb{N}$.

Since $P_0 \circ (\hat{U}_{(n)})_{n\in\mathbb{N}} = P \circ (U_{(n)})_{n\in\mathbb{N}}$, we easily obtain from (5.9) and Proposition 2 the assertion.

To prove Corollary 6, we remark that

$$\{(2B_n \operatorname{Log}_2 B_n)^{-1/2} \sum_1^n \xi_k : n\in\mathbb{N}\} = \pi_1(\{U_{(n)} : n\in\mathbb{N}\})$$

and

$$\{\Gamma^{1/2} x : |x| \leqq 1\} = \pi_1(\mathscr{F}_\Gamma), \qquad \text{where } \pi_1 : C_d[0,1] \to \mathbb{R}^d$$

is (trivially) continuous.

6. Lemmas

Lemma 1. *Let $\xi_1, \ldots, \xi_n$: $\Omega \to \mathbb{R}^d$ be random vectors such that $E[|\xi_k|^s] < \infty$ for some $s > 3$ $(1 \leqq k \leqq n)$. Set $B_n := \sum_1^n E[|\xi_k|^2]$. Then we have for any given $\bar{s} \in [3, s]$*

$$\text{(a)} \qquad B_n^{-\bar{s}/2} \sum_1^n E[|\xi_k|^{\bar{s}}] \leqq (B_n^{-s/2} \sum_1^n E[|\xi_k|^s])^{(\bar{s}-2)/(s-2)}.$$

If we additionally assume that $E[\xi_k] = 0$ $(1 \leqq k \leqq n)$, we have furthermore

$$\text{(b)} \quad \lambda_n^{-\bar{s}/2} \sum_1^n E[|\xi_k|^{\bar{s}}] \leqq \left(\frac{\Lambda_n}{\lambda_n}\right)^{\bar{s}/2} d^{(s-\bar{s})/(s-2)} \left(\lambda_n^{-s/2} \sum_1^n E[|\xi_k|^s]\right)^{(\bar{s}-2)/(s-2)}$$

where $\lambda_n(\Lambda_n)$ is the smallest (largest) eigenvalue of $\Gamma_n := \operatorname{cov}(\xi_1 + \ldots + \xi_n)$.

Proof. (a) follows from Lemma 2, [12] Chap. VI (applied to $X_k = |\xi_k|$, $1 \leqq k \leqq n$). Note that this lemma is formulated in [12] for r.v.'s with zero means. But it can be easily seen from the proof in [12] that this lemma is also applicable in our situation.

(b) follows from (a), since $d\lambda_n \leqq B_n \leqq d\Lambda_n$.

Lemma 2. *Let $\xi_1, \ldots, \xi_n$: $\Omega \to \mathbb{R}^d$ be integrable random vectors with zero means. Set for a fixed $\delta > 0$:*

$$\bar{\xi}_k := \xi_k 1\{|\xi_k| \leqq \delta\}, \quad \tilde{\xi}_k := \bar{\xi}_k - E[\bar{\xi}_k], \quad 1 \leqq k \leqq n.$$

Then we have:

$$P\left\{\max_{1\leqq k\leqq n} \left|\sum_{m=1}^k \xi_m - \sum_{m=1}^k \tilde{\xi}_m\right| \geqq 2\delta\right\} \leqq \delta^{-2} \sum_{k=1}^n E[|\xi_k|^2 1\{|\xi_k| > \delta\}].$$

Proof. W.l.o.g. we assume $\delta^{-2} \sum_1^n E[|\xi_k|^2 1\{|\xi_k| > \delta\}] \leqq 1$. In this case we obtain:

$$\sum_{k=1}^n |E[\bar{\xi}_k]| \leqq \sum_{k=1}^n E[|\xi_k - \bar{\xi}_k|] \leqq \delta^{-1} \sum_{k=1}^n E[|\xi_k|^2 1\{|\xi_k| > \delta\}] \leqq \delta,$$

hence

$$P\left\{\max_{1\leqq k\leqq n}\left|\sum_{m=1}^{k}\xi_m-\sum_{m=1}^{k}\tilde{\xi}_m\right|\geqq 2\delta\right\}\leqq P\left\{\max_{1\leqq k\leqq n}\left|\sum_{m=1}^{k}\xi_m-\sum_{m=1}^{k}\bar{\xi}_m\right|\geqq\delta\right\}$$

$$\leqq\sum_{k=1}^{n}P\{\xi_k\neq\bar{\xi}_k\}\leqq\delta^{-2}\sum_{k=1}^{n}E[|\xi_k|^2\,1\{|\xi_k|>\delta\}].$$

Lemma 3. *Let ξ_k: $\Omega\to\mathbb{R}^d$, $1\leqq k\leqq n$, be independent random vectors with zero means. Let $s>2$. Let Λ_n be the largest eigenvalue of $\Gamma_n:=\operatorname{cov}(\xi_1+\ldots+\xi_n)$. Then we have for $t>0$:*

$$P\left\{\max_{1\leqq k\leqq n}\left|\sum_{m=1}^{k}\xi_m\right|\geqq t\right\}\leqq\bar{c}_1\left(\exp\left(-\frac{\bar{c}_2 t^2}{\Lambda_n}\right)+t^{-s}\sum_{k=1}^{n}E[|\xi_k|^s]\right)$$

with positive constants $\bar{c}_1,\bar{c}_2$ depending only on s and d.

Proof. W.l.o.g. we assume $t\geqq 3\sqrt{d}\sqrt{\Lambda_n}$.

(If $t<3\sqrt{d}\sqrt{\Lambda_n}$, the estimate is trivial, since in this case $\exp\left(-\frac{\bar{c}_2 t^2}{\Lambda_n}\right)\geqq\exp(-9d\bar{c}_2)$.)

Let $\xi_k=(\xi_{k,1},\ldots,\xi_{k,d})$, $1\leqq k\leqq n$. Since we have

$$P\left\{\max_{1\leqq k\leqq n}\left|\sum_{m=1}^{k}\xi_m\right|\geqq t\right\}\leqq\sum_{i=1}^{d}P\left\{\max_{1\leqq k\leqq n}\left|\sum_{m=1}^{k}\xi_{m,i}\right|\geqq t/\sqrt{d}\right\},$$

we obtain from Theorem 12 ([12], Chap. III)

$$P\left\{\max_{1\leqq k\leqq n}\left|\sum_{m=1}^{k}\xi_m\right|\geqq t\right\}\leqq 2\sum_{i=1}^{d}P\left\{\left|\sum_{m=1}^{n}\xi_{m,i}\right|\geqq t/(2\sqrt{d})\right\}.$$

The assertion follows from the last inequality by an application of Corollary 4 ([9], p. 653).

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