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Probability Theory and Related Fields

Uniform Convexity and the Distribution of the Norm for a Gaussian Measure

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Summary. We show that if a Banach space E has a norm $\|\cdot\|$ such that the modulus of uniform convexity is bounded below by a power function, then for each Gaussian measure μ on E the distribition of the norm for μ has a bounded density with respect to Lebesgue measure. This result is optimum in the following sense:

If (a_n) is an arbitrary sequence with $a_n \to 0$, there exists a uniformly convex norm $N(\cdot)$ on the standard Hilbert space, equivalent to the usual norm such that the modulus of convexity of this norm satisfies $\alpha(\varepsilon) \geqq \varepsilon^n$ for $\varepsilon \geqq a_n$, and a Gaussian measure μ on E such that the distribution of the norm for μ does not have a bounded density with respect to Lebesgue measure.

1. Introduction and Results

Consider a Banach space E and μ a centered Gaussian measure on E, that is a Radon measure on E such that for each $x^* \in E^*$ the law of x^* is centered normal. For $a \in E$, $t \in R^+$, let

$$B(a,t) = \{x \in E; \|x-a\| \leqq t\}, \quad \text{and} \quad \psi_a(t) = \mu(B(a,t)).$$

The function $\psi_a(t)$ has remarkable properties. C. Borell showed that $\log \psi_a$ is concave. A remarkable improvement of this result has been obtained recently by A. Ehrhard [1] who showed that $\Psi_a = \Phi^{-1} \cdot \psi_a$ is concave, where

$$\Phi(u) = (2\pi)^{-1/2} \int_{-\infty}^{u} e^{-x^2/2}\, dx.$$

It follows that for each $t_0 > 0$, there is a constant C such that

$$|\Psi_a(t) - \Psi_a(u)| \leqq C|t-u| \quad \text{for } t, u \geqq t_0.$$

Since Φ is Lipschitz, it follows that

$$|\psi_a(t) - \psi_a(u)| \leqq C|t-u| \quad \text{for } t, u \geqq t_0.$$

In other words, the distribution of $x \to \|x-a\|$ has a bounded density with respect to Lebesgue measure on each interval $[t_0, \infty]$. In this paper, we consider the problem of whether this density is bounded on $[0, \infty[$, that is whether there is a constant C such that

$$\text{for } 0 \leqq u \leqq t, \quad \mu\{x: u < \|x-a\| < t\} \leqq C(t-u). \tag{$*$}$$

A first positive result has been obtained by J. Kuelbs and T. Kurtz [3]. They showed that condition $(*)$ holds for each Gaussian μ when $E = l^2(\mathbb{N})$, with the usual norm and $a=0$. V.I. Paulauskas extended their result to some special Gaussian measures on l^p [5]. The best result known at present seems to be due to F. Gotze [2]. He assumes that the norm has strong differentiable properties and uses a technical condition, which is, roughly speaking, a global way of saying that the ball is round, at least with respect to the measure μ.

In the opposite direction it has been shown independently by V.I. Paulauskas and the authors [7] that condition $(*)$ fails in general. Paulauskas' example is on $c_0(\mathbb{N})$ with the usual norm, while the example of the authors is on $l^2(\mathbb{N})$, for a norm equivalent to the usual norm. A further example by the authors exhibits a C^∞ renorming of $l^2(\mathbb{N})$, such that all the differentials of the norm remain bounded on the unit sphere, and still condition $(*)$ fails for this renorming [8]. It is hence not possible to ensure condition $(*)$ by assuming that the Banach space is very regular and the norm is very smooth. However, the crucial point in these various examples is that the unit sphere for the norms which fails $(*)$ has flat areas, as is very clear in [8]. It should be noted that Gotze's condition is a global way to forbid the existence of such flat areas. Hence, the crucial fact is the shape of the unit ball. So it is natural to investigate what type of geometrical conditions will force condition $(*)$. These conditions have to force the ball to be "round" in some sense.

Given a Banach space E, and $0 < \varepsilon < 2$, let

$$\alpha(\varepsilon) = 1 - \sup\{\|(x+y)/2\|; \|x\|, \|y\| \leqq 1, \|x-y\| \geqq \varepsilon\}. \tag{1}$$

The Banach space E is called uniformly convex if $\alpha(\varepsilon) > 0$ whenever $\varepsilon > 0$, and the function α is called the modulus of uniform convexity. For example, if E is $l^p(\mathbb{N})$, then $\alpha(\varepsilon)$ is of order ε^2 for $p \leqq 2$ and ε^p for $p \geqq 2$. See also [6].

