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Probability Theory and Related Fields

On the Absolute Continuity of Infinite Product Measure and Its Convolution

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Summary. Let $X=\{X_k\}$ be an I.I.D. random sequence and $Y=\{Y_k\}$ be a symmetric independent random sequence which is also independent of X. Then X and $X+Y=\{X_k+Y_k\}$ induce probability measures μ_X and μ_{X+Y} on the sequence space, respectively. The problem is to characterize the absolute continuity of μ_X and μ_{X+Y} and give applications to the absolute continuity of stochastic processes; in particular we give a sufficient condition for the absolute continuity of the sum of Brownian motion and an independent process with respect to the Brownian motion.

We assume that the distribution of X_1 is equivalent to the Lebesgue measure and the density function f satisfies

$$\text{(C)} \qquad \int_{-\infty}^{+\infty} \frac{f''(x)^2}{f(x)}\, dx < +\infty .$$

Under this condition we shall give some sufficient conditions and necessary conditions for $\mu_X \sim \mu_{X+Y}$. The critical condition is $\sum_k \mathbb{E}[|Y_k|^2 : |Y_k| \leqq \varepsilon]^2 < +\infty$ for some $\varepsilon > 0$. In particular in the case where X is Gaussian, we shall give finer results. Finally we shall compare the condition (C) with the Shepp's condition:

$$\text{(A)} \qquad \int_{-\infty}^{+\infty} \frac{f'(x)^2}{f(x)}\, dx < +\infty .$$

1. Introduction

Let $X=\{X_k\}$ be an I.I.D. random sequence and $Y=\{Y_k\}$ be a random sequence which is independent of X. Then X and $X+Y=\{X_k+Y_k\}$ induce probability measures μ_X and μ_{X+Y} on the sequence space, respectively. The problem is

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to characterize the absolute continuity of μ_X and μ_{X+Y} and give applications to the absolute continuity of stochastic processes.

Throughout the paper we assume that the distribution of X_1 is mutually absolutely continuous with respect to the Lebesgue measure and denote the density function by f.

When Y is a deterministic sequence $\{y_k\}$, the following theorem due to L.A. Shepp is wellknown.

Theorem 1 (L.A. Shepp [9]). *Let $X=\{X_k\}$ be an I.I.D. random sequence with the distribution which is equivalent to the Lebesgue measure and $Y=\{y_k\}$ be a deterministic sequence. Then the following statements are equivalent.*

(A) *f is absolutely continuous with respect to the Lebesgue measure and the Radon-Nikodym derivative f' satisfies*

$$\int_{-\infty}^{+\infty} \frac{f'(x)^2}{f(x)} dx < +\infty .$$

(B) *$\mu_X \sim \mu_{X+Y}$ (mutually absolutely continuous) if and only if*

$$\sum_k |y_k|^2 < +\infty .$$

As a corollary of Theorem 1, it is easy to prove that if (A) is satisfied and $\sum_k |Y_k|^2 < +\infty$, a.s., then we have $\mu_X \sim \mu_{X+Y}$. But the converse is not true. In fact it is known that if both of X and Y are centered Gaussian and Y_k's are independent, then $\mu_X \sim \mu_{X+Y}$ if and only if

$$\sum_k |Y_k|^4 < +\infty , \text{ a.s.,} \tag{1}$$

(Yu.V. Rozanov [5] and X. Fernique [2]), and if X is centered Gaussian and Y_k's are independent and $P(Y_k=a_k)=P(Y_k=-a_k)=\frac{1}{2}$ where $\{a_k\}$ is a real sequence, then (1) implies $\mu_X \sim \mu_{X+Y}$ (H. Sato [7]). Therefore, interesting is the role of (1) concerning $\mu_X \sim \mu_{X+Y}$.

In this paper we treat the case where $Y=\{Y_k\}$ is an independent random sequence and assume the symmetry of distributions of Y_k's unless the contrary is explicitly stated. We consider a condition similar to (A) as follows:

(C) *f is differentiable, the derivative f' is absolutely continuous with respect to the Lebesgue measure, and the Radon-Nikodym derivative f'' satisfies*

$$\int_{-\infty}^{+\infty} \frac{f''(x)^2}{f(x)} dx < +\infty .$$

In Sect. 2, under some additional contitions to (C) we prove that (1) implies $\mu_X \sim \mu_{X+Y}$ (Theorem 3), and more generally

$$\sum_k \mathbb{E}[Y_k^2 : |Y_k| \leqq \varepsilon]^2 < +\infty \tag{2}$$

and

$$\sum_k \mathbb{P}(|Y_k| > \varepsilon) < +\infty, \tag{3}$$

for some $\varepsilon > 0$, which are slightly weaker than (1), imply $\mu_X \sim \mu_{X+Y}$ (Theorem 4). Conversely if f is twice continuously differentiable and

$$\lim_{k \to +\infty} Y_k = 0, \text{ a.s.}, \tag{4}$$

then $\mu_X \sim \mu_{X+Y}$ implies (2) for some $\varepsilon > 0$ (Theorem 5).

Section 3 is devoted to the case where X is standard Gaussian, that is, the distribution of X_1 is Gaussian with mean zero and variance 1. In this case (1) implies $\mu_X \sim \mu_{X+Y}$ (Theorem 7). More generally (2) and (3) for some $\varepsilon > 0$ imply $\mu_X \sim \mu_{X+Y}$ and conversely $\mu_X \sim \mu_{X+Y}$ implies (2) and

$$\sum_k \mathbb{P}(|Y_k| > \varepsilon)^2 < +\infty, \tag{5}$$

for every $\varepsilon > 0$ (Theorem 9). However neither (2) and (5), nor (2) and (3) are necessary and sufficient for $\mu_X \sim \mu_{X+Y}$. We shall show two counter examples. It is remarkable that there exist examples such that $\mu_X \sim \mu_{X+Y}$ and $\lim_k |Y_k| = +\infty$, a.s.

In Sect. 4 we give some applications to the absolute continuity for stochastic processes. In particular we give a sufficient condition for the absolute continuity of the sum of the Brownian motion and an independent process (Theorem 11).

Concerning the relation between (A) and (C), in Sect. 5 we prove that (C) implies (A) if $f = f(x)$ is monotone for large $|x|$.

In the proof of Theorem 1, where Y is deterministic, Shepp made use of Kakutani's criterion [4], which is based on the Hellinger integral, for the absolute continuity of infinite product measures and the Fourier transform. But in our case, where Y is a random sequence, it is difficult to apply them. We make use of Sato's criterion [8] for the absolute continuity of infinite product measures.

