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Probability Theory and Related Fields

Existence of Quantum Diffusions

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Summary. A quantum diffusion (A, A', j) comprises of unital *-algebras A and A' and a family of identity preserving *-homomorphisms $j=(j_t\colon t\geqq 0)$ from A into A'. Also j satisfies a system of quantum stochastic differential equations $dj_t(x_0)=j_t(\mu^i_j(x_0))\,dM^j_i, j_0(x_0)=x_0\otimes I$ for all $x_0\in A$ where μ^i_j, $1\leqq i,j\leqq N$ are maps from A to itself and are known as the structure maps. In this paper an existence proof is given for a class of quantum diffusions, for which the structure maps are bounded in the operator norm sense. A solution to the system of quantum stochastic differential equations is first produced using a variation of the Picard iteration method. Another application of this method shows that the solution is a quantum diffusion.

1. Introduction

In several recent papers (e.g., [1, 2]) the concept of a quantum diffusion has been introduced and many of their algebraic properties investigated. A quantum diffusion is a quantum stochastic process (A, A', j), where A, A' are unital *-algebras and j is a family of identity preserving *-homomorphisms indexed by the positive real line. In our case A is a *-subalgebra of $B(H_0)$ where H_0 is a Hilbert space and $A'=B(H_0\otimes H)$, where H is the noise space of N dimensional quantum stochastic calculus. In addition j satisfies a stochastic differential equation of the type

$$dj_t(x_0)=\sum_{i,j=0}^{N} j_t(\mu^i_j(x_0))\,dM^i_j,$$

where x_0 is an arbitrary member of A, μ^i_j are maps from *-algebra to itself and M^j_i are families of operators in H which are the integrands of the theory. The purpose of this paper is to construct a class of diffusions in which the maps μ^i_j are bounded.

In Sect. 2 we introduce relevant definitions and results from quantum stochastic calculus [3]. Section 3 defines quantum diffusions and looks at the properties of the structure maps. In Sect. 4 we prove the existence of maps j_t, by methods similar to those of [3] Proposition 7.1. In Sect. 5 these are shown to have all the properties of a quantum diffusion. Section 6 looks at an example of a class of diffusions in which the diffusions are given by unitary conjugation.

Inner products are conjugate linear on the left. Two operators E, F are mutually adjoint on a domain D if $\langle Eu, v\rangle = \langle u, Fv\rangle$ for all u, v in D. The algebraic tensor product is denoted by $\underline{\otimes}$. For $f \in L^\infty([0,\infty))$, M_f is the operator on $L^2([0,\infty))$ given by $(M_f g)(t) = f(t)g(t)$ for $g \in L^2([0,\infty))$.

2. Quantum Stochastic Calculus

Let h be a Hilbert space. Then $H = \Gamma(h)$, the Boson Fock Space of h, is the Hilbert space determined up to unitary equivalence by the total set $\{\psi(f): f \in h\}$ of exponential vectors where $\langle \psi(f), \psi(g)\rangle = \exp\langle f, g\rangle$. *The annihilation* $a(f)$, *creation* $a^\dagger(f)$ *and conservation* $\Lambda(T)$ *operators* $(f \in h, T \in B(h))$ are defined on the span of the exponential vectors by linear extension of

$$a(f)\psi(g) = \langle f, g\rangle \psi(g)$$

$$a^\dagger(f)\psi(g) = \frac{d}{d\varepsilon}\psi(g+\varepsilon f)\bigg|_{\varepsilon=0}$$

$$\Lambda(T)\psi(g) = \frac{d}{d\varepsilon}\psi(e^{\varepsilon T}g)\bigg|_{\varepsilon=0}.$$

Since the exponential vectors are linearly independent, these operators are well defined. Also $a(f)$ and $a^\dagger(f)$ are mutually adjoint as are $\Lambda(T^*)$ and $\Lambda(T)$.

If h is the direct sum of h_1 and h_2 then H is the tensor product of $\Gamma(h_1)$ and $\Gamma(h_2)$ and if $f = f_1 \oplus f_2 \in h$, $\psi(f) = \psi(f_1) \otimes \psi(f_2)$.

In this paper we are interested in the case $h = L^2([0,\infty); \mathbb{C}^N)$ where N is a fixed positive integer.