Our first result will be

Theorem A. *Assume E is uniformly convex Banach space, and that there exists $\beta > 0$ and $p \in \mathbb{N}$ such that the modulus of uniform continuity $\alpha(\cdot)$ satisfies $\alpha(\varepsilon) \geqq \beta \varepsilon^p$ for $0 < \varepsilon < 2$. Then condition $(*)$ holds for each a and each μ.*

One might wonder if it would not be enough to assume a weaker condition on the modulus of continuity $\alpha(\cdot)$. To show that this is essentially impossible, we prove the following:

Theorem B. *Let (a_n) be a sequence with $a_n \to 0$. Then there exists a uniformly convex norm $N(\cdot)$ on the standard Hilbert space H such that the modulus of uniform continuity $\alpha(\cdot)$ of $N(\cdot)$ satisfies $\alpha(\varepsilon) \geqq \varepsilon^n$ for $n \geqq 3$ and $10^{-2} \geqq \varepsilon \geqq a_n$, and a Gaussian measure μ on E, such that $(*)$ fails for $a=0$.*

In particular, conditions of the type $\alpha(\varepsilon)\geqq\beta\varepsilon^{a(\varepsilon)}$ where $a(\varepsilon)\to\infty$ when ε goes to zero, are not sufficient to imply (*).

Using the techniques of [8], there is no doubt that in the statement of Theorem B one can also force the norm to be infinitely differentiable, and have bounded differentials on the unit sphere; this however increases the complexity of the construction, and we have not carried out the details, since the result of [8] already mentioned has shown that this type of restriction on the norm is largely irrelevant in the study of condition (*).

This use of condition (*) was initiated by the study of the rate of convergence in the Central Limit Theorem. Minor modifications of the construction we give for Theorem B enable to perform the construction such that there is no rate of convergence in the CLT for the new norm $N(\cdot)$. In other words, given a sequence $\xi_n\to 0$ one can construct an E-valued random variable X, which is bounded, has expectation zero, and such that if X_n denotes an i.i.d. sequence distributed like X, and μ the Gaussian measure with the same covariance as X, the inequality

$$\sup_t\Big(P\big(N(n^{-1/2}\sum_{i\leqq n}X_i)\leqq t\big)-\mu\{x;\|x\|\leqq t\}\Big)\geqq\xi_n$$

occurs for infinitely many n's. The techniques, which are described in detail in [8], will not be reproduced here.

2. Proof of Theorem A

We can suppose that E is the support of μ, and that E is infinite dimensional. We fix an integer n. The first lemma is classical. We include its proof for completeness. Let $n\in\mathbb{N}$.

Lemma 1. *We can write E as a product $E_1\times E_2$, where $\dim E_1=n$, such that there exist two Gaussian measures μ_1 and μ_2 on E_1 and E_2 such that μ identifies with $\mu_1\times\mu_2$.*

Proof. Let E_1^* be the unit ball of E^*, provided with the weak* topology. Each sequence (x_n^*) of E_1^* has a subsequence $x_{n_k}^*$ which converges weak*, in particular μ a.e., to some $x^*\in E_1^*$. It follows that $x_{n_k}^*$ goes to x^* in probability, hence in $L^2(\mu)$. Since E_1^* identified as a subset of $L^2(\mu)$ is Hausdorff, it is compact, and the L^2-topology coincides with the weak*-topology.

Let F_1 be an n-dimensional subspace of E^*. Let

$$F_2=\{y^*\in E^*;\ \text{for } x^*\in F_1,\ \int x^*(x)\,y^*(x)\,d\mu(x)=0\},$$

that is, $F_2=F_1^{\perp}\cap E^*$ in $L^2(\mu)$. Then $F_2\cap E_1^*$ is closed in $L^2(\mu)$, so is weak*-compact. Banach's Theorem shows that F_2 is weak*-closed. So we have written $E^*=F_1\oplus F_2$, where F_1 and F_2 are orthogonal in $L^2(\mu)$ and weak*-closed. It follows that $E=E_1\oplus E_2$, where

$$E_1=\{x\in E;\ \text{for } x^*\in F_2,\ x^*(x)=0\}$$

$$E_2=\{x\in E;\ \text{for } x^*\in F_1,\ x^*(x)=0\}.$$

Let S_1 (resp. S_2) be the projection of E onto E_1 (resp. E_2), and let μ_2 be the image of μ under S_1 (resp. S_2). It is straightforward to check on the convariances that μ identifies with $\mu_1 \times \mu_2$. The lemma is proved. Of course, $\|S_1\|$ and $\|S_2\|$ can be very large and depend on n. But this won't be a problem.