2. General Case

Let $X = \{X_k\}$ be an I.I.D. random sequence with density function f ($f > 0$, a.e. (dx)), and $Y = \{Y_k\}$ be an *independent* random sequence which is also *independent* of X. We assume that the distributions of Y_k's are symmetric in this section. For every k denote the distribution of X_k and $X_k + Y_k$ by m_k and ν_k, respectively. Then ν_k is absolutely continuous with respect to m_k and we have $\frac{d\nu_k}{dm_k}(x)$

$=\dfrac{\mathbb{E}^Y[f(x+Y_k)]}{f(x)}$ where $\mathbb{E}^Y[\ ]$ is the expectation with respect to Y. Define for every k

$$Z(X_k)=\frac{\mathbb{E}^Y[f(X_k+Y_k)]}{f(X_k)}-1. \tag{6}$$

Our starting point is the following theorem.

Theorem 2. *In the above situation the following three statements are equivalent.*

(Q) $\mu_X \sim \mu_{X+Y}$.

(S) $\sum_k Z(X_k)$ *converges almost surely.*

(K) $$\sum_k \mathbb{E}[Z(X_k): Z(X_k)\geqq 1]<+\infty, \tag{$K-1$}$$

$$\sum_k \mathbb{E}[Z(X_k)^2: |Z(X_k)|<1]<+\infty. \tag{$K-2$}$$

Proof. The equivalence of (Q) and (S) is derived from Theorem 3 of H. Sato [8], and that of (S) and (K) from Kolmogorov's three series theorem since $Z(X_k)$'s are independent, $Z(X_k)>-1$, a.s., and we have

$$\begin{aligned}\sum_k \mathbb{E}[Z(X_k): Z(X_k)\geqq 1]&=-\sum_k \mathbb{E}[Z(X_k): |Z(X_k)|<1]\\ &=\sum_k |\mathbb{E}[Z(X_k): |Z(X_k)|<1]|. \quad \text{Q.E.D.}\end{aligned}$$

For every $\varepsilon>0$ and k, decompose $Z(X_k)$ as follows.

$$\begin{aligned}Z(X_k)&=V_\varepsilon(X_k)+W_\varepsilon(X_k)\\ &=\{\mathbb{E}^Y[f(X_k+Y_k): |Y_k|>\varepsilon]/f(X_k)-\mathbb{P}(|Y_k|>\varepsilon)\}\\ &\quad+\mathbb{E}^Y[f(X_k+Y_k)-f(X_k): |Y_k|\leqq\varepsilon]/f(X_k).\end{aligned} \tag{7}$$

Then we have the following lemma.

Lemma 1. *If* (3) *is satisfied for some* $\varepsilon>0$, *then* $\sum_k V_\varepsilon(X_k)$ *absolutely converges almost surely.*

Proof. Fubini's theorem implies

$$\begin{aligned}&\mathbb{E}^Y[\sum_k |V_\varepsilon(X_k)|]\\ &\leqq \sum_k \int_{-\infty}^{+\infty} \mathbb{E}^Y[f(x+Y_k): |Y_k|>\varepsilon]\,dx+\sum_k \mathbb{P}(|Y_k|>\varepsilon)\\ &\leqq \sum_k \mathbb{E}^Y\left[\int_{-\infty}^{+\infty} f(x+Y_k)\,dx: |Y_k|>\varepsilon\right]+\sum_k \mathbb{P}(|Y_k|>\varepsilon)\\ &=2\sum_k \mathbb{P}(|Y_k|>\varepsilon),\end{aligned}$$

and (3) proves the lemma. Q.E.D.

Our first result is the following.

Theorem 3. *If f satisfies* (C), 3 *times differentiable, $f^{(3)}$ is absolutely continuous with respect to the Lebesgue measure and the Radon-Nikodym derivative $f^{(4)}$ satisfies*

$$\int_{-\infty}^{+\infty} |f^{(4)}(x)|\,dx < +\infty, \tag{8}$$

then (1) *implies $\mu_X \sim \mu_{X+Y}$.*

Proof. (C) implies the integrability of f'' as

$$\left\{\int_{-\infty}^{+\infty} |f''(x)|\,dx\right\}^2 \leqq \int_{-\infty}^{+\infty} \frac{f''(x)^2}{f(x)}\,dx \int_{-\infty}^{+\infty} f(y)\,dy < +\infty,$$

and by Theorem 2 of E.F. Beckenbach and R. Bellman [1], Chap. 5, Sect. 3, f' is also integrable and we have

$$f'(+\infty) = \int_{-\infty}^{+\infty} f''(x)\,dx = 0.$$

On the other hand, by Kolmogorov's three series theorem, (1) implies (3) for $\varepsilon = 1$ and

$$\sum_k \mathbb{E}[Y_k^4 : |Y_j| \leqq 1] < +\infty. \tag{9}$$

Then the almost sure convergence of $\sum_k V_1(X_k)$ is derived from (3) by Lemma 1.

On the other hand, by Taylor expansion, for every k we have

$$\begin{aligned} W_1(X_k) = \frac{1}{f(X_k)} \mathbb{E}^Y [& f'(X_k)\,Y_k + \tfrac{1}{2} f''(X_k)\,Y_k^2 + \tfrac{1}{6} f^{(3)}(X_k)\,Y_k^3 \\ & + \tfrac{1}{6} \int_0^1 (1-s)^3 f^{(4)}(X_k + s\,Y_k)\,ds\,Y_k^4 : |Y_k| \leqq 1] \end{aligned} \tag{10}$$

and by the symmetricity of the distributions of Y_k's we have

$$\begin{aligned} W_1(X_k) &= \tfrac{1}{2} \mathbb{E}^Y [Y_k^2 : |Y_k| \leqq 1] \frac{f''(X_k)}{f(X_k)} \\ &\quad + \tfrac{1}{6} \int_0^1 (1-s)^3\,ds\,\mathbb{E}^Y \left[\frac{f^{(4)}(X_k + s\,Y_k)}{f(X_k)}\,Y_k^4 : |Y_k| \leqq 1\right] \\ &= \tfrac{1}{2} Q(X_k) + \tfrac{1}{6} R(X_k). \end{aligned}$$

Since we have for every k

$$\mathbb{E}[Q(X_k)] = \mathbb{E}^Y [Y_k^2 : |Y_k| \leqq 1] \int_{-\infty}^{+\infty} f''(x)\,dx = 0,$$

and $Q(X_k)$'s are independent, in order to show the almost sure convergence of $\sum_k Q(X_k)$, it is enough to show $\sum_k \mathbb{E}[Q(X_k)^2] < +\infty$. In fact (9) implies

$$\begin{aligned}\sum_k \mathbb{E}[Q(X_k)^2] &\leqq A^2 \sum_k \mathbb{E}^Y[Y_k^2 : |Y_k| \leqq 1]^2 \\ &\leqq A^2 \sum_k \mathbb{E}^Y[Y_k^4 : |Y_k| \leqq 1] < +\infty,\end{aligned}$$

where $A = \int_{-\infty}^{+\infty} |f''(x)|\, dx$.