Let $\{e_i\}_{i=1}^N$ and $\{E_j^i\}_{i,j=1}^N$ be the canonical bases of $\mathbb{C}^N$, M^N respectively. Thus $f \in h = L^2([0,\infty)) \otimes \mathbb{C}^N$, $L \in B(h) = B(L^2([0,\infty)) \otimes M^N$ can be written as $f = \sum_{i=1}^N f^i \otimes e_i$, $L = \sum_{i,j=1}^N L_i^j \otimes E_j^i$. Also $Lf = \sum_{i,j=1}^N L_i^j f^i \otimes e_j$.

Let S be the dense subspace in h of locally bounded functions.

For each $t \geq 0$, h is the direct sum of $h_t = L([0,t]; \mathbb{C}^N)$ and $h^t = L((t,\infty); \mathbb{C}^N)$, so H the Boson Fock space of h, is the tensor product of $H_t = \Gamma(h_t)$ and $H^t = \Gamma(h^t)$. Denote by ξ, ξ_t and ξ^t the span of the exponential vectors $\psi(f)$, $\psi(f_t)$ and $\psi(f^t)$ $(f = f_t \oplus f^t)$ in H, H_t and H^t respectively.

Let G, G_t and G^t be the dense subspaces of H, H_t and H^t spanned by the exponential vectors $\psi(f)$, $\psi(f_t)$ and $\psi(f^t)$ $(f = f_t \oplus f^t \in S)$.

We are interested in the Hilbert space $\tilde{H} = H_0 \otimes H$ where H_0 is a Hilbert space known as the initial space.

Definition. A family of operators $E = (E(t): t \geq 0)$ *in* $\tilde{H}$ is *an adapted process* if for all t the domain of $E(t)$ contains $H_0 \underline{\otimes} G_t \underline{\otimes} H^t$ and its restriction to

$H_0 \otimes G_t \underline{\otimes} H^t$ is the algebraic ampliation of an operator in $H_0 \underline{\otimes} H_t$ with domain $H_0 \underline{\otimes} G_t$.

An adapted process is respectively *simple, continuous, square integrable, bounded* if it is piecewise constant, the map $t \to E(t) u \otimes \psi(f)$ is strongly continuous for all $u \in H_0 f \in S$, the map $t \to E(t) u \otimes \psi(f)$ is strongly measurable and $\int_0^t \|E(s) u \otimes \psi(f)\|^2 ds < \infty$ for *all* $t > 0\, u \in H_0 f \in S$, each $E(t)$ is a bounded operator on $\tilde{H}$.

The basic processes of the theory are $M_j^i\, 0 \leqq i, j \leqq N$ where

$$M_j^i = \begin{cases} \Lambda(M_{\chi_{[0,t]}} \otimes E_j^i) & 1 \leqq i, j \leqq N \\ a(\chi_{[0,t]} \otimes e_i) & 1 \leqq i \leqq N, j = 0 \\ a^\dagger(\chi_{[0,t]} \otimes e_j) & 1 \leqq j \leqq N, i = 0 \\ tI & i = 0 = j. \end{cases}$$

These are all continuous adapted processes and will be the integrators of the theory. The stochastic integral $M(t) = \int_0^t \sum_{i,j=0}^N F_j^i dM_i^j$ is first defined for simple adapted processes $F_j^i\ 0 \leqq i, j \leqq N$ such that for $u, v \in H_0$, $g \in S, f \in h$

$$\begin{aligned} &\langle u \otimes \psi(f), M(t) v \otimes \psi(g) \rangle \\ &= \int_0^t \sum_{i,j=0}^N f_i(s) g^j(s) \langle u \otimes \psi(f), F_j^i(s) v \otimes \psi(g) \rangle ds \end{aligned} \tag{2.1}$$

where $f_i = \bar{f}^i\ \ 1 \leqq i \leqq N$, $g^0 = 1 = f_0$. If also $M'(t) = \int_0^t \sum_{i,j=0}^N F'^i_j dM_i^j$ then for $u, v \in H_0, f, g \in S$

$$\begin{aligned} &\langle M'(t) u \otimes \psi(f), M(t) v \otimes \psi(g) \rangle \\ &= \int_0^t \sum_{i,j=0}^N f_i(s) g^j(s) \{\langle M'(s) u \otimes \psi(f), F_j^i(s) v \otimes \psi(g) \rangle \\ &\quad + \langle F'^j_i(s) u \otimes \psi(f), M(s) v \otimes \psi(g) \rangle \\ &\quad + \sum_{k=1}^N \langle F'^k_i u \otimes \psi(f), F_j^k v \otimes \psi(g) \rangle\} ds. \end{aligned} \tag{2.2}$$

Putting $M = M'$, $u = v$ and $f = g$ in (2.2) produces the following inequality:

$$\|M(t) u \otimes \psi(f)\|^2 \leqq (2N^2 + 3N + 1) \alpha(T)^2 \int_0^t e^{t-s} \sum_{i,j=0}^N \|F_j^i(s) u \otimes \psi(f)\|^2 ds \tag{2.3}$$

where $0 \leqq t \leqq T$, $\alpha(T) = \max\{1, \sup\{\|f(s)\|^2 : 0 \leqq s \leqq T\}\}$.

