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Probability Theory and Related Fields

On Characterizations of Distributions via Moments of Record Values

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Summary. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. random variables having continuous distribution $F(x)$ with $E|X|^{l+\varepsilon}<\infty$ for some positive integer l and for some $\varepsilon>0$. It is shown that for any fixed integer $N\geqq 0$ the sequence of moments of record values $\{E(X_{L(n)})^l\}_{n=N}^{\infty}$ characterizes F. Furthermore, this result is applied to the weak convergence of continuous distributions.

1. Introduction

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed (i.i.d.) random variables having continuous distribution $F(x)$. Define the sequence of record times $\{L(n)\}_{n=0}^{\infty}$ by $L(0)=1$ and $L(n)=\min\{j \mid X_j>X_{L(n-1)},\ j>L(n-1)\}$, $n\geqq 1$. Then the sequence $\{X_{L(n)}\}_{n=0}^{\infty}$ is called the sequence of record values of $\{X_n\}_{n=1}^{\infty}$. For applications of record values see Glick (1978) and the references in Gupta (1984).

Recently, Kirmani and Beg (1984) proved that for any fixed integer $N\geqq 0$, the set of expected record values $\{EX_{L(n)}\}_{n=N}^{\infty}$ characterizes the continuous distribution $F(x)$ provided $E|X|^p<\infty$ for some $p>1$. In this note, we extend the result above to the higher-order moment case (Theorem 1), and then study the relationship between the weak convergence of continuous distributions and the convergence of moments of record values (Theorems 2 and 3).

2. Lemmas

In order to prove the main results in Sects. 3 and 4, we need the following lemmas.

Lemma 1. *Let X be distributed by a continuous distribution $F(x)$ with $E|X|^s<\infty$ for some $s>0$, then $E|X_{L(n)}|^r<\infty$ for all $r\in(0, s)$ and $n\geqq 0$.*

Proof. Define the inverse function of a distribution $F(x)$ by $F^{-1}(t) = \inf\{x \mid F(x) \geqq t\}$, $t \in (0, 1)$. Since

$$E|X_{L(n)}|^r = \frac{1}{n!} \int_0^1 |F^{-1}(t)|^r \left(\log \frac{1}{1-t}\right)^n dt$$

(Nagaraja 1978 or Kirmani and Beg 1984), we show by Hölder's inequality that for all $r \in (0, s)$ and $n \geqq 0$,

$$\begin{aligned}\int_0^1 |F^{-1}(t)|^r &\left(\log \frac{1}{1-t}\right)^n dt \\ &\leqq \left\{\int_0^1 |F^{-1}(t)|^s \, dt\right\}^{r/s} \left\{\int_0^1 \left(\log \frac{1}{1-t}\right)^{ns/(s-r)} dt\right\}^{(s-r)/s} \\ &= (E|X|^s)^{r/s} \left\{\int_0^\infty x^{ns/(s-r)} e^{-x} \, dx\right\}^{(s-r)/s} \\ &= (E|X|^s)^{r/s} \{\Gamma(ns/(s-r)+1)\}^{(s-r)/s} < \infty.\end{aligned}$$

For the case $s > r = 1$, Lemma 1 reduces to a result of Nagaraja (1978, Lemma 1), and for the case $r = \sqrt{s} > 1$, it reduces to a result of Kirmani and Beg (1984, p. 464).

Lemma 2. *For continuous distributions F and G, $F = G$ iff $(F^{-1}(t))^l = (G^{-1}(t))^l$ a.e. (almost everywhere) on $(0, 1)$ for some positive integer l.*

Proof. Since the necessity is trivial, we only consider the sufficiency. For odd integer l the result is also trivial. For even l the result follows from the crucial point that the inverse function of a continuous distribution is strictly increasing and left continuous on $(0, 1)$.

Lemma 3. *For every $p > 1$, the sequence of functions $\left\{\left(\log \frac{1}{1-t}\right)^n\right\}_{n=0}^\infty$ is complete in $L_p(0, 1)$. Namely, if $f \in L_p(0, 1)$ satisfies*

$$\int_0^1 f(t) \left(\log \frac{1}{1-t}\right)^n dt = 0, \qquad n = 0, 1, 2, \ldots, \tag{1}$$

then $f(t) = 0$, a.e. on $(0, 1)$.