Finally by Fubini's theorem we have

$$\begin{aligned}&\mathbb{E}[\sum_k |R(X_k)|] \\ &\leqq \sum_k \int_{-\infty}^{+\infty} (1-s)^3\, ds\, \mathbb{E}^Y\left[\mathbb{E}^X\left[\frac{f^{(4)}(X_k + sY_k)}{f(X_k)}\right] Y_k^4 : |Y_k| \leqq 1\right] \\ &\leqq B \sum_k \mathbb{E}^Y[Y_k^4 : |Y_k| \leqq 1] < +\infty,\end{aligned}$$

where $B = \int_{-\infty}^{+\infty} |f^{(4)}(x)|\, dx$.

Therefore $\sum_k W_1(X_k) = \frac{1}{2}\sum_k Q(X_k) + \frac{1}{6}\sum_k R(X_k)$ converges almost surely so that (S) of Theorem 2 is verified. Q.E.D.

Remark. In the above theorem we did not assume the Shepp's condition (A) but (C). We shall show by the following example that (C) can not be replaced by (A).

Let $f_0(x)$ be a C^4-function on the real line such that

$$f_0(x) = \begin{cases} \sin^2 x, & |x| \leqq \frac{\pi}{2}, \\ \exp\left[-\frac{x^2}{2}\right], & |x| \geqq 2, \\ \text{positive}, & \text{otherwise}, \end{cases}$$

and define $f(x) = cf_0(x)$ where $c = [\int f_0(x)\, dx]^{-1}$. Then f satisfies (A) but not (C). Let $X = \{X_k\}$ be an I.I.D. sequence with the density function f and $Y = \{Y_k\}$ be an independent sequence such that $\mathbb{P}(Y_k = a_k) = \mathbb{P}(Y_k = -a_k) = \frac{1}{2}$ where $\{a_k\}$ is a sequence of positive numbers such that $\sup_k a_k < \frac{1}{2}$ and $\sum_k a_k^4 < +\infty$. Then we have $\sum_k |Y_k|^4 < +\infty$, a.s.

Assume that X and Y are independent and that $\mu_{X+Y} \sim \mu_X$. Then by Theorem 2,

$$\sum_k Z(X_k = \sum_k \left\{ \frac{\mathbb{E}^Y[f(X_k+Y_k)]}{f(X_k)} - 1 \right\}$$
$$= \sum_k \left\{ \frac{\frac{1}{2}[f(X_k+a_k)+f(X_k-a_k)]}{f(X_k)} - 1 \right\}$$

converges almost surely. Since $\{Z(X_k)\}$ is an independent sequence, by the Kolmogorov's three series theorem, we have

$$+\infty > \sum_k \mathbb{P}(|Z(X_k)|>1)$$
$$= \sum_k \mathbb{P}(Z(X_k)>1)$$
$$\geqq \sum_k \mathbb{P}(Z(X_k)>1, |X_k| \leqq 1)$$
$$= \sum_k \mathbb{P}\left(\sin^2 X_k < \frac{\sin^2 a_k}{1+2\sin^2 a_k}, |X_k| \leqq 1 \right)$$
$$\geqq \sum_k \mathbb{P}(|X_k| < \kappa a_k, |X_k| \leqq 1)$$

for some positive constant less than 1 and

$$= \sum_k \int_{-\kappa a_k}^{\kappa a_k} \sin^2 x \, dx \geqq \text{const.} \sum_k a_k^3.$$

This is impossible if $\sum_k a_k^3 = +\infty$.

We give another sufficient condition for $\mu_{X+Y} \sim \mu_X$ as follows.

Theorem 4. *If f satisfies* (C) *and moreover there exists $\varepsilon > 0$ such that*

$$\int_{-\infty}^{+\infty} \sup_{|z|<\varepsilon} \frac{f''(x+z)^2}{f(x)} dx < +\infty, \tag{11}$$

then (2) *and* (3) *for this ε are sufficient for $\mu_X \sim \mu_{X+Y}$.*

Proof. We shall show (S) of Theorem 2.

The almost sure convergence of $\sum_k V_\varepsilon(X_k)$ is derived from (3) and Lemma 1.

In order to show the almost sure convergence of $\sum_k W_\varepsilon(X_k)$, since $\mathbb{E}[W_\varepsilon(X_k)] = 0$ for every k and $W_\varepsilon(X_k)$'s are independent, it is sufficient to show

$\sum_k \mathbb{E}[W_\varepsilon(X_k)^2] < +\infty$. In fact, by the symmetry of distributions of Y_k's, for every k we have

$$W_\varepsilon(X_k) = \tfrac{1}{2}\mathbb{E}^Y[f(X_k - |Y_k|) + f(X_k + |Y_k|) - 2f(X_k) : |Y_k| \leqq \varepsilon]/f(X_k),$$

and by Taylor's expansion

$$\begin{aligned}
&\mathbb{E}[W_\varepsilon(X_k)^2] \\
&= \tfrac{1}{4}\int_{-\infty}^{+\infty} \mathbb{E}^Y[f(x - |Y_k|) + f(x + |Y_k|) - 2f(x) : |Y_k| \leqq \varepsilon]^2/f(x)\,dx \\
&= \tfrac{1}{4}\int_{-\infty}^{+\infty} \mathbb{E}^Y\left[\tfrac{1}{2}\int_0^1 s\,ds \int_{-1}^1 f''(x - ts|Y_k|)\,dt\,|Y_k|^2 : |Y_k| \leqq \varepsilon\right]^2 \Big/ f(x)\,dx \\
&\leqq \tfrac{1}{8}\int_{-\infty}^{+\infty} \sup_{|z| \leqq \varepsilon} \frac{f''(x+z)^2}{f(x)}\,dx\,\mathbb{E}^Y[Y_k^2 : |Y_k| \leqq \varepsilon]^2.
\end{aligned}$$

Therefore (2) implies $\sum_k \mathbb{E}[Z_2(X_k)^2] < +\infty$ and the theorem is proved. Q.E.D.

Corollary. *If f satisfies the hypothesis of Theorem 4, then* (1) *implies* $\mu_X \sim \mu_{X+Y}$.

Proof. It is easy to show that (1) implies (2) and (3) for every $\varepsilon > 0$, and Theorem 4 proves the corollary. Q.E.D.

Conversely we give a necessary condition for $\mu_X \sim \mu_{X+Y}$.

Theorem 5. *If f is twice continuously differentiable and* (4) *is satisfied, then* $\mu_X \sim \mu_{X+Y}$ *implies* (2) *for some* $\varepsilon > 0$.