Using this estimate it is possible to extend stochastic integration to $M(t) = \int_0^t \sum_{i,j=0}^N F_j^i \, dM_j^i$ where F_j^i are square integrable adapted processes. This is done in such a way that (2.1), (2.2) and (2.3) remain true. Also M will be a continuous adapted process and thus square integrable.

If 2 stochastic integrals M, M' are such that for all $t>0$

$$\sup\{\|M(s)\|, \|M'(s)\|, \|F_j^i(s)\|, \|F_j'^i(s)\| : 0 \leqq i, j \leqq N, 0 \leqq s \leqq t\} < \infty \tag{2.4}$$

then from (2.2) one obtains:

$$d(M'M) = (dM')M + M'dM + dM' \cdot dM \tag{2.5}$$

where dM_j^i commutes with adapted processes and the Ito correction $dM' \cdot dM$ is evaluated by linear extrapolation of the following formula

$$dM_j^i \cdot dM_l^k = \delta_l'^i \, dM_j^k \tag{2.6}$$

$$\text{where } \delta_l'^i = \begin{cases} 1 & \text{if } i = l \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark. If $M(t)$ is constant for all t then $dM(t) = 0$ a.e. t, and since integrands are independent this implies $F_j^i(t) = 0$ a.e. t $0 \leqq i, j \leqq N$; in particular if they are continuous this will be true for all $t \geqq 0$.

3. Quantum Diffusions

Let A be a unital *-subalgebra of $B(H_0)$.

Definition. *A quantum diffusion* on A is a family $j = (j_t;\, t \geqq 0)$ of identity preserving *-homomorphisms from A into $B(\tilde{H})$ such that for all $x_0 \in A$:

(1) $j_0(x_0) = x_0 \otimes I \ldots (3.1)$

(2) $x = (x(t) = j_t(x_0);\, t \geqq 0)$ is an adapted process.

(3) there exist maps $\mu_j^i\colon A \to A$, $0 \leqq i, j \leqq N$ such that $x(t)$ satisfies the stochastic differential equation

$$dj_t(x_0) = \sum_{i,j=0}^N j_t(\mu_j^i(x_0))\, dM_i^j \tag{3.2}$$

μ_j^i will be referred to as the structure maps.

If such a j exists, since $j_t(x_0)$ is a stochastic integral it is a continuous adapted process and therefore square integrable. Also by the remark at the end of Sect. 2 and (3.1), the structure maps inherit the following properties from j. They are linear in $x_0 \in A$, vanish on the identity, and $\mu_j^i(x_0)^* = \mu_i^j(x_0^*) \ldots (3.3)$.

If in addition for each $x_0 \in A_j \sup\{\|j_s(x_0)\| : 0 \leqq s \leqq t\} < \infty$, for all $t > 0$, then (2.5) is applicable to $j_t(x_0) j_t(y_0) (= j_t(x_0 y_0))$ and by equating coefficients we obtain the following structural equations:

$$\mu^i_j(x_0)\, y_0 + x_0\, \mu^i_j(y_0) + \sum_{k=1}^{N} \mu^i_k(x_0)\, \mu^k_j(y_0) = \mu^i_j(x_0 y_0). \tag{3.4}$$

This will be true in particular if each j_t is contractive, for example if A is closed under holomorphic functional calculus.

