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Probability Theory and Related Fields

Almost Surely Convergent Random Variables with Given Laws

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Summary. Let $\mu_n \to \mu$ be a weakly converging sequence of Borel probability measures on a topological space X. We prove the existence of an almost surely converging sequence of random variables $\xi_n \to \xi$ which obey this laws, if a certain μ-dependent countability property of the topology holds. Especially this is the case if

(a) X is second countable,

(b) X is first countable and μ has countable support,

(c) X is metrizable and μ is τ-smooth.

A final example disproves the existence of such random variables for (tight) measures on a Lusin space.

0. Introduction

One of the classical marginal problems (see [7]) of probability theory can be stated as follows. Let X be a topological space, let $P_\sigma(X)$ be the space of Borel probability measures endowed with the weak topology (see [14]) and $\mu_n \to \mu$ a converging sequence in $P_\sigma(X)$. Do there exist a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ and random variables ξ_n, ξ such that $\mathbb{P} \circ \xi_n^{-1} = \mu_n$, $\mathbb{P} \circ \xi^{-1} = \mu$ and $\xi_n \to \xi$ almost surely? It is well known, that this is the case if X is metrizable and μ is τ-smooth or, equivalently, has separable support (see [2], [4], [12], [15]).

We extend this result in the following way. Looking at the proof of Dudley in [2], one observes easily that the problem has a simple solution in case of Dirac measures μ on first countable spaces. The reduction of the general problem to this case is obvious if there are kernels T_n: $X \to P_\sigma(X)$ such that

$$\int T_n(x, .)\, \mu(dx) = \mu_n$$

and

$$T_n(x, .) \to \varepsilon_x$$

for μ-almost all $x \in X$. The existence of these kernels under weak countability conditions on the topology of X will be proved by a previous result of the author which establishes the openness of convex addition in the cone $P_\sigma(X)$ (see [10]).

In [5] Fernique proved a global realization theorem: Given a Polish space X there exists a random variable $X(\nu)$ for each Borel probability ν on X such that for each converging sequence $\mu_n \to \mu$ the random variables $X(\mu_n)$ converge to $X(\mu)$ on a set of probability 1. Though this theorem may be straightforeward extended to the case of τ-smooth measures on wider classes of metrizable spaces (for instance those which are Borel subsets of a completition), it does not hold in the non-metrizable setting of this paper: Consider a set $C \subset [0, 1]$ of outer Lebesgue measure 1 and inner Lebesgue measure 0. Let $X = [0, 1]$ be endowed with the topology generated by the Euclidean open sets and C. In this case, open sets in X are of type $G_1 \cup (G_2 \cap C)$ for Euclidean open sets G_i and Borel sets of type $(B_1 \setminus C) \cup (B_2 \cap C)$ for Euclidean Borel sets B_i. Define probabilities on X by

$$\nu_i((B_1 \setminus C) \cup (B_2 \cap C)) = \lambda(B_i),$$

where λ denotes the Lebesgue measure (cf. [6], p. 71 (2)). Observe that for G_i as above

$$\nu_1(G_1 \cup (G_2 \cap C)) = \lambda(G_1) \leqq \lambda(G_1 \cup G_2) = \nu_2(G_1 \cup (G_2 \cap C)),$$

which implies that the constant sequence $\mu_n = \nu_2$ converges to ν_1. Since X is second countable, Theorem (2.6) yields a realizing sequence of random variables $\xi_n \to \xi$. However, the random variables ξ_n may not be chosen identically since this would imply $\xi_n = \xi$ almost surely and so $\nu_1 = \nu_2$. In other words, we have constructed a space X without global realization where each converging sequence may be realized.

Applications of the realization theorem may be found in [1], [3] or [8].

1. Notations and Preliminary Results

Given a topological space X, denote by

$\mathscr{G}(X)$ the class of open sets in X,

$\mathscr{B}(X)$ the class of Borel sets in X,

$P_\sigma(X)$ the set of Borel probability measures on X.

Endow $P_\sigma(X)$ with the weak or narrow topology, i.e. the weakest topology such that the mapping

$$\mu \mapsto \mu(G)$$

is lower semi-continuous for each $G \in \mathscr{G}(X)$ (see e.g. [11], [14]). The Dirac measure in a point $x \in X$ will be abbreviated by ε_x. For the definition of special properties of Borel measures we refer to [14]. Finally we call a map $T: X \to P_\sigma(X)$ a kernel, if for each $B \in \mathscr{B}(X)$ the function

$$x \mapsto T(x, B)$$

is Borel measurable.