We now fix $0<u<t\leqq 1$, and let

$$A=\{x; u\leqq \|x-a\| \leqq t\}.$$

For $y\in E_2$, let

$$A_y=\{x\in E_1; x+y\in A\}.$$

We will estimate $\mu(A)$ using Fubini's Theorem:

$$\mu(A)=\int \mu_1(A_y)\, d\mu_2(y). \tag{2}$$

so we need to estimate $\mu_1(A_y)$. For $y\in E_2$, let

$$N(y)=\operatorname{Inf}\{\|x+y-a\|; x\in E_1\}.$$

In other words, $N(y)$ is the norm of $y-a$ in E/E_1.

Lemma 2. *Let $N(y)<t$ and let $x'\in E_1$ with $\|x'+y-a\|=N(y)$. Then for $x\in A_y$, we have*

$$\|x-x'\| \leqq C_1 t(1-t^{-1}N(y))^{1/p}$$

for a constant C_1.

Proof. Let $x\in A_y$. Let $z_1=x+y$ and $z_2=x'+y$.

We have $\|(z_1+z_2)/2-a\| \geqq N(y)$. If we use (1) with $t^{-1}(z_1-a)$ and $t^{-1}(z_2-a)$, we have

$$\begin{aligned}\beta t^{-p}\|x'-x\|^p &\leqq \alpha(t^{-1}\|x'-x\|)\\ &\leqq 1-t^{-1}\|(z_1+z_2)/2-a\|\\ &\leqq 1-t^{-1}N(y).\end{aligned}$$

So $\|x'-x\|^p \leqq C_1^p t^p(1-t^{-1}N(y))$ with $b^{-1}=C_1^p$.

Lemma 3. *Let $y\in E_2$ with $N(y)<u$. Let $x'\in E_1$ with $\|y+x'-a\|=N(y)$. Let $z\in E_1$ with $\|z\|=1$. Let*

$$B_z=\{s\in \mathbb{R}^+; x'+sz\in A_y\}$$

then diam $B_z\leqq 2(t-u)/(u-N(y))$.

Proof. Let $s_1\in\mathbb{R}^+$ be the smallest s with $x'+sz\in A_y$ and s_2 be the largest s with the same property. Since the function

$$\theta(s)=\|y+x'-a+sz\|$$

is continuous, and since its value at zero is $<u$, it follows that $\theta(s_1)=u$. Indeed we must have $\theta(s_1)\in\{u,t\}$. If $\theta(s_1)=t$, then there is $s\in[0,s_1[$ with $\theta(s)=u$, which contradicts the definition of s_1. We also have $\theta(s_2)=t$. Otherwise, since $\lim\theta(s)=\infty$, there is $s>s_2$ with $\theta(s)=t$. Let $y'=y+x'-a$. We have

$$\|y'\|=N(y),\quad \|y'+s_1 z\|=u,\quad \|y'+s_2 z\|=t \quad\text{and}\quad \|uy'/N(y)\|=u.$$

We also have $\|ut^{-1}y'+ut^{-1}s_2z\|=u$. It follows that for $\gamma\in[0,1]$ we have

$$\|(\gamma u/N(y)+(1-\gamma)ut^{-1})y'+(1-\gamma)ut^{-1}s_2z\|\leqq u.$$

Let us pick γ such that

$$(1-\gamma)ut^{-1}s_2=s_1(\gamma u/N(y)+(1-\gamma)ut^{-1})$$

that is

$$\gamma=(s_2-s_1)ut^{-1}/((s_2-s_1)ut^{-1}+s_1u/N(y)).$$

We get

$$(\gamma u/N(y)+(1-\gamma)ut^{-1})\|y'+s_1z\|\leqq u$$

and since $\|y'+s_1z\|=u$, we have

$$\gamma u/N(y)+(1-\gamma)ut^{-1}\leqq 1.$$

It follows by straight computation that

$$s_2\leqq s_1(t-N(y))/(u-N(y)).$$

Moreover, since $t\leqq 1$, we get $\|s_1z\|<\|x'+y-a\|+1$, so $|s_1|\leqq 2$.