Define $\sigma: A \to A \otimes M_N$, $\alpha: A \to A \otimes \mathbb{C}^N$ and $\tau: A \to A$ by

$$\sigma(x_0) = \sum_{i,j=1}^{N} (\mu^i_j(x_0) - \delta^i_j x_0) \otimes E^i_j, \qquad \alpha(x_0) = \sum_{i=1}^{N} \mu^i_0(x_0) \otimes e_i,\ \tau(x_0) = \mu^0_0(x_0).$$

From (3.3) and (3.4) we find that σ is a *-homomorphism, $\alpha(x_0 y_0) = \alpha(x_0)\, y_0 + \sigma(x_0)\, \alpha(y_0)$ (where multiplication in the first term is "scalar" and in the second is "matrix"), and $\tau(x_0 y_0) = \tau(x_0)\, y_0 + x_0\, \tau(y_0) + \alpha^\dagger(x_0)\, \alpha(y_0)$. These equations and their cohomological properties were investigated in [1, 2]. They will play no part in the existence proof, we only assume properties of the structure maps already stated and the additional assumption that the structure maps are bounded, in the sense that there exists an M such that for all $x_0 \in A$, $0 \leqq i, j \leqq N$ $\|\mu^i_j(x_0)\| \leqq M \|x_0\| \ldots (3.5)$.

4. Existence of j

j is constructed by the Picard iteration method, imitating Proposition 7.1 of [3].

Proposition 4.1. *For all $x_0 \in A$, there exist adapted processes $x^{(n)}(t) = (j^{(n)}_t(x_0) : t \geqq 0)$ satisfying*

$$x^{(0)}(t) = j^{(0)}_t(x_0) = x_0 \otimes I \qquad \text{for } n \geqq 1$$

$$x^{(n)}(t) = j^{(n)}_t(x_0) = x_0 \otimes I + \int_0^t \sum_{i,j=0}^{N} j^{(n-1)}_t(\mu^i_j(x_0))\, dN^j_i \tag{4.1}$$

such that for $u \in H_0$, $f \in S$, $T \geqq 0$ and $0 \leqq t \leqq T$

$$\|(x^{(n)}(t) - x^{(n-1)}(t))\, u \otimes \psi(f)\|^2 \leqq e^{T + \|f\|^2} \frac{\beta(T)^n}{n!} \|x_0\|^2 \|u\|^2 \tag{4.2}$$

where $\beta(T) = (N+1)^2 (2N^2 + 3N + 1)\, \alpha(T)^2\, T M^2$.

Proof. Clearly $x^{(0)}$ is a square integrable adapted process for all $x_0 \in A$, assume that $x^{(n-1)}$ is also, then

$$\int_0^t \sum_{i,j=0}^{N} \|j^{(n-1)}_s(\mu^i_j(x_0))\, u \otimes \psi(f)\|^2\, ds < \infty$$

since $\mu^i_j(x_0) \in A$ for *all* $0 \leqq i, j \leqq N$.

Hence $x^{(n)}$ is well defined and as it is stochastic integral it will be a square integrable adapted process. By induction, there exist adapted processes $x^{(n)}$ satisfying (4.1) for all $x_0 \in A$, $n \geqq 0$. And

$$x^{(n)}(t) - x^{(n-1)}(t) = \int_0^t \sum_{i,j=0}^N j_s^{(n-1)}(\mu_j^i(x_0)) - j_s^{(n-2)}(\mu_j^i(x_0))\, dM_i^j.$$

So for $u \in H_0, f \in S$ by (2.3)

$$\begin{aligned}
&\|(x^{(n)}(t) - x^{(n-1)}(t))\, u \otimes \psi(f)\|^2 \\
&\leqq (2N^2+3N+1)\,\alpha(T)^2 \int_0^t e^{t-s} \sum_{i,j=0}^N \|(j_s^{(n-1)}(\mu_j^i(x_0)) - j_s^{(n-2)}(\mu_j^i(x_0)))\, u \otimes \psi(f)\|^2\, ds \\
&\leqq (N+1)^2 (2N^2+3N+1)\, e^T \alpha(T)^2 \int_0^t e^{-s} \max\{\|j_s^{(n-1)}(\mu_j^i(x_0)) \\
&\quad - j_s^{(n-2)}(\mu_j^i(x_0)))\, u \otimes \psi(f)\|^2;\ 0 \leqq i, j \leqq N\}\, ds
\end{aligned}$$

from which (4.2) follows by induction. □

Corollary 4.2. *For each $x_0 \in A$ there exists an adapted process $x = (x(t)\colon t \geqq 0)$ such that $x(t)$ is the strong limit of $x^{(n)}(t)$ for all $t \geqq 0$. Furthermore defining maps j_t, $t \geqq 0$ from A to $B(\tilde{H})$ by $j_t(x_0) = x(t)$ for $x_0 \in A$, we have that $j = (j_t\colon t \geqq 0)$ satisfies the system of stochastic differential equations* (3.2).