Proof. Since f'' is continuous, $f(x) > 0$, a.e. (dx), and f is integrable, there exists $\gamma > 0$ such that $\{x; f''(x) < -\gamma\}$ includes an interval. Define for every $h > 0$

$$\Gamma(h) = \left\{x; \frac{1}{h^2}[f(x+h) + f(x-h) - 2f(x)] < -\gamma\right\}.$$

Then, from the continuity of f'', there exist $\varepsilon > 0$, and a compact interval $\mathbb{I}$ such that $\mathbb{I} \subset \Gamma(h)$ for every $0 \leqq h \leqq \varepsilon$. Define for every k

$$\mathbb{G}_k = \{x; \mathbb{E}^Y[f(x + |Y_k|) + f(x - |Y_k|) : |Y_k| \leqq \varepsilon] \leqq 2f(x)\}.$$

Then for every $x \in \mathbb{I}$ and every $0 \leqq h \leqq \varepsilon$ we have $[f(x+h) + f(x-h) - 2f(x)] < -\gamma h^2$ so that $\mathbb{E}^Y[f(x + |Y_k|) + f(x - |Y_k|) : |Y_k| \leqq \varepsilon] < 2f(x)$. Therefore we have $x \in \mathbb{G}_k$ for every k and $\mathbb{I} \subset \bigcap_k \mathbb{G}_k$.

Let $Z(X_k) = V_\varepsilon(X_k) + W_\varepsilon(X_k)$ be the decomposition in (7). Then (4) implies (3) for the ε and by Lemma 1, $\sum_k V_\varepsilon(X_k)$ converges almost surely. On the other hand by Theorem 2, $\mu_X \sim \mu_{X+Y}$ implies the almost sure convergence of $\sum_k Z(X_k)$ so that $\sum_k W_\varepsilon(X_k) = \sum_k Z(X_k) - \sum_k V_\varepsilon(X_k)$ converges almost surely.

Since $\mathbb{E}[W_\varepsilon(X_k)] = 0$ and $W_\varepsilon(X_k) > -1$, a.s., Theorem 2, (K) is applicable to $\{W_\varepsilon(X_k)\}$ and by $\{|W_\varepsilon(X_k)| \leqq 1\} = \{W_\varepsilon(X_k) \in \mathbb{G}_k\}$ for every k, by Kolmogorov's three series theorem we have

$$
\begin{aligned}
+\infty &> \sum_k \mathbb{E}[W_\varepsilon(X_k)^2 : |W_\varepsilon(X_k)| \leqq 1] \\
&= \sum_k \int_{\mathbb{G}_k} \mathbb{E}^Y[f(x+|Y_k|) + f(x-|Y_k|) - 2f(x) : |Y_k| \leqq \varepsilon]^2 \frac{dx}{f(x)} \\
&\geqq \sum_k \int_{\mathbb{I}} \mathbb{E}^Y\left[\frac{f(x+|Y_k|) + f(x-|Y_k|) - 2f(x)}{|Y_k|^2} |Y_k|^2 : |Y_k| \leqq \varepsilon\right]^2 \frac{dx}{f(x)} \\
&\geqq \frac{1}{\max_{x \in \mathbb{I}} f(x)} \gamma^2 \lambda(\mathbb{I}) \sum_k \mathbb{E}^Y[|Y_k|^2 : |Y_k| \leqq \varepsilon]^2,
\end{aligned}
$$

where λ is the Lebesgue measure, and this proves the theorem. Q.E.D.

Combining Theorems 4 and 5, we have the following theorem.

Theorem 6. *Assume that f is twice continuously differentiable, satisfies* (C) *and* (11) *for some $\varepsilon > 0$, and assume* (4) *for $Y = \{Y_k\}$. Then* (2) *for the ε is a necessary and sufficient condition for $\mu_X \sim \mu_{X+Y}$.*

3. Gaussian Case

Let $X = \{X_k\}$ be a standard Gaussian sequence, that is, an I.I.D. random sequence with density function $g(x) = \sqrt{2\pi}^{-1} \exp\left[-\frac{x^2}{2}\right]$, $-\infty < x < +\infty$, and $Y = \{Y_k\}$ be an independent (not necessarily symmetric) random sequence which is also independent of X. Then g satisfies (C), (8) and (11) so that all the above results are applicable. Furthermore, since we know the explicit form of the density function g, we have the following theorems.

Theorem 7. *Let $X = \{X_k\}$ be a standard Gaussian sequence and $Y = \{Y_k\}$ be an independent (not necessary symmetric) random sequence which is also independent of X, and satisfies*

$$\sum_k \mathbb{E}^Y[Y_k : |Y_k| \leqq \varepsilon]^2 < +\infty, \tag{13}$$

for some $\varepsilon > 0$. Then (1) *implies $\mu_X \sim \mu_{X+Y}$.*

Proof. Since g satisfies (8), the arguments in the proof of Theorem 3 are applicable and we have only to show the almost sure convergence of

$$\sum_k \frac{1}{g(X_k)} \mathbb{E}^Y[g'(X_k) Y_k : |Y_k| \leqq \varepsilon],$$

and

$$\sum_k \frac{1}{g(X_k)} \mathbb{E}^Y[g^{(3)}(X_k) Y_k^3 : |Y_k| \leqq \varepsilon],$$

in the Taylor expansion (10) where "$|Y_k| \leqq 1$" is replaced by "$|Y_k| \leqq \varepsilon$". In fact for every k we have $g'(X_k)/g(X_k) = -X_k$ and $g^{(3)}(X_k)/g(X_k) = 3X_k - X_k^3$, which form mean zero, square integrable independent sequences. Then it is enough to show the convergences of the sums of the square of coefficients of them. Since (1) implies (9), where "$|Y_k| \leqq 1$" is replaced by "$|Y_k| \leqq \varepsilon$", we have by Hölder's inequality

$$\begin{aligned}\sum_k \mathbb{E}^Y[Y_k^3 : |Y_k| \leqq \varepsilon]^2 &\leqq \sum_k \mathbb{E}^Y[Y_k^4 : |Y_k| \leqq \varepsilon]^{\frac{3}{2}} \\ &\leqq \{\sum_k \mathbb{E}^Y[Y_k^4 : |Y_k| \leqq \varepsilon]\}^{\frac{2}{3}} < +\infty,\end{aligned}$$

and (9) and (13) proves the theorem. Q.E.D.