We now prove Theorem A. We apply the preceding construction and estimates with $n=p+1$. We identify E_1 with R^n. Then μ_1 has bounded density with respect to Lebesgue's measure. Moreover on E_1 the norm of E and the Euclidean norm are equivalent.

For each y, we estimate $\mu_1(A_y)$. Suppose first that

$$N(y)\geqq t-2(t-u)=2u-t.$$

then Lemma 2 shows that A_y is contained in a ball centered at x' and of radius $\leqq C_1t(1-t^{-1}N(y))^{1/p}\leqq C_1t(2(1-u/t))^{1/p}$. The volume of this ball is less than $C_2t^n(1-u/t)\leqq C_2(t-u)$, since $n\geqq p$. So $\mu_1(A_y)\leqq C_3(t-u)$.

Suppose that $N(y)\leqq 2u-t$. We compute the volume of A_y using polar coordinates of center x'. It follows from Lemma 2 and 3 that this volume is less than

$$H=C_4t^{n-1}(1-t^{-1}N(y))^{n-1/p}(t-u)/(u-N(y)),$$

since $n-1=p$, for $t\leqq 1$, $H\leqq C_4(t-u)(t-N(y))/(u-N(y))$ and since $N(y)\leqq 2u-t$, $\mu_1(A_y)\leqq 2C_5(t-u)$.

In any case, we have $\mu_1(A_y)\leqq C_5(t-u)$. The Theorem then follows from (2).

Theorem C. *If E is a super-reflexive Banach space, there is an equivalent norm on E which satisfies $(*)$ for each Gaussian measure.*

3. Proof of Theorem B

The proof will use the ideas of [8]. We first need an auxiliary norm on $H=l^2(N)$.

Theorem D. *Let $0<\eta\leqq 1$. Let $(a_n)_{n\geqq 3}$ be a sequence of numbers between 0 and 1 with $a_n\to 0$. Then there is a norm N^0 on H and a sequence $b_n>0$ with the follow-*

ing properties:

(a) *for* $x \in H$, $\|x\| \leqq N^0(x) \leqq (1+\eta)\|x\|$.

(b) *The modulus of uniform continuity* α *of* N^0 *satisfies*

$$\alpha(\varepsilon) \geqq \varepsilon^n \quad \text{for } 1/16 \geqq \varepsilon \geqq a_n \text{ and } n \geqq 3.$$

(c) *Whenever* $x=(x_1, \ldots, x_n, 0, \ldots)$ *and* $y=(0, \ldots, y_{n+1}, 0, \ldots)$ *satisfy* $N^0(y)=1$ *and* $N^0(x) \leqq b_n$, *then*

$$1 \leqq N^0(x+y) \leqq 1 + \|x\|^n.$$

Proof. We can assume (a_n) decreasing and $a_3 \leqq 1/16$. We set $a_2 = 1/16$. For $q \geqq 3$, let n_q be the smallest integer such that $(1-\varepsilon^2/16)^{1/n_q} \geqq 1-\varepsilon^q$. Then $\lim n_q = \infty$. Let δ_n be a sequence with $\delta_n < 1$, such that $n > n_q \Rightarrow a_q^2/4 \geqq 4(1-\delta_n^2)$. Now let $\beta_n = (1+\delta_n)/2\delta_n > 1$. We can assume $\beta_n \leqq 1+\eta$. Let for $n \geqq 3$,

$$N_n(x) = (1/2(\sum_{i \neq n} x_i^2)^{n/2} + \beta_n |x_n|^n)^{1/n}$$

and $N^0(x) = \sup\{\|x\|, N_n(x), n \geqq 3\}$. First, since $N_n(x) \leqq \beta_n^{1/n}\|x\|$, (a) holds. Now we show that

$$|x_n| \leqq \delta_n \|x\| \Rightarrow N_n(x) \leqq \|x\|. \tag{3}$$

Indeed, we have

$$\begin{aligned} N_n^n(x) &\leqq 1/2(\sum_{i \neq n} x_i^2)^{n/2} + 1/2|x_n|^n + (\beta_n - 1/2)|x_n|^n \\ &\leqq \|x\|^n (1/2 + \delta_n(\beta_n - 1/2)) \leqq \|x\|^n. \end{aligned}$$