Next we cite a result, which is the crucial basis of this paper.

(1.1) **Theorem.** *Let* $\lambda \in [0, 1]$. *Then the mapping*

$$P_\sigma(X) \times P_\sigma(X) \ni (\mu, \nu) \mapsto \lambda \mu + (1-\lambda) \nu \in P_\sigma(X)$$

is open.

Proof. See [10]. □

The following corollary will be the main tool to get the above mentioned decomposition lemma.

(1.2) **Corollary.** *Let* $\lambda \in [0, 1]$. *Let* $\mu, \nu, \rho \in P_\sigma(X)$ *such that* $\lambda \mu + (1-\lambda) \nu = \rho$. *If* ρ *has a countable neighbourhood base in* $P_\sigma(X)$, *then for each sequence* $(\rho_n)_{n \in \mathbb{N}}$ *converging to* ρ *in* $P_\sigma(X)$, *there exist sequences* $(\mu_n)_{n \in \mathbb{N}}$ and $(\nu_n)_{n \in \mathbb{N}}$ *such that*

$$\lambda \mu_n + (1-\lambda) \nu_n = \rho_n$$

and

$$\mu_n \to \mu \quad \text{and} \quad \nu_n \to \nu.$$

Proof. Observe first that μ and ν have countable neighbourhood bases too. Let $(\Delta_k)_{k \in \mathbb{N}}$ and $(\Gamma_k)_{k \in \mathbb{N}}$ be decreasing neighbourhood bases of μ and ν respectively. By Theorem (1.1) the sets $\lambda \Delta_k + (1-\lambda) \Gamma_k$ are open in $P_\sigma(X)$. So there are integers N_k such that for each $n \geqq N_k$

$$\rho_n \in \lambda \Delta_k + (1-\lambda) \Gamma_k.$$

We may choose the N_k increasing and converging to infinity. For each integer $n \geqq N_1$ denote by k_n the largest integer such that $n \geqq N_{k_n}$, choose $\mu_n \in \Delta_{k_n}$, $\nu_n \in \Gamma_{k_n}$ such that $\rho_n = \lambda \mu_n + (1-\lambda) \nu_n$ and the proof is complete. □

2. The Theorem

Since the existence of almost surely converging random variables which realize a weak convergent sequence $\mu_n \to \mu$, depends mainly on the limiting measure μ, the following definition is adequate.

(2.1) **Definition.** *We call* $\mu \in P_\sigma(X)$ *realizable iff for each sequence* μ_n *which converges in* $P_\sigma(X)$ *to* μ, *there exist a probability space* $(\Omega, \mathscr{A}, \mathbb{P})$ *and random variables* ξ_n, ξ *such that*

(i) $$\mathbb{P} \circ \xi_n^{-1} = \mu_n, \qquad \mathbb{P} \circ \xi^{-1} = \mu$$

and

(ii) $$\{\xi_n \to \xi\} \in \mathscr{A}, \qquad \mathbb{P}(\xi_n \to \xi) = 1.$$

To be able to handle three relevant cases simultaneously, we give the following

(2.2) **Definition.** *Let $\mu \in P_\sigma(X)$. X is called μ-countable iff there is a Borel subset X_0 in X and a countable class $\mathscr{G}$ of open sets, such that $\mu(X_0)=1$ and $\mathscr{G}$ includes a neighbourhood base of x for each $x \in X_0$.*

We state some simple facts.

(2.3) *Remark. X is μ-countable if*
 (a) *X is second countable,*
 (b) *X is first countable and $\mu = \sum_{i \in \mathbb{N}} \lambda_i \varepsilon_{x_i}$,*
 (c) *X is metrizable and μ is τ-smooth.*

Proof. (a) and (b) are obvious. For (c) observe that τ-smooth measures in metrizable spaces have separable supports. □

(2.4) *Remark. If X is μ-countable then*
 (a) *μ has a countable neighbourhood base in $P_\sigma(X)$,*
 (b) *μ is τ-smooth.*

Proof. The observation that the subspace X_0 in Definition (2.2) is second countable gives (b). The construction of a countable neighbourhood base of μ from the set $\mathscr{G}$ is straightforeward (see e.g. [14], Theorem 11.2.(iii)). □