Proof. It is known that the sequence $\{x^n e^{-x}\}_{n=0}^\infty$ is complete in $L_2(0, \infty)$ (Goffman 1965, p. 193). In fact, by similar argument we can see that the sequence is complete in $L_p(0, \infty)$ for every $p > 1$, and hence for every $\lambda > 0$, the sequence $\{x^n e^{-\lambda x}\}_{n=0}^\infty$ is also complete in $L_p(0, \infty)$. Thus if $f \in L_p(0, 1)$ satisfies (1), then

$$\int_0^\infty f(1-e^{-x}) \, e^{-x/p} x^n e^{-(p-1)x/p} \, dx = 0, \qquad n = 0, 1, 2, \ldots. \tag{2}$$

Let $g(x) = f(1-e^{-x}) \, e^{-x/p}$, $x \in (0, \infty)$. Clearly $g \in L_p(0, \infty)$ and hence by (2), g vanishes a.e. on $(0, \infty)$. Therefore, $f(1-e^{-x}) = 0$, a.e. on $(0, \infty)$ or, equivalently, $f(t) = 0$, a.e. on $(0, 1)$. The proof is complete.

3. Characterizations of Continuous Distributions

We now extend the result of Kirmani and Beg (1984) to the following

Theorem 1. *For any two fixed integers* $N \geqq 0$ *and* $l > 0$, *the sequence of moments of record values* $\{E(X_{L(n)})^l\}_{n=N}^{\infty}$ *characterizes the continuous distribution* $F(x)$ *provided* $E|X|^{l+\varepsilon} < \infty$ *for some* $\varepsilon > 0$.

Proof. Let Y be distributed by a continuous G with $E|Y|^{l+\varepsilon} < \infty$ and $E(X_{L(n)})^l = E(Y_{L(n)})^l$ for every $n \geqq N$. Then

$$\begin{aligned} 0 &= E(X_{L(n)})^l - E(Y_{L(n)})^l \\ &= \frac{1}{n!} \int_0^1 h(t) \left(\log \frac{1}{1-t}\right)^{n-N} dt, \qquad n = N,\ N+1, \ldots, \end{aligned} \tag{3}$$

where $h(t) \equiv [(F^{-1}(t))^l - (G^{-1}(t))^l] \left(\log \frac{1}{1-t}\right)^N$. Note that $p \equiv (l + \frac{1}{2}\varepsilon)/l > 1$ and that $E|X|^{l+\varepsilon}$ and $E|Y|^{l+\varepsilon}$ are both finite, thus $h \in L_p(0, 1)$ by Hölder inequality. Therefore, by (3) and Lemma 3, $h(t) = 0$, a.e. on $(0, 1)$, and hence $(F^{-1}(t))^l = (G^{-1}(t))^l$, a.e. on $(0, 1)$. The desired result follows from Lemma 2.

4. Weak Convergence of Continuous Distributions

In this section, we shall apply Theorem 1 to the weak convergence of continuous distributions. Let $X, X^{(1)}, X^{(2)}, \ldots$ be a sequence of random variables having continuous distributions $F, F_1, F_2, \ldots$, respectively. Recall that the sequence $\{F_m\}_{m=1}^{\infty}$ converges weakly to a continuous F (denoted by $F_m \xrightarrow[m\to\infty]{w} F$) iff $F_m(x) \xrightarrow[m\to\infty]{} F(x)$ for every $x \in (-\infty, \infty)$ (Billingsley (1968)). We first consider the necessary condition of the weak convergence of $\{F_m\}_{m=1}^{\infty}$ (Theorem 2), and then the sufficient conditions (Theorem 3).

Theorem 2. *If* $F_m \xrightarrow[m\to\infty]{w} F$ *and* $E|X^{(m)}|^{l+\varepsilon} \xrightarrow[m\to\infty]{} E|X|^{l+\varepsilon} < \infty$ *for some positive integer* l *and for some* $\varepsilon > 0$, *then* $E(X_{L(n)}^{(m)})^l \xrightarrow[m\to\infty]{} E(X_{L(n)})^l$ *for all* $n \geqq 0$.

Proof. Note that $F_m \xrightarrow[m\to\infty]{w} F$ implies $F_m^{-1}(t) \xrightarrow[m\to\infty]{} F^{-1}(t)$, a.e. on $(0, 1)$ (Serfling 1980, p. 21), and hence by the assumptions,

$$\int_0^1 |(F_m^{-1}(t))^l - (F^{-1}(t))^l|^{(l+\varepsilon)/l}\, dt \xrightarrow[m\to\infty]{} 0 \tag{4}$$