Proof. From (4.2) for $u \in H_0, f \in S$ and $T \geqq 0$

$$\begin{aligned}
&\|x_0 u \otimes \psi(f)\| + \sum_{n=1}^{\infty} \sup\{\|(j_t^{(n)}(x_0) - j_t^{(n-1)}(x_0))\, u \otimes \psi(f)\|;\ 0 \leqq t \leqq T\} \\
&\leqq e^{1/2(T + \|f\|^2)} \|x_0\|\, \|u\| \sum_{n=1}^{\infty} \frac{\beta(T)^{n/2}}{(n!)^{1/2}} \\
&= M_{f,T} \|x_0\|\, \|u\| < \infty
\end{aligned} \tag{4.3}$$

where $M_{f,T} (\in [1, \infty))$ does not depend on x or u. Therefore $x^{(n)}(t)\, u \otimes \psi(f)$ converges in $\tilde{H}$ uniformly in each finite interval $[0,T]$ to say $x(t)\, u \otimes \psi(f) \cdot x(t)$ will be linear in $u \in H_0$, and so defines an operator with domain $H_0 \otimes G \cdot x$ being the strong limit of adapted processes will itself be an adapted process. Letting $j_t(x_0) = x(t)$ for $x_0 \in A$, using uniformity of convergence (in x as well as in t) and (2.3), take strong limits on both sides of (4.1) to conclude that j satisfies (3.2). □

5. Multiplicativity and Other Properties of *j*

The following properties of j are all derived from induction on $j^{(n)}$ and properties of the structure maps;

(1) j_t is linear in $x_0 \in A$ ($\mu_j^i(x_0)$ is linear).

(2) $j_0(x_0) = x_0 \otimes I$

(3) $j_t(I) = I$ for all t $(\mu^i_j(I) = 0)$

(4) $j_t(x_0^*) = j_t(x_0)^\dagger$ $(\mu^i_j(x_0)^* = \mu^j_i(x_0^*),\ M^{j\dagger}_i = M^i_j)$.

All that remains to be shown is multiplicativity of j and that $j_t(x_0) \in B(\tilde{H})$ for *all* $x_0 \in A$. Before proving that j is multiplicative note that from (4.3) have that

$$\sup\{\|j_t(x_0)u \otimes \psi(f)\|^2 : 0 \leqq t \leqq T\} \leqq M_{f,T} \|x_0\|\ \|u\|. \tag{5.1}$$

As $j_t(x_0)$ is not necessarily bounded, $j_t(x_0)j_t(y_0)$ is not defined, so we look at

$$\langle j_t(x_0^*)u \otimes \psi(f), j_t(y_0)v \otimes \psi(g)\rangle - \langle u \otimes \psi(f), j_t(x_0 y_0)v \otimes \psi(g)\rangle.$$

Proposition 5.1. *For all* $u, v \in H_0, f, g \in S, n \geqq 0$

$$\begin{aligned}|\langle j_t(x_0^*)u \otimes \psi(f), j_t(y_0)v \otimes \psi(g)\rangle - \langle u \otimes \psi(f), j_t(x_0 y_0)v \otimes \psi(g)\rangle| \\ \leqq \frac{\delta(T)^n}{n!} M_{f,g,T} \|x_0\|\ \|y_0\|\ \|u\|\ \|v\|\end{aligned} \tag{5.2}$$

where $\delta(T) = (N+1)^2(2+N)T\max\{M^2, 1\}\ \sup\{\|f(s)g(s)\|, \|f(s)\|, \|g(s)\|, 1\}$ *and* $M_{f,g,T} = (M_{f,T} + e^{\|f\|^2/2}) M_{g,T}$.

Proof. By (2.1) and (2.2)