In the same idea with that of Theorem 7, we can prove:

Theorem 8. *Let $X = \{X_k\}$ be a standard Gaussian sequence and $Y = \{Y_k\}$ be an independent (not necessary symmetric) random sequence which is also independent of X, and satisfies*

$$\sum_k \mathbb{E}^Y[Y_k \exp(-\tfrac{1}{2} Y_k^2)]^2 < +\infty. \tag{14}$$

Then (1) *implies $\mu_X \sim \mu_{X+Y}$.*

Proof. The Taylor expansion

$$\begin{aligned}Z(X_k) &= \mathbb{E}^Y\left[\exp(-\tfrac{1}{2} Y_k^2)\left\{\exp(X_k Y_k) - \exp\frac{Y_k^2}{2}\right\} : |Y_k| \leqq 1\right] \\ &= \mathbb{E}^Y\Bigg[\exp(-\tfrac{1}{2} Y_k^2)\Bigg\{X_k Y_k + \frac{Y_k^2}{2}(X_k^2 - 1) + \tfrac{1}{6}(X_k Y_k)^3 \\ &\quad + \int_0^1 \frac{(1-t)^3}{6} \exp(t X_k Y_k)\, dt (X_k Y_k)^4 \\ &\quad - \int_0^1 (1-t) \exp\left(t \frac{Y_k^2}{2}\right) dt \frac{(Y_k)^2}{2} : |Y_k| \leqq 1\Bigg]\end{aligned}$$

and the similar estimations in the proofs of Theorems 3 and 7 prove the theorem. Q.E.D.

Theorem 9. *Let $X=\{X_k\}$ be a standard Gaussian sequence and $Y=\{Y_k\}$ be an independent symmetric random sequence which is also independent of X. Then* (2) *and* (3) *for some $\varepsilon>0$ imply $\mu_X \sim \mu_{X+Y}$. Conversely $\mu_X \sim \mu_{X+Y}$ implies* (2) *and* (5) *for every $\varepsilon>0$.*

Proof. Since the Gaussian density function g satisfies (C) and (11) for every $\varepsilon>0$, the first part of the theorem is derived immediately from Theorem 4.

We shall prove the remaining part of the theorem. In fact by the symmetry of the distributions of Y_k's, for every k we have

$$Z(X_k)=\mathbb{E}^Y[\exp(-\tfrac{1}{2}Y_k^2)\cosh(X_k|Y_k|)]-1.$$

Assume $\mu_X \sim \mu_{X+Y}$. Then by Theorem 2 two series $(K-1)$ and $(K-2)$ converge.
For every k, since the function

$$\psi_k(u)=\mathbb{E}^Y[\exp(-\tfrac{1}{2}Y_k^2)\cosh(u|Y_k|)] \tag{15}$$

is continuous, strictly increasing for $0\leqq u<+\infty$, $0\leqq\psi_k(1)\leqq\sqrt{e}<2$ and $\lim_{u\to+\infty}\psi_k(u)=+\infty$, there exists unique $\alpha_k>1$ such that $\psi_k(\alpha_k)=2$.
On the other hand, since we have for every $0\leqq u\leqq\frac{1}{2}$ and $0\leqq t<+\infty$

$$1-\exp(-\tfrac{1}{2}t^2)\cosh\tfrac{1}{2}u\geqq\tfrac{1}{2}\{1-\exp(-\tfrac{1}{2}t^2)\}\geqq 0,$$

$(K-2)$ implies

$$\begin{aligned}+\infty&>\sum_k \mathbb{E}[Z(X_k)^2:|Z(X_k)|\leqq 1]\\&=\sum_k \mathbb{E}[\mathbb{E}^Y[\exp(-\tfrac{1}{2}Y_k^2)\cosh(X_k|Y_k|)-1]^2:|X_k|\leqq\alpha_k],\end{aligned}$$

and since $\alpha_k>1$,

$$\begin{aligned}&\geqq 2\sum_k\int_0^{\frac{1}{2}}\mathbb{E}^Y[\exp(-\tfrac{1}{2}Y_k^2)\cosh(t|Y_k|)-1]^2\,g(t)\,dt\\&\geqq\tfrac{1}{2}\sum_k\int_0^{\frac{1}{2}}\mathbb{E}^Y[1-\exp(-\tfrac{1}{2}Y_k^2)]^2\,g(t)\,dt\\&=C\sum_k\mathbb{E}^Y[1-\exp(-\tfrac{1}{2}Y_k^2)]^2,\end{aligned}$$

where C is a constant independent of Y and k. Therefore we have

$$\sum_k\mathbb{E}^Y[1-\exp(-\tfrac{1}{2}Y_k^2)]^2<+\infty \tag{16}$$

and it is easy to show that this implies (2) and (5) for every $\varepsilon>0$. Q.E.D.

In the above theorem neither (2) and (3) are necessary nor (2) and (5) are sufficient for $\mu_X \sim \mu_{X+Y}$. Furthermore it is interesting that (4) is not necessary. We shall give illuminating examples as follows.

Let $X=\{X_k\}$ be a standard Gaussian sequence and $Y=\{Y_k\}$ be an independent symmetric random sequence, which is also independent of X, with the distribution

$$\mathbb{P}(Y_k=a_k)=\mathbb{P}(Y_k=-a_k)=\tfrac{1}{2}p_k,$$
$$\mathbb{P}(Y_k=0)=1-p_k, \tag{17}$$

for every k, where $\{a_k\}$ and $\{p_k\}$ are sequences of positive numbers such that $\lim_k a_k=+\infty$ and

$$\sum_k p_k=+\infty, \quad \text{and} \quad \sum_k p_k^2<+\infty. \tag{18}$$

Then Y satisfies (2) and (5) but not (3) and (4), and we have

$$\begin{aligned}\psi_k(u)&=\mathbb{E}^Y[\exp(-\tfrac{1}{2}Y_k^2)\cosh(u|Y_k|)]\\&=p_k\exp(-\tfrac{1}{2}a_k^2)\cosh(a_k u)+(1-p_k).\end{aligned}$$

The root α_k of the equation $\psi_k(u)=2$ is given by

$$\alpha_k=\frac{1}{a_k}\log\left\{\frac{(1+p_k)}{p_k}\exp(\tfrac{1}{2}a_k^2)+\left(\frac{(1+p_k)^2}{p_k^2}\exp(a_k^2)-1\right)^{\frac{1}{2}}\right\}.$$

Define

$$\begin{aligned}\gamma_k&=\frac{1}{a_k}\log\left\{2\frac{(1+p_k)}{p_k}\exp(\tfrac{1}{2}a_k^2)\right\}\\&=\frac{1}{a_k}\log\left\{2\frac{(1+p_k)}{p_k}\right\}+\tfrac{1}{2}a_k.\end{aligned}$$

Then simple estimations show $|\gamma_k-\alpha_k|\leqq p_k^2\exp(-a_k^2)$. By reformulating Theorem 2, it is not difficult to show the following lemma.