Now let ε with $a_{q-1} \geqq \varepsilon > a_q$ and $q \geqq 3$. Let $x, y \in H$ with $N^0(y) \leqq 1$, and $N^0(y) \leqq 1$ and $N^0(x-y) \geqq \varepsilon$. From (a), we have $\|x-y\| \geqq \varepsilon/2$, so we have

$$\|(x+y)/2\|^2 \leqq 1 - \varepsilon^2/16 \leqq 1 - \varepsilon^q.$$

Assume that there is a n such that

$$N_n((x+y)/2) \geqq \|(x+y)/2\|.$$

Then $\|x+y\| \geqq |x_n + y_n| \geqq 2\delta_n$. Since $\|x\|$ and $\|y\|$ are $\leqq 1$, we have

$$\varepsilon^2/4 \leqq \|x-y\|^2 \leqq 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 \leqq 4(1-\delta_n^2).$$

It follows that $n \leqq n_q$. We have

$$\begin{aligned} N_n^n((x+y)/2) \leqq\ & 1/2(1/2(\sum_{i \neq n} x_i^2 + \sum_{i \neq n} y_i^2) - \sum_{i \neq n} ((x_i - y_i)/2)^2)^{n/2} \\ & + \beta_n (1/2(x_n^2 + y_n^2) - ((x_n - y_n)/2)^2)^{n/2}. \end{aligned}$$

Using the inequality $|a-b|^n \leqq a^n - b^n$ for $0 \leqq b \leqq a$, we get

$$\begin{aligned} N_n^n((x+y)/2) \leqq\ & 1/2(1/2(\sum_{i \neq n} x_i^2 + \sum_{i \neq n} y_i^2))^{n/2} + \beta_n(1/2(x_n^2 + y_n^2))^{n/2} \\ & - 1/2(\sum_{i \neq n} ((x_i - y_i)/2)^2)^{n/2} - \beta_n |(x_n - y_n)/2|^n. \end{aligned}$$

Now we use the fact that $\psi(a,b)=(1/2a^{n/2}+\beta_n b^{n/2})^{2/n}$ is convex for a, $b\geqq 0$, and that $N_n(x)\leqq 1$ and $N_n(y)\leqq 1$:

$$N_n^n((x+y)/2)\leqq 1-1/2(\sum_{i\neq n}((x_i-y_i)/2)^2)^{n/2}-\beta_n|(x_n-y_n)/2|^n.$$

Since $\operatorname{Sup}\{\sum_{i\neq n}((x_i-y_i)/2)^2, ((x_n-y_n)/2)^2\}\geqq \varepsilon^2/8$, and $n\leqq n_q$, we get

$$N_n((x+y)/2)\leqq(1-\varepsilon^2/16)^{1/n}\leqq 1-\varepsilon^q.$$

This shows that $\alpha(\varepsilon)\geqq\varepsilon^q$.

To prove (c), notice that $|y_{n+1}|=\beta_{n+1}-1/(n+1)$. So

$$N_{n+1}(x+y)\leqq(1+1/2\|x\|^{n+1})^{1/(n+1)}\leqq 1+\|x\|^{n+1}.$$

Also

$$\|x+y\|=(\beta_{n+1}^{-2/(n+1)}+\|x\|^2)^{1/2}\leqq 1 \qquad \text{for } \|x\|\leqq b_n$$

if b_n is small enough. Finally, if $b_n\leqq 1/2$, then for all $q\neq n$,

$$N_q^q(x+y)\leqq 1/2\|x+y\|^{q/2}+\beta_q|x_q|^q\leqq 1/2+2|x_q|^q\leqq 1.$$

Theorem D is proved.

Proof of Theorem B. The basic observation is as follows. Let γ_n be the canonical Gaussian measure on l_n^2 (that is the product of n standard normal). Then the one-dimensional CLT shows that the law of $\|x\|^2$ is asymptotically $N(n, 3n^{-1/2})$. In particular, γ_n becomes very concentrated around the sphere of radius $\sqrt{n}$.

By induction over n, we construct a sequence $k(n)$ of integers, two sequences $\eta(n)$ and $\gamma(n)$ of reals and a Gaussian measure v_n on $l_{k(n)}^2$. Denote $q(n)=\sum_{i<n}k(i)$ and μ_n be the product measure of $v_1,\dots,v_{n-1}$ on $l_{q(n)}^2$, where the later space is identified with the Hilbertial sum $l_{k(1)}^2\oplus\dots\oplus l_{k(n-1)}^2$. We make this construction such that the following conditions are satisfied:

$$\eta(n)\leqq\gamma(n)/2, \quad \gamma(n)\leqq\eta(n-1)/4, \quad \eta(1)\leqq 1, \tag{4}$$