Conditions (a) and (b) are not sufficient for X to be μ-countable: Let $X = [0,1]$ endowed with the right half-open interval topology (counterexample 51 of [13]). Since for $G \in \mathscr{G}(X)$ there is a countable set C such that $G \setminus C$ is open in the Euclidean topology on $[0,1]$, the Borel sets of X are the same as in the Euclidean case and we may take μ to be the Lebesgue measure on X. Since a neighbourhood base of a point $x \in X$ consists of at least countably many sets of the form $[x, x+\varepsilon[$, a neighbourhood base of an uncountable subset of X is uncountable itself, which implies that X is not μ-countable. (a) and (b) are holding since, as mentioned above, each open set in X includes an open set of the Euclidean topology with same μ-measure.

The following decomposition lemma enables us to reduce the realization problem to limiting measures ε_x.

(2.5) **Lemma.** *Let X be μ-countable. Then for each sequence $(\mu_n)_{n \in \mathbb{N}}$ converging to μ there exist kernels $T_n\colon X \to P_\sigma(X)$ and a Borel set X^0 such that $\mu(X^0)=1$,*

(iii) $$\int T_n(x, B)\, \mu(dx) = \mu_n(B) \quad \textit{for each } B \in \mathscr{B}(X)$$

and

(iv) $$T_n(x, .) \to \varepsilon_x \quad \textit{for each } x \in X^0.$$

Proof. Choose X_0 and $\mathscr{G} = \{G_m : m \in \mathbb{N}\}$ according to (2.2). Let $B^{m1}, \ldots, B^{mr_m}$ be those atoms of the algebra generated by $G_1, \ldots, G_m$, which possess positive μ-measure. By (1.2) and (2.4) there exist measures $\mu_n^{mk} \in P_\sigma(X)$ such that

$$\mu_n^{mk} \to \frac{1}{\mu(B^{mk})} 1_{B^{mk}} \mu, \qquad m \in \mathbb{N}, k \leqq r_m,$$

and

$$\sum_{k=1}^{r_m} \mu(B^{mk})\, \mu_n^{mk} = \mu_n, \qquad n \in \mathbb{N}.$$

Choose $N_m \uparrow \infty$ such that $N_1 = 1$ and for each $n \geqq N_m$, $m' \leqq m$ and $k \leqq r_m$

$$\mu_n^{mk}(G_{m'}) \geqq \frac{1}{\mu(B^{mk})}\, \mu(B^{mk} \cap G_{m'}) - \frac{1}{m}.$$

Call m_n the largest integer such that $n \geqq N_{m_n}$. Define

$$T_n(x,.) = \begin{cases} \mu_n^{m_n k} & \text{if } x \in B^{m_n k},\ 1 \leqq k \leqq r_{m_n} \\ \text{arbitrary} & \text{else.} \end{cases}$$

Then T_n is a kernel and (iii) follows by

$$\int T_n(x, B)\, \mu(dx) = \sum_{k=1}^{r_{m_n}} \mu_n^{m_n k}(B)\, \mu(B^{m_n k}) = \mu_n(B).$$

To get (iv) set $X^0 = X_0 \cap \bigcap_{m \in \mathbb{N}} \bigcup_{k=1}^{r_m} B^{mk}$, which is obviously a Borel set of μ-measure 1. For $x \in X^0$ and $x \in G_{m'}$ we have to show

$$\liminf T_n(x, G_{m'}) = 1.$$

Since for each n such that $m_n \geqq m'$ there exists an integer k_n such that $x \in B^{m_n k_n} \subset G_{m'}$, we have

$$\begin{aligned} \liminf T_n(x, G_{m'}) &= \liminf \mu_n^{m_n k_n}(G_{m'}) \\ &\geqq \liminf \frac{1}{\mu(B^{m_n k_n})}\, \mu(B^{m_n k_n}) - \frac{1}{m_n} \\ &= 1. \end{aligned}$$

This finishes the proof. □

It is worth mentioning that the kernels appearing in Lemma (2.5) may be chosen continuous under certain conditions (for instance if X is compact and metrizable), which can be derived by application of Corollary (2.2) of [9].

The main result can now be established.

(2.6) **Theorem.** *If $\mu \in P_\sigma(X)$ and X is μ-countable then μ is realizable.*

Proof. Choose X_0 and $\mathcal{G} = \{G_m \colon m \in \mathbb{N}\}$ according to (2.2). We may and do assume that each $G \in \mathcal{G}$ appears infinitely often in the sequence $(G_m)_{m \in \mathbb{N}}$.