Lemma 2. *In the above situation,* $\mu_X\sim\mu_{X+Y}$ *if and only if the following two series are convergent.*

$$\sum_k p_k \int_{\gamma_k-a_k}^{\gamma_k}\exp(-\tfrac{1}{2}u^2)\,du<+\infty. \tag{G-1}$$

$$\sum_k p_k^2 \int_{-2a_k}^{\gamma_k-2a_k}\exp(a_k^2-\tfrac{1}{2}u^2)\,du<+\infty. \tag{G-2}$$

Proof. We have $Z(X_k)=p_k\{\exp(-\frac{1}{2}a_k^2)\cosh(a_k u)-1\}$ and, using $|\gamma_k-\alpha_k| \leqq p_k^2\exp(-a_k^2)$, (K-1) and (K-2) of Theorem 2 are reformulated as

$$\sum_k \int_{\gamma_k}^{+\infty} p_k\{\exp(-\tfrac{1}{2}a_k^2)\cosh(a_k u)-1\}\exp(-\tfrac{1}{2}u^2)\,du<+\infty, \tag{K'-1}$$

$$\sum_k \int_0^{\gamma_k} p_k^2\{\exp(-\tfrac{1}{2}a_k^2)\cosh(a_k u)-1\}^2\exp(-\tfrac{1}{2}u^2)\,du<+\infty. \tag{K'-2}$$

Then we have

$$\begin{aligned}
&\int_{\gamma_k}^{+\infty}\{\exp(-\tfrac{1}{2}a_k^2)\cosh(a_k u)-1\}\exp(-\tfrac{1}{2}u^2)\,du\\
&=\tfrac{1}{2}\left(\int_{\gamma_k-a_k}^{+\infty}+\int_{\gamma_k+a_k}^{+\infty}-2\int_{\gamma_k}^{+\infty}\right)\exp(-\tfrac{1}{2}u^2)\,du\\
&=\tfrac{1}{2}\left\{\int_{\gamma_k-a_k}^{\gamma_k}-\int_{\gamma_k}^{\gamma_k+a_k}\right\}\exp(-\tfrac{1}{2}u^2)\,du.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\int_0^{\gamma_k}\{\exp(-\tfrac{1}{2}a_k^2)\cosh(a_k u)-1\}^2\exp(-\tfrac{1}{2}u^2)\,du\\
&=\tfrac{1}{4}\exp(a_k^2)\left\{\int_{-2a_k}^{\gamma_k-2a_k}+\int_{2a_k}^{\gamma_k+2a_k}\right\}\exp(-\tfrac{1}{2}u^2)\,du\\
&\quad+\left\{\int_{-a_k}^{\gamma_k-a_k}+\int_{a_k}^{\gamma_k+a_k}\right\}\exp(-\tfrac{1}{2}u^2)\,du+\{1+\exp(-a_k^2)\}\int_0^{\gamma_k}\exp(-\tfrac{1}{2}u^2)\,du.
\end{aligned}$$

Using these estimations, it is not difficult to show that (K'-1) and (K'-2) are equivalent to (G-1) and (G-2) under the condition (18). Q.E.D.

Now we shall show the examples.

Example 1. In (17) define $a_k=\sqrt{\log k}$ and $p_k=(\frac{1}{2}k-1)^{-1}$ for $k\geqq 5$. Then we have $\gamma_k=\frac{3}{2}\sqrt{\log k}$,

$$\sum_k p_k\int_{\gamma_k-a_k}^{\gamma_k}\exp(-\tfrac{1}{2}u^2)\,du\leqq 6\sum_k k^{-\frac{9}{8}}\sqrt{\log k}<+\infty,$$

and

$$\sum_k p_k^2\int_{-2a_k}^{\gamma_k-2a_k}\exp(a_k^2-\tfrac{1}{2}u^2)\,du\leqq 54\sum_k k^{-\frac{9}{8}}\sqrt{\log k}<+\infty.$$

Therefore by Lemma 2 we have $\mu_X\sim\mu_{X+Y}$ but Y does not satisfy (3).

Example 2. In (17) define $a_k=\sqrt{\log k}$ and $p_k=(\frac{1}{2}k^{\frac{1}{2}+\delta}-1)^{-1}$ for some fixed $\delta>0$ where $k\geqq 16$. Then we have $\gamma_k=(1+\delta)\sqrt{\log k}$, and using the inequality

$$\int_t^{t+\frac{1}{t}} \exp(-\tfrac{1}{2}u^2)\,du \geqq \frac{1}{2t}\exp(-\tfrac{1}{2}t^2)$$

for $t\geqq 1$, we have

$$\sum_k p_k \int_{\gamma_k-a_k}^{\gamma_k} \exp(-\tfrac{1}{2}u^2)\,du \geqq \sum_k k^{-(\frac{1}{2}+\delta)}\frac{1}{(1+\delta)\sqrt{\log k}}\exp(-\tfrac{1}{2}\delta^2\log k)$$

$$=\sum_k k^{-\frac{1}{2}(1+\delta)^2}\frac{1}{2\sqrt{\log k}}=+\infty,$$

for $\delta\leqq\sqrt{2}-1$. Therefore Y satisfies (2) and (5) but by Lemma 2 μ_X and μ_{X+Y} are singular.

4. Applications to Stochastic Processes

Let $\mathbb{X}=\{X(t)\}_{t\in T}$ and $\mathbb{Y}=\{Y(t)\}_{t\in T}$ be mutually independent stochastic processes, and $P_{\mathbb{X}}$ and $P_{\mathbb{X}+\mathbb{Y}}$ be probability measures on the function space induced by $\mathbb{X}$ and $\mathbb{X}+\mathbb{Y}=\{X(t)+Y(t)\}_{t\in T}$, respectively. In the information theory they treat the absolute continuity of $P_{\mathbb{X}+\mathbb{Y}}$ with respect to $P_{\mathbb{X}}$ (denoted by $P_{\mathbb{X}+\mathbb{Y}}\ll P_{\mathbb{X}}$) and the estimation of the entropy

$$H(P_{\mathbb{X}+\mathbb{Y}}\,|\,P_{\mathbb{X}})=\mathbb{E}\left[\frac{dP_{\mathbb{X}+\mathbb{Y}}}{dP_{\mathbb{X}}}\log\frac{dP_{\mathbb{X}+\mathbb{Y}}}{dP_{\mathbb{X}}}\right],$$

where $\mathbb{E}[\;]$ is the expectation with respect to $P_{\mathbb{X}}$ (S. Ihara [3]). However, unless both of $\mathbb{X}$ and $\mathbb{Y}$ are Gaussian, there are few tools in analysing them.