(Royden 1968, p. 118). Now, for all $n \geqq 0$ and for all m sufficiently large,

$$\begin{aligned} &|E(X_{L(n)}^{(m)})^l - E(X_{L(n)})^l| \\ &\quad \leqq \frac{1}{n!} \int_0^1 |(F_m^{-1}(t))^l - (F^{-1}(t))^l| \left(\log \frac{1}{1-t}\right)^n dt \\ &\quad \leqq \frac{1}{n!} \left\{ \int_0^1 |(F_m^{-1}(t))^l - (F^{-1}(t))^l|^{(l+\varepsilon)/l}\, dt \right\}^{l/(l+\varepsilon)} \\ &\qquad \cdot \{\Gamma(n(l+\varepsilon)/\varepsilon + 1)\}^{\varepsilon/(l+\varepsilon)} \xrightarrow[m\to\infty]{} 0, \end{aligned}$$

in which the second inequality follows from Hölder's inequality, and the last assertion from (4).

Theorem 3. *Assume that* $\{E|X^{(m)}|^{l+\varepsilon}\}_{m=1}^{\infty}$ *is bounded for some positive integer* l *and for some* $\varepsilon>0$, *and that* $E(X_{L(n)}^{(m)})^l \xrightarrow[m\to\infty]{} \mu_n$ *for all* $n \geqq N \geqq 0$, *where* N *is a fixed integer. In addition, the distributions* $\{F_m\}_{m=1}^{\infty}$ *are equicontinuous. Then there exists a random variable* X *distributed by a continuous* F *such that* $F_m \xrightarrow[m\to\infty]{w} F$ *and* $\mu_n = E(X_{L(n)})^l$ *for all* $n \geqq N$.

Proof. The boundedness of the $l+\varepsilon$-th absolute moments of $\{X^{(m)}\}_{m=1}^{\infty}$ implies (Loève 1977, p. 186) that each subsequence $\{F_{m'}\}$ of $\{F_m\}$ contains another subsequence $\{F_{m''}\}$ converging weakly to a distribution F, say, obeyed by a random variable X. Since $\{F_{m''}\}$ are equicontinuous, F should be also continuous (see Royden (1968, p. 178)). Further, applying the well-known Moment Convergence Theorem (Chow and Teicher 1978, p. 254) we know

$$E|X^{(m'')}|^{l+\varepsilon/2} \xrightarrow[m''\to\infty]{} E|X|^{l+\varepsilon/2},$$

and hence $E(X_{L(n)})^l = \mu_n$ for all $n \geqq N$ by Theorem 2. This means that each converging subsequence of $\{F_m\}$ has the same limiting continuous distribution $F(x)$ by Theorem 1. Therefore $F_m \xrightarrow[m\to\infty]{w} F$ and $E(X_{L(n)})^l = \mu_n$ for all $n \geqq N$.

5. Remarks

Let $X_{1,n} \leqq X_{2,n} \leqq \ldots \leqq X_{n,n}$ be the order statistics of i.i.d. random variables $\{X_i\}_{i=1}^{n}$ from *any* distribution $F(x)$ with finite mean. It is known (Huang 1975) that the set of expected maximal order statistics $\{EX_{n_j,n_j}\}_{j=1}^{\infty}$ characterizes F if

$$\sum_{j=1}^{\infty} 1/n_j = \infty \quad \text{and} \quad 0 < n_1 < n_2 < \ldots. \tag{5}$$

It is natural to ask whether the result of Kirmani and Beg (1984) can be extended to a subsequence $\{EX_{L(n_j)}\}_{j=1}^{\infty}$, where $\{n_j\}_{j=1}^{\infty}$ satisfies (5). The answer is negative due to Lin and Huang (1986). They also point out that the result of Kirmani and Beg (1984) cannot be extended to a wider class of distributions containing discrete case.

On the other hand, the set of all even moments of all order statistics

$$\{E(X_{k,n})^l \mid n \geqq 1, 1 \leqq k \leqq n, l = 2, 4, 6, 8, \ldots\}$$

cannot characterize distribution F due to the following example: Let $P(X=1) = P(Y=-1) = 1$, then $E(X_{k,n})^l = E(Y_{k,n})^l = 1$ for all k, n and even l. Hwang and Lin (1984a) proved that for any fixed even l, the set

$$\{EX\} \cup \{E(X_{n_j,n_j})^l\}_{j=1}^{\infty} \tag{6}$$

characterizes any distribution F with $EX^l < \infty$, where $\{n_j\}_{j=1}^{\infty}$ satisfies (5). However, the mean EX in (6) is redundant for characterizing continuous distribution. This is a simple application of Lemma 2 and Müntz-Szász Theorem (Hwang and Lin 1984b).

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