(1) **The Case $\mu=\varepsilon_x$ for $x\in X_0$.** Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $P_\sigma(X)$ converging to ε_x and $U_k=G_{m_k}$ a decreasing neighbourhood base of x. Choose integers $N_k\uparrow\infty$ such that $N_1=1$ and for each $n\geqq N_k$

$$v_n(U_k)\geqq 1-\frac{1}{k}.$$

Call again k_n the largest integer such that $n\geqq N_{k_n}$. Denoting by $\bar{Z}$ the complement of a set $Z\subset X$, we set

$$\Omega=X^{\mathbb{N}}\times X,$$
$$\tilde{\mathscr{A}}=(\otimes_{\mathbb{N}}\mathscr{B}(X))\otimes\mathscr{B}(X),$$
$$\mathbb{P}_{nt}=\begin{cases}\dfrac{1}{v_n(U_{k_n})}1_{U_{k_n}}v_n & \text{if } t\leqq v_n(U_{k_n}),\\ \dfrac{1}{v_n(\overline{U_{k_n}})}1_{\overline{U_{k_n}}}v_n & \text{else,}\end{cases}$$
$$\mathbb{P}_t=(\otimes_{\mathbb{N}}\mathbb{P}_{nt})\otimes\varepsilon_x,$$
$$\tilde{\mathbb{P}}=\int_{]0,1[}\mathbb{P}_t\lambda(dt),$$
$$(\mathscr{A},\mathbb{P})=\text{completion of }(\tilde{\mathscr{A}},\tilde{\mathbb{P}})$$

(the completition is necessary since we have to guarantee the measurability of the set $\{\xi_n\to\xi\}\cap\{\xi\notin X_0\}$). As random variables ξ_n and ξ we choose the projections. Then (i) follows by

$$\mathbb{P}\circ\xi^{-1}=\varepsilon_x,$$
$$\mathbb{P}\circ\xi_n^{-1}=\int\mathbb{P}_{nt}\lambda(dt)=v_n.$$

To get (ii) we observe first that $\{\xi_n\to x\}\in\tilde{\mathscr{A}}$. For $t<1$ we get that $\mathbb{P}_t$-almost surely holds $\xi_n\in U_{k_n}$ eventually, since $v_n(U_{k_n})\to 1$. So

$$\mathbb{P}_t(\xi_n\to x)=1$$

and

$$\mathbb{P}(\xi_n\to x)=1,$$

which completes the proof of (1).

(2) **The General Case.** Let $\mu_n\to\mu$. Choose T_n, $n\in\mathbb{N}$ and X^0 according to (2.5). For $x\in X_0\cap X^0$ set $v_n=T_n(x)$ in (1) and choose the m_k as measurable functions of x (Define increasing sequences $l_k(x)$ such that

$$\{m\in\mathbb{N}: x\in G_m\}=\{l_1(x), l_2(x), \ldots\}$$

and set by recurrence

$$m_1(x)=l_1(x),$$
$$m_{k+1}(x)=\min\{n>m_k(x): x\in G_n\subset G_{m_k(x)}\cap G_{l_{k+1}(x)}\}).$$

In this case N_k, k_n and $\hat{\mathbb{P}}$ are measurable functions or kernels. Define

$$\hat{\mathbb{P}}(A)=\int \hat{\mathbb{P}}(x, A)\,\mu(dx), \qquad A\in\tilde{\mathscr{A}},$$

where $\hat{\mathbb{P}}(x, .)$ can be chosen arbitrarily for $x\notin X_0\cap X^0$. Complete for the same reasons as above $(\tilde{\mathscr{A}}, \hat{\mathbb{P}})$ to $(\mathscr{A}, \mathbb{P})$. We get (i) by an application of part (1):

$$\begin{aligned}\mathbb{P}\circ\xi_n^{-1}(B)&=\int \hat{\mathbb{P}}(x, \xi_n^{-1}(B))\,\mu(dx)\\&=\int T_n(x, B)\,\mu(dx)\\&=\mu_n(B),\\\mathbb{P}\circ\xi^{-1}(B)&=\int \hat{\mathbb{P}}(x, \xi^{-1}(B))\,\mu(dx)\\&=\int \varepsilon_x(B)\,\mu(dx).\\&=\mu(B).\end{aligned}$$

For (ii) observe first that $\{\xi_n\to\xi\}$ equals up to a $\mathbb{P}$-negligible set

$$A=\{\xi\in G_m\Rightarrow\xi_n\in G_m \text{ eventually}\}\in\tilde{\mathscr{A}}.$$

We compute finally, using again part (1)

$$\mathbb{P}(\xi_n\to\xi)=\mathbb{P}(A)=\int \hat{\mathbb{P}}(x, A)\,\mu(dx)=1. \quad \square$$

We conclude with two examples showing the assumption in (2.6) to be essential but not necessary.