On the other hand, let $\mathscr{X}$ and $\mathscr{Y}$ be Polish spaces (i.e. complete separable metric spaces), μ and ν be probability measures on $\mathscr{X}$, and φ be an injective map of $\mathscr{X}$ into $\mathscr{Y}$ defined μ-almost surely. Then $\nu\ll\mu$ implies $\varphi(\nu)\ll\varphi(\mu)$ and the Radon-Nikodym derivative is given by $\dfrac{d\varphi(\nu)}{d\varphi(\mu)}=\dfrac{d\nu}{d\mu}\cdot\varphi^{-1}$, $\varphi(\mu)$-a.s. Conversely, let P be a probability measure on $\mathscr{Y}$ such that $P\ll\varphi(\mu)$. Then we have $\varphi^{-1}(P)\ll\mu$ and $\dfrac{d\varphi^{-1}(P)}{d\mu}=\dfrac{dP}{d\varphi(\mu)}\circ\varphi$, μ-a.s.

The typical examples of the above scheme are stochastic processes given by random Fourier series. In particular we define a class of stochastic processes which have similar properties with Gaussian processes (H. Sato [6]) as follows.

Let $\mathscr{W}$ be the Banach space of all continuous functions on [0, 1] which vanish at 0, P a probability measure on $\mathscr{W}$. We call a stochastic process $\mathbb{X}=\{X(t)\}_{t\in[0,1]}$ an *$\mathscr{E}$-process* if there exist a sequence $\{x_k\}$ in $\mathscr{W}$ and a sequence

$\{\xi_k\}$ in $\mathscr{W}^*$, the topological dual space of $\mathscr{W}$, such that $\langle x_k, \xi_j\rangle = \delta_{kj}$, $k, j = 1, 2, 3, \ldots$, and for which

$$X(t) = \sum_k \langle x, \xi_k\rangle\, x_k(t), \qquad t \in [0, 1] \tag{19}$$

converges uniformly P-almost surely. Then the random sequence $X = \{\langle x, \xi_k\rangle\}$ on $(\mathscr{W}, P)$ induces a probability measure μ_X on the sequence space. On the other hand let Q be another probability measure on $\mathscr{W}$. Then the random sequence $\mathbb{Z} = \{\langle x, \xi_k\rangle\}$ on $(\mathscr{W}, Q)$ induces a probability measure $\mu_{\mathbb{Z}}$ and we have the following theorem.

Theorem 10. *In the above situation, we have $Q \ll P$ if and only if $\mu_{\mathbb{Z}} \ll \mu_X$ and the Radon-Nikodym derivative is given by*

$$\frac{dQ}{dP}(x) = \frac{d\mu_{\mathbb{Z}}}{d\mu_X}(\{\langle x, \xi_k\rangle\}), \qquad P\text{-a.s.} \tag{20}$$

The most illuminating example of the above theorem is the Brownian motion.

Example 3. Let $P_{\mathbb{B}}$ be the Wiener measure on $\mathscr{W}$ and define $\varphi_0(t) = 1$, $\varphi_k(t) = \sqrt{2}\cos \pi k t$, $k = 1, 2, 3, \ldots$, $x_k(t) = \int_0^t \varphi_k(s)\, ds$, $k = 0, 1, 2, \ldots$, and a sequence of Radon measure on $(0, 1]$ by $d\xi_0(t) = \delta_1$, $d\xi_k(t) = \sqrt{2}\pi k \sin \pi k t\, dt + \sqrt{2}(-1)^k \delta_1$, $k = 1, 2, 3, \ldots$, where δ_1 is the Dirac measure concentrated on $t = 1$. Then the Brownian motion $\mathbb{B} = \{B(t)\}_{t \in [0, 1]}$ is expanded in an almost surely uniformly convergent sequence

$$B(t) = \sum_{k=0}^{+\infty} \langle x, \xi_k\rangle\, x_k(t), \tag{21}$$

where $\langle x, \xi_k\rangle = \int_{(0,1]} x(s)\, d\xi_k(s)$ is also written as the stochastic integral $\int_0^1 \varphi_k(s)\, dB(s)$, $k = 0, 1, 2, \ldots$.

On the other hand, let $\mathbb{Y} = \{Y_k\}$ be a symmetric independent random sequence which is also independent of $\mathbb{B}$ and satisfy (2) and (3) for some $\varepsilon > 0$. Then by Theorem 7

$$Y(t) = \sum_{k=0}^{+\infty} Y_k\, x_k(t), \tag{22}$$

converges uniformly almost surely, $\mathbb{B} + \mathbb{Y} = \{B(t) + Y(t)\}_{t \in [0, 1]}$ induces a probability measure $P_{\mathbb{B}+\mathbb{Y}} \ll P_{\mathbb{B}}$, and the Radon-Nikodym derivative is given by

$$\frac{dP_{\mathbb{B}+\mathbb{Y}}}{dP_{\mathbb{B}}}(x) = \prod_{k=0}^{+\infty} \mathbb{E}^Y[\exp(-\tfrac{1}{2} Y_k^2 + Y_k\langle x, \xi_k\rangle)], \qquad P_{\mathbb{B}}\text{-a.s.}, \tag{23}$$

or equally

$$\frac{dP_{\mathbb{B}+\mathbb{Y}}}{dP_{\mathbb{B}}}=\prod_{k=0}^{+\infty}\mathbb{E}^{Y}\left[\exp\left(-\tfrac{1}{2}Y_k^2+Y_k\int_0^1\varphi_k(s)\,dB(s)\right)\right],\qquad P_{\mathbb{B}}\text{-a.s.}\tag{24}$$

In particular, let $\{a_k\}$ be a sequence of numbers such that $\sum_k a_k^4<+\infty$ and assume that $\mathbb{P}(Y_k=a_k)=\mathbb{P}(Y_k=-a_k)=\frac{1}{2}$ for every k. Then, using the inequality $\log(\cosh t)\leqq\frac{1}{2}t^2$ for $t\geqq 0$, we have

$$H(P_{\mathbb{B}+\mathbb{Y}}\,|\,P_{\mathbb{B}})\leqq\tfrac{1}{2}\sum_k a_k^4<+\infty.$$

We formulate the above example as a theorem.

Theorem 11. *Let $\mathbb{Y}=\{Y_k\}$ be an independent symmetric random sequence independent of a Brownian motion $\mathbb{B}=\{B(t)\}_{t\in[0,1]}$ and satisfy* (2) *and* (3) *for some $\varepsilon>0$. Then $Y(t)=\sum_{k=0}^{+\infty}Y_k x_k(t)$, $0\leqq t\leqq 1$, uniformly converges almost surely and $\mathbb{B}+\mathbb{Y}=\{B(t)+Y(t)\}_{t\in[0,1]}$ induces a probability measure $P_{\mathbb{B}+\mathbb{Y}}\ll P_{\mathbb{B}}$ on $\mathscr{W}$ with the Radon-Nikodym derivative* (24).