$$\begin{aligned}&\langle j_t(x_0^*)u \otimes \psi(f), j_t(y_0)v \otimes \psi(g)\rangle - \langle u \otimes \psi(f), j_t(x_0 y_0)v \otimes \psi(g)\rangle \\ &= \int_0^t \sum_{i,j=0}^N f_i(s) g^j(s) \{\langle j_s(x_0^*)u \otimes \psi(f), j_s(\mu^i_j(y_0))v \otimes \psi(g)\rangle \\ &\quad + \langle j_s(\mu^j_i(x_0^*))u \otimes \psi(f), j_s(y_0)v \otimes \psi(g)\rangle \\ &\quad + \sum_{k=1}^N \langle j_s(\mu^k_i(x_0^*))u \otimes \psi(f), j_s(\mu^k_j(y_0)v \otimes \psi(g)\rangle \\ &\quad - \langle u \otimes \psi(f), j_s(\mu^i_j(x_0 y_0)v \otimes \psi(g)\rangle\}\, ds \\ &= \int_0^t \sum_{i,j=0}^N f_i(s) g^j(s) \{\langle j_s(x_0^*)u \otimes \psi(f), j_s(\mu^i_j(y_0)v \otimes \psi(g)\rangle \\ &\quad - \langle u \otimes \psi(f), j_s(x_0 \mu^i_j(y_0)v \otimes \psi(g)\rangle \\ &\quad + \langle j_s(\mu^i_j(x_0)^*)u \otimes \psi(f), j_s(y_0)v \otimes \psi(g)\rangle - \langle u \otimes \psi(f), j_s(\mu^i_j(x_0)\, y_0)v \otimes \psi(g)\rangle \\ &\quad + \sum_{k=1}^N \langle j_s(\mu^i_k(x_0)^*)u \otimes \psi(f), j_s(\mu^k_j(y_0))v \otimes \psi(g)\rangle \\ &\quad - \langle u \otimes \psi(f), j_s(\mu^i_k(x_0)\, \mu^k_j(y_0))v \otimes \psi(g)\rangle\}\, ds.\end{aligned}$$

The right hand side is made up of $(N+1)^2(2+N)$ pairs each of which is of the form

$$\langle j_s(w^*)u \otimes \psi(f), j_s(z)v \otimes \psi(g)\rangle - \langle u \otimes \psi(f), j_s(wz)v \otimes \psi(g)\rangle$$

where $w, z \in A$. In modulus each term is less than or equal to

$$\gamma(T)\max\{|\langle j_s(v(x_0)^*)u\otimes\psi(f), j_s(v(y_0))v\otimes\psi(g)\rangle - \langle u\otimes\psi(f), j_s(v_i(x_0)v_j(y_0)v\otimes\psi(g)\rangle|: 0\leqq i,j\leqq(N+1)^2\}$$

where

$$v_0(x_0)=x_0 \quad\text{and}\quad v_i(x_0)=\mu_k^{i-(N+1)k}(x_0) \quad (N+1)k<i\leqq(N+1)(k+1),\ 0\leqq k\leqq N$$

and

$$\gamma(T)=\sup\{\|f(s)\|\ \|g(s)\|,\ \|f(s)\|,\ \|g(s)\|,\ 1;\ 0\leqq s\leqq T\}.$$

Therefore

$$\begin{aligned}
&|\langle j_t(x_0^*)u\otimes\psi(f), j_t(y_0)v\otimes\psi(g)\rangle - \langle u\otimes\psi(f), j_t(x_0y_0)v\otimes\psi(g)\rangle|\\
&\leqq(N+1)^2(2N+1)\gamma(T)\int_0^t \max\{|\langle j_s(v_i(x_0)^*)u\otimes\psi(f), j_s(v_j(y_0))v\otimes\psi(g)\rangle\\
&\quad -\langle u\otimes\psi(f), j_s(v_i(x_0)v_j(y_0))v\otimes\psi(g)\rangle|;\ 0\leqq i,j\leqq(N+1)^2\}\,ds\\
&\leqq((N+1)^2(2N+1)\gamma(T)^n\int_0^t\dots\int_0^{t_{n-1}}\max\{|\langle j_{t_n}(v_{i_n}\dots v_{i_1}(x_0)^*)\\
&\quad\cdot u\otimes\psi(f), j_{t_n}(v_{j_n}\dots v_{j_1}(y_0))v\otimes\psi(g)\rangle\\
&\quad -\langle u\otimes\psi(f), j_{t_n}(v_{i_n}\dots v_{i_1}(x_0)v_{j_n}\dots v_{j_1}(y_0))\\
&\quad\cdot v\otimes\psi(g)\rangle|: 0\leqq i_k, j_k\leqq(N+1)^2, 0\leqq k\leqq n\}\,dt_n\dots dt_1\\
&\leqq((N+1)^2(2N+1)\gamma(T))^n\int_0^t\dots\int_0^{t_{n-1}}\max\{\|j_{t_n}(v_{i_n}\dots v_{i_1}(x_0)^*)\\
&\quad\cdot u\otimes\psi(f)\|\ \|j_{t_n}(v_{j_n}\dots v_{j_1}(y_0))v\otimes\psi(g)\|\\
&\quad+\|u\otimes\psi(f)\|\ \|j_{t_n}(v_{i_n}\dots v_{i_1}(x_0)v_{j_n}\dots v_{j_1}(y_0))\\
&\quad\cdot v\otimes\psi(g)\|: \leqq i_k, j_k\leqq(N+1)^2, 0\leqq k\leqq n\}\,dt^n\dots dt_1\\
&\leqq((N+1)^2(2N+1)\gamma(T))^n\int_0^t\dots\int_0^{t_{n-1}}\max\{M_{f,T}\|v_{i_n}\dots v_{i_1}(x_0)\|\\
&\cdot\ \|u\|M_{g,T}\|v_{j_n}\dots v_{j_1}(y_0)\|\ \|v\|\\
&\quad+\|u\otimes\psi(f)\|M_{g,T}\|v_{i_n}\dots v_{i_1}(x_0)v_{j_n}\dots v_{j_1}(y_0)\|:\\
&\quad\cdot 0\leqq i_k, j_k\leqq(N+1)^2, 0\leqq k\leqq n\}\,dt_n\dots dt_1\\
&\leqq((N+1)^2(2N+1)\gamma(T))^n(\max\{1, M^2\})^n\\
&\quad\cdot\|x_0\|\ \|u\|\ \|y_0\|\ \|v\|\left(M_{f,T}+e^{\|f\|^2/2}M_{g,T}\frac{T^n}{n!}\right).\quad\square
\end{aligned}$$