(2.7) *Example. There exist (tight) measures on Lusin spaces which are not realizable.*

Proof. We use counterexample 98 of [13]. Let $X=\mathbb{N}\cup\{\infty\}$ and define a topology on X such that every subset of $\mathbb{N}$ is open and for $\infty\in G\subset X$

$$G\in\mathscr{G}(X)\Leftrightarrow\frac{1}{n}\,|\{1, \ldots, n\}\cap G|\to 1.$$

Choose $\mu=\varepsilon_\infty$ and $\mu_n=\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i$. Obviously $\mu_n\to\mu$ while there is no sequence in $\mathbb{N}$ converging to ∞: If $x_n\to\infty$ there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k}>10^k$ for each $k\in\mathbb{N}$. But then the set $\{x_{n_k}: k\in\mathbb{N}\}$ is closed in contradiction to $x_{n_k}\to\infty$. $\square$

(2.8) **Example.** *There exist (tight) measures μ on Lusin spaces X, such that μ is realizable, though X is not μ-countable.*

Proof. We use counterexample 26 of [13]. Let $X=\mathbb{N}\times\mathbb{N}\cup\{\infty\}$. Define a topology on X such that each subset of $\mathbb{N}\times\mathbb{N}$ is open and for $\infty\in G\subset X$

$$G\in\mathscr{G}(X)\Leftrightarrow|\{m\in\mathbb{N}: |\{n\in\mathbb{N}: (n, m)\notin G\}|\nless\infty\}|<\infty.$$

(1) Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $P_\sigma(X)$ converging to ε_∞. We will show $\mu_n(\{\infty\})\to 1$. Assuming the contrary and, if necessary, considering a subsequence, there is a real $\alpha>0$ such that for each $n\in\mathbb{N}$ holds $\mu_n(\overline{\{\infty\}})>\alpha$. Denoting the projections on $\mathbb{N}\times\mathbb{N}$ by π_1 and π_2, we proceed by recursion ($n_1=1$, $m_1=0$) for $k\in\mathbb{N}$:

Choose $x_1^k, \ldots, x_{r_k}^k$ such that $\pi_2(x_i^k)>m_k$ and

$$\mu_{n_k}(\{x_i^k\colon i\leqq r_k\})>\alpha.$$

Define

$$m_{k+1}=\max\{\pi_2(x_i^k)\colon i\leqq r_k\}.$$

Since $\mu_n(\mathbb{N}\times\{1, \ldots, m_{k+1}\})\to 0$ there is an integer n_{k+1} such that

$$\mu_{n_{k+1}}(\mathbb{N}\times\{m_{k+1}+1, \ldots\})>\alpha.$$

Setting now $F=\{x_i^k\colon k\in\mathbb{N},\ i\leqq r_k\}$, we derive a contradiction by observing that $\bar{F}$ is an open neighbourhood of ∞ and

$$\liminf \mu_n(\bar{F})\leqq 1-\alpha.$$

(2) ε_∞ is realizable, since:

Set

$$\mathbb{P}_{nt}=\begin{cases}\varepsilon_\infty & \text{if } t\leqq\mu_n(\{\infty\}),\\ \dfrac{1}{\mu_n(\overline{\{\infty\}})}\,1_{\overline{\{\infty\}}}\,\mu_n & \text{else}\end{cases}$$

in the construction of the probability measure in the proof of (2.6). Part (1) yields for $t<1$

$$\mathbb{P}_t(\xi_n=\infty \text{ eventually})=1$$

and the proof of (2) is complete.

(3) The point ∞ possesses no countable neighbourhood base, since in this case their would be a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $\mathbb{N}\times\mathbb{N}$ converging to ∞. This would yield $\varepsilon_{x_n}\to\varepsilon_\infty$, which is a contradiction to (1). □

The same method as above yields that the Dieudonné measure μ on the space of countable ordinals X (see [6], p. 231, (10)) is realizable though X is not μ-countable.

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