5. Does (C) Imply (A)?

The aim of this section is to study the relation between the condition (C) and (A) and prove the following theorem.

Theorem 12. *If f satisfies* (C) *and is monotone for large $|x|$, then f satisfies* (A).

Proof. Assume that f satisfies (C). Then, since f is continuously differentiable and monotone for large $|x|$, there are real numbers S, $T(-\infty<S\leqq T<+\infty)$ and sequences of disjoint open finite intervals $\{(a_n,b_n)\}$ and $\{(c_n,d_n)\}$ such that

$$\mathscr{F}_+=\{x: f(x)>0\}=\{\bigcup_n(a_n,b_n)\}\cup(-\infty,S),$$

$$\mathscr{F}_-=\{x: f(x)<0\}=\{\bigcup_n(c_n,d_n)\}\cup(T,+\infty),$$

and

$$f'(a_n)=f'(b_n)=f'(c_n)=f'(d_n)=f'(S)=f'(T)=0,$$

for every n.

On every (a_n,b_n), we have

$$\int_{a_n}^{b_n}\frac{f'(x)^2}{f(x)}dx=\int_{a_n}^{b_n}\frac{dx}{f(x)}\left|\int_{a_n}^{x}f''(y)\,dy\right|^2$$

$$\leqq\int_{a_n}^{b_n}\frac{dx}{f(x)}\int_{a_n}^{x}f(z)\,dz\int_{a_n}^{x}\frac{f''(y)^2}{f(y)}dy,$$

and, since $f=f(z)$ is monotone increasing in (a_n, b_n),

$$\leqq \int_{a_n}^{b_n} (x-a_n)\,dx \int_{a_n}^{b_n} \frac{f''(y)^2}{f(y)}\,dy$$

$$\leqq (b_n-a_n)^2 \int_{a_n}^{b_n} \frac{f''(x)^2}{f(x)}\,dx$$

$$\leqq (T-S)^2 \int_{a_n}^{b_n} \frac{f''(x)^2}{f(x)}\,dx.$$

In a similar manner on every (c_n, d_n) we have

$$\int_{c_n}^{d_n} \frac{f'(x)^2}{f(x)}\,dx = \int_{c_n}^{d_n} \frac{dx}{f(x)} \left| \int_x^{d_n} f''(y)\,dy \right|^2$$

$$\leqq \int_{c_n}^{d_n} \frac{dx}{f(x)} \int_x^{d_n} f(z)\,dz \int_x^{d_n} \frac{f''(y)^2}{f(y)}\,dy,$$

and, since $f=f(z)$ is monotone decreasing in (c_n, d_n),

$$\leqq (T-S)^2 \int_{c_n}^{d_n} \frac{f''(x)^2}{f(x)}\,dx.$$

On the other hand, define $h=f''-f$. Then as a solution of the differential equation with the boundary condition $f(\pm\infty)=0$, f is expressed in the form

$$f(x) = -\tfrac{1}{2}\left\{e^{-x} \int_{-\infty}^{x} e^y h(y)\,dy + e^x \int_x^{+\infty} e^{-y} h(y)\,dy\right\}, \qquad -\infty < x < +\infty.$$

Evidently we have $\int \frac{h^2}{f} < +\infty$, and $\int \frac{f'^2}{f} < +\infty$ if either $\int \frac{(f+f')^2}{f} < +\infty$ or $\int \frac{(f-f')^2}{f} < +\infty$.

On $(-\infty, S)$, $f=f(y)$ is monotone increasing and we have

$$\int_{-\infty}^{S} \frac{(f(x)-f'(x))^2}{f(x)}\,dx = \int_{-\infty}^{S} \frac{dx}{f(x)} \left| e^{-x} \int_{-\infty}^{x} e^y h(y)\,dy \right|^2$$

$$\leqq \int_{-\infty}^{S} \frac{dx}{f(x)} \int_{-\infty}^{x} e^{z-x} f(z)\,dz \int_{-\infty}^{x} e^{y-x} \frac{h(y)^2}{f(y)}\,dy$$

$$\leqq \int_{-\infty}^{S} e^{-x} \left\{ \int_{-\infty}^{x} e^y \frac{h(y)^2}{f(y)}\,dy \right\} dx,$$

and by partial integration

$$= \int_{-\infty}^{S} \left\{ -e^{x-S} \frac{h(x)^2}{f(x)} + \frac{h(x)^2}{f(x)} \right\} dx$$

$$\leqq \int_{-\infty}^{S} \frac{h(x)^2}{f(x)} dx < +\infty .$$

Similarly on $(T, +\infty)$, $f = f(x)$ is monotone decreasing and we have

$$\int_T^{+\infty} \frac{(f(x)+f'(x))^2}{f(x)} dx = \int_T^{+\infty} \frac{dx}{f(x)} \left| e^x \int_x^{+\infty} e^{-y} h(y)\, dy \right|^2$$

$$\leqq \int_T^{+\infty} \frac{h(x)^2}{f(x)} dx < +\infty .$$

Summing up the above estimations we have

$$\int_{-\infty}^{+\infty} \frac{f'(x)^2}{f(x)} dx = \int_{\mathscr{F}_+} \frac{f'(x)^2}{f(x)} dx + \int_{\mathscr{F}_-} \frac{f'(x)^2}{f(x)} dx$$

$$= \sum_n \int_{a_n}^{b_n} \frac{f'(x)^2}{f(x)} dx + \int_{-\infty}^{S} \frac{f'(x)^2}{f(x)} dx$$

$$+ \sum_n \int_{c_n}^{d_n} \frac{f'(x)^2}{f(x)} dx + \int_T^{+\infty} \frac{f'(x)^2}{f(x)} dx$$

$$= (T-S)^2 \sum_n \int_{a_n}^{b_n} \frac{f''(x)^2}{f(x)} dx + \int_{-\infty}^{S} \frac{f''(x)^2}{f(x)} dx$$

$$+ (T-S)^2 \sum_n \int_{c_n}^{d_n} \frac{f''(x)^2}{f(x)} dx + \int_T^{+\infty} \frac{f''(x)^2}{f(x)} dx$$

$$\leqq [1+(T-S)^2] \left\{ \int_{\mathscr{F}_+} \frac{f''(x)^2}{f(x)} dx + \int_{\mathscr{F}_-} \frac{f''(x)^2}{f(x)} dx \right\}$$

$$\leqq [1+(T-S)^2] \int_{-\infty}^{+\infty} \frac{f''(x)^2}{f(x)} dx < +\infty . \quad \text{Q.E.D.}$$

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