Letting $n\to\infty$ in (5.2) the r.h.s. tends to zero and so

$$\langle j_t(x_0^*)u\otimes\psi(f), j_t(y_0)v\otimes\psi(g)\rangle=\langle u\otimes\psi(f), j_t(x_0y_0)v\otimes\psi(g)\rangle$$

and so by linearity for any $h_1, h_2 \in H_0 \underline{\otimes} G$

$$\langle j_t(x_0^*) h_1, j_t(y_0) h_2 \rangle = \langle h_1, j_t(x_0 y_0) h_2 \rangle. \tag{5.3}$$

For an arbitrary non-zero element $h = \sum_{i=1}^{L} u_i \otimes \psi(f_i)$ of $H_0 \underline{\otimes} G$ we have that for $0 \leqq t \leqq T$

$$\begin{aligned}
\|j_t(x_0) h\|^2 &= \langle j_t(x_0) h, j_t(x_0) h \rangle \\
&= \langle h, j_t(x_0^* x_0) h \rangle \\
&\leqq \|h\| \, \|j_t(x_0^* x_0) h\| \\
&\leqq \|h\|^{2-1/2^n} \|j_t((x_0^* x_0)^{2^n} h\|^{1/2^n} \\
&\leqq \|h\|^{2-1/2^n} \left(\sum_{i=1}^{L} \|j_t((x_0^* x_0)^{2^n}) u_i \psi(f_i)\| \right)^{1/2^n} \\
&\leqq \|h\|^{2-1/2^n} \left(\sum_{i=1}^{L} M_{f_i, T} \|u_i\| \, \|x_0^* x_0\|^{2^n} \right)^{1/2^n} \\
&= \|h\|^{2-1/2^n} \|x_0\|^2 \left(\sum_{i=1}^{L} M_{f_i, T} \|u_i\| \right)^{1/2^n}.
\end{aligned}$$

Letting $n \to \infty$, the r.h.s. converges to $\|h\|^2 \|x_0\|^2$, therefore $\|j_t(x_0)\| \leqq \|x_0\| \, \|h\|$. As $H_0 \underline{\otimes} G$ is dense in $\tilde{H}$, this implies that $j_t(x_0)$ can be uniquely extended to a bounded operator on $\tilde{H}$ with bound less than or equal to $\|x_0\|$. From (5.3) and continuity of inner products, we deduce that j_t is multiplicative.

We now have all the information required and state it as a theorem.