$$v_n\{x\in l_{k(n)}^2: \gamma(n)-\eta(n)<\|x\|\leqq\gamma(n)+\eta(n)\}\geqq 1-2^{-n-1}, \tag{5}$$

$$n\eta(n)\leqq\mu_n\{x\in l_{q(n)}^2; \|x\|\leqq 1/4\, b_{q(n)+1}\,\gamma(n)(\eta(n))^{1/(q(n)+1)}\}, \tag{6}$$

$$\text{the support of } v_n \text{ is } l_{k(n)}^2. \tag{7}$$

Notice that for some $a>0$,

$$\mu_n\{x\in l_{q(n)}^2; \|x\|\leqq t\}\geqq a\,t^{q(n)}$$

for $t\leqq 1$, since (7) implies that μ_n is equivalent to Lebesgue's measure. So the construction is possible by picking $\gamma(n)=\eta(n-1)/4$, then $\eta(n)$ small enough to force (6) and $k(n)$ large enough to ensure the existence of v_n in (5).

Let μ be the product measure on $\prod_n l^2_{k(n)}$. We identify H with the Hilbertian sum $\oplus l^2_{k(n)}$. So $x \in H$ is written $x=(x_n)$ with $x_n \in l^2_{k(n)}$. Since $\gamma(n) \leqq 2^{-n}$, $\eta(n) \leqq 2^{-n}$ from (4), (5) show that

$$\mu\{x \in H;\text{ for every } n,\ \|x_n\| \leqq 2^{-n+1}\} > 0.$$

So it follows that $\mu(H)=1$ by the $0-1$ law. For $x \in H$, let $\theta(x) \in H$, given by $\theta(x)_q = \|x_n\|$ if q is of the type $1+q(n)$, and $\theta(x)_q = 0$ otherwise. Let $N(x) = N^0(\theta(x))$. Then Theorem D ensures that the modulus α of uniform continuity satisfies the condition of Theorem B. We now estimate $\mu(D_n)$, where

$$D_n = \{x; \gamma(n) - 2\eta(n) \leqq N(x) \leqq \gamma(n) + 3\eta(n)\}.$$

Let

$$A = \{x; \|(x_1, \ldots, x_{n-1})\| \leqq 1/4\, b_{q(n)+1}\, \gamma(n)(\eta(n))^{1/(q(n)+1)}\}$$

$$B = \{x; \gamma(n) - \eta(n) \leqq \|x_n\| \leqq \gamma(n) + \eta(n)\}$$

and

$$C = \{x; \|(0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots)\| \leqq \eta(n)\}.$$

Then (6) implies that $\mu(A) > n\eta(n)$. From (5), we have

$$\mu(B) \geqq 1 - 2^{-n-1} \geqq 1/2,$$

and we also have since $\gamma(n+k) + \eta(n+k) \leqq 2\gamma(n+k) \leqq 2^{-k}\eta(n)$:

$$\mu(C) \geqq \prod_{k \geqq n+1} (1 - 2^{-k-1}) \geqq 1/2.$$

Since A, B and C are independent, we have $\mu(A \cap B \cap C) \geqq n/4\eta(n)$. We now show that $A \cap B \cap C \subset D_n$ which will finish the proof since the width of D_n is $5\eta(n)$.

Let $x \in A \cap B \cap C$, $x' = (x_1, \ldots, x_{n-1}, 0, \ldots)$ and $y = (0, \ldots, x_n, 0, \ldots)$. We have

$$\begin{aligned} N(x') &\leqq 2\|x'\| \leqq b_{q(n)+1}(\gamma(n)/2)\,\eta(n)^{1/(q(n)+1)} \\ &\leqq b_{q(n)+1}\,\eta(n)^{1/(q(n)+1)}\|y\| \\ &\leqq b_{q(n)+1}\,\eta(n)^{1/(q(n)+1)} N(y). \end{aligned}$$

Thus Theorem D, (c) implies

$$\begin{aligned} N(y) \leqq N(x'+y) &\leqq N(y) + (N(x')/N(y))^{q(n)+1} \\ &\leqq N(y) + (b_{q(n)+1}\,\eta(n)^{1/(q(n)+1)})^{q(n)+1} \\ &\leqq N(y) + \eta(n). \end{aligned}$$

Since $N(x - x' - y) \leqq \eta(n)$,

$$N(y) - \eta(n) \leqq N(x) \leqq N(y) + 2\eta(n).$$

The result follows.

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