Theorem. *Given structure maps μ_j^i: $A \to A$, $0 \leqq i, j \leqq N$ satisfying*

(1) *Linearity*

(2) $\mu_i^j(x_0^*)^* = \mu_j^i(x_0)$

(3) $\mu_j^i(I) = 0$

(4) $\mu_j^i(x_0 y_0) = \mu_j^i(x_0) y_0 + x_0 \mu_j^i(y_0) + \sum_{k=1}^{N} \mu_k^i(x_0) \mu_j^k(y_0)$

(5) $\|\mu_j^i(x_0)\| \leqq M \|x_0\|$

*there exists a unique family $j = (j_t; t \geqq 0)$ of identity preserving contractive *-homorphisms from A to $B(\tilde{H})$ such that:*

(1) $j_0(x_0) = x_0 \otimes I$

(2) $x = (x(t) = j_t(x_0)$ *is an adapted process*

(3) *j satisfies the stochastic differential equation*

$$dj_t(x) = j_t(\mu_j^i(x_0)) \, dM_i^j$$

Proof of Uniqueness. If j' is another such family then as in Proposition 4.1

$$\begin{aligned}
&\|(j_t(x_0)-j'_t(x_0)u\otimes\psi(f)\|^2\\
&\leqq((N+1)^2(N^2+3N+1)\alpha(T)^2)^n e^t\int_0^t\dots\int_0^{t_{n-1}}\max\{\|j_t(\mu^{i_n}_{j_n}\dots\mu^{i_n}_{j_n}(x_0))\\
&\quad -j'_t(\mu^{i_n}_{j_n}\dots\mu^{i_1}_{j_1}(x_0))u\otimes\psi(f)\|^2 : 0\leqq i_k, j_{k\leqq N}1\leqq k\leqq n\}\,dt_n\dots dt_1\\
&\leqq((N+1)^2(N^2+3N+1)\alpha(T)^2)^n e^T\frac{T^p}{p!}2M^{2n}\|x_0\|^2\,\|u\otimes\psi(f)\|^2
\end{aligned}$$

as $n\to\infty$ r.h.s. $\to 0$ so $j_t(x_0)=j'_t(x_0)$ this is true for all x_0 and t, so j is unique. □

6. Unitary Diffusions

Let σ, α, τ defined in Sect. 3 be given by

$$\begin{aligned}
&\sigma(x_0)\\
&\quad =Wx_0\otimes IW^*,\ \alpha(x_0)=Lx_0-Wx_0W^*L,\ \tau(x_0)=-\tfrac{1}{2}(L^\dagger Lx_0-2L^\dagger WxW^*L)\\
&\quad\quad +i[H,x_0]
\end{aligned}$$

where W is a unitary member of $A\otimes M_N$, $L\in A\otimes\mathbb{C}^N$ with $L^\dagger$ being the corresponding element of $A\otimes\mathbb{C}^{N*}$ and H a self-adjoint element of A. Writing $W=\sum_{i,j=1}^{N}L^i_j\otimes E^j_i$ and $L=\sum_{i=1}^{N}L^i\otimes e_i$ define L^i_j by

$$L^i_j=\begin{cases} W^i_j-\delta^i_j I & 1\leqq i,j\leqq N\\ L^i & 1\leqq i\leqq N, j=0\\ -\sum_{i=1}^{N}W^i_j L^{i*} & 1\leqq j\leqq N, i=0\\ iH-\tfrac{1}{2}\sum_{i=1}^{N}L^{j*}L^j. & i=0=j.\end{cases}$$

These satisfy the following equations

$$L^i_j+L^{j*}_i+\sum_{k=1}^{N}L^{k*}_i L^k_j=0$$

$$L^i_j+L^{j*}_i+\sum_{k=1}^{N}L^i_k L^{j*}_k=0.$$

These are an N-dimensional generalisation of the unitarity conditions given in Sect. 7 of [3]. Similarly the stochastic differential equation

$$dU = U\left(\sum_{i,j=0}^{N} L^i_j \, dM^j_i\right), \qquad U(0) = I$$

will have a unique unitary solution. From which we can describe the diffusion j with σ, α and τ as above by

$$j_t(x_0) = U(t)\, x_0\, U^*(t),$$

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References

1. Evans, M.P., Hudson, R.L.: Multidimension diffusions. In: Accardi, L., Waldenfels, W. von (eds.) Quantum probability III. Proceedings, Oberwolfach 1987. (Lect. Notes Math., vol. 1303, pp. 69–88). Berlin Heidelberg New York Tokyo: Springer 1987
2. Hudson, R.L.: Quantum diffusions and cohomology of algebras. In: Prohorov, Y., Sazonov, V.V. (eds.) Proceedings of 1st World Congress of Bernoulli Society, vol. 1, pp. 479–485. Utrecht: VNU Science Press 1987
3. Hudson, R.L., Parthasarathy, K.R.: Quantum Ito's formula and stochastic evolutions. Comm. Math. Phys. **93**, 301–323 (1984)

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