Probab. Th. Rel. Fields 80, 315–322 (1988)

Probability Theory and Related Fields

Autoregressive Representations of Multivariate Stationary Stochastic Processes *

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Abstract. Consider a q-variate weakly stationary stochastic process $\{X_n\}$ with the spectral density W. The problem of autoregressive representation of $\{X_n\}$ or equivalently the autoregressive representation of the linear least squares predictor of X_n based on the infinite past is studied. It is shown that for every W in a large class of densities, the corresponding process has a mean convergent autoregressive representation. This class includes as special subclasses, the densities studied by Masani (1960) and Pourahmadi (1985). As a consequence it is shown that the condition $W^{-1} \in L^1_{q \times q}$ or minimality of $\{X_n\}$ is dispensable for this problem. When W is not in this class or when W has zeros of order 2 or more, it is shown that $\{X_n\}$ has a mean Abel summable or mean compounded Cesáro summable autoregressive representation.

1. Introduction

While it is well-known that every purely nondeterministic q-variate weakly stationary stochastic process (SSP) $\{X_n\}$ with the spectral density W has an (infinite order) one-sided moving average representation, not every such process can have a mean convergent (infinite order) autoregressive representation (ARR) and the problem of ARR of such processes has not received the attention which it deserves. Due to the importance of ARR in prediction theory, and particularly in the statistical theory of multivariate time series, this paper is devoted to the problem of finding the weakest condition on W which guarantees the existence of an infinite order ARR for $\{X_n\}$.

To be more precise, we say that the SSP $\{X_n\}$ has a mean convergent (summable) ARR if there exists a sequence $\{A_k\}_{k=1}^{\infty}$ of constant $q \times q$ matrices such that, for each n,

$$X_n = \sum_{k=1}^{\infty} A_k X_{n-k} + \varepsilon_n, \tag{1.1}$$

* Research supported by the NSF Grant MCS-8301240 and the AFOSR Grant F49620 82 C0009. This work was done while the author was visiting Center for Stochastic Processes, University of North Carolina, Chapel Hill

where $\{\varepsilon_n\}$ is the innovation process of $\{X_n\}$ and the infinite series $\sum_{k=1}^{\infty} A_k X_{n-k}$ is to be convergent (summable) in the mean. *This representation of the process* $\{X_n\}$ *as an infinite order stochastic difference equation can also be regarded as the inversion of the one-sided moving average representation of* $\{X_n\}$. *Such inversion of the one-sided moving average representation of a q-variate* SSP *plays a vital role in the statistical estimation of the parameters of* $\{X_n\}$. For the notation and definitions see [9, 10, 13].

It is obvious that the problem of ARR $\{X_n\}$, cf. (1.1), is equivalent to the problem of ARR of $\hat{X}_{n|n-1}$ (the linear least squares predictor of X_n based on $\{X_{n-k}; k\geqq 1\}$):

$$\hat{X}_{n|n-1}=\sum_{k=1}^{\infty} A_k X_{n-k}, \tag{1.2}$$

which has been studied by Wiener and Masani [16], and Masani [7].

It follows from the isomorphism between the time and spectral domains of $\{X_n\}$ that the infinite series (1.1) or (1.2) is mean convergent (summable), if and only if the isomorph of ε_n in $L^2(W)$ has a convergent (summable) Fourier series in the norm of $L^2(W)$. For a purely nondeterministic full rank SSP $\{X_n\}$ with the spectral density $W=\Phi\Phi^*$ and G the one-step ahead prediction error matrix, it is well-known [16, II, p. 115] that the function $G^{1/2}\Phi^{-1}e^{in\theta}$ in $L^2(W)$ is the isomorph of ε_n in $M(X)$. Thus, the series in (1.1) is mean convergent (summable), if and only if the Fourier series of Φ^{-1} is convergent (summable) to Φ^{-1} in the norm of $L^2(W)$. Also, it can be shown that the $(v+1)$-step ahead $(v\geqq 0)$ linear least squares predictor $\hat{X}_{n+v|n-1}$ based on $\{X_{n-k}; k\geqq 1\}$ has a mean convergent (summable) ARR, if and only if the Fourier series of Φ^{-1} is convergent (summable) to Φ^{-1} in the norm of $L^2(W)$. (This latter assertion for the univariate processes is proved by Miamee and Salehi [11] in the spectral domain and by Bloomfield [3] in the time domain.)

From the previous discussion *the convergence (summability) of the Fourier series of* Φ^{-1} *to* Φ^{-1} *in the norm of* $L^2(W)$ *emerges as the spectral necessary and sufficient condition for the existence of a mean convergent (summable)* ARR *of* $\{X_n\}$. Although this condition is not concrete in terms of W, it is useful in obtaining some concrete sufficient conditions in terms of W for the ARR of $\{X_n\}$. These conditions are stated and proved in Sect. 2 by using some techniques from harmonic analysis. In the following we state and discuss the implication of these conditions for the problem of ARR of $\{X_n\}$.

The condition

$$W\in L^{\infty}_{q\times q}, \qquad W^{-1}\in L^{1}_{q\times q}, \tag{1.3}$$

is known to be sufficient for the existence of a mean convergent ARR of $\{X_n\}$. It was pointed out by Masani [7, p. 143] that the condition $W\in L^{\infty}_{q\times q}$ is unduly strong and it would be worthwhile to relax it, see also Feldman [4].

In [13] the author has shown that, indeed, the restriction $W\in L^{\infty}_{q\times q}$ is dispensable. This is done by employing the equivalence between the convergence of Fourier series of all functions in $L^2(W)$ and the positivity of the angle θ between the "past and present" subspace and the "future" subspace of $\{X_n\}$, cf. Theorem 2.1. Thus, from Theorem 2.1 $\{X_n\}$ has a mean convergent ARR if

$$\theta > 0 (\text{or } \rho(W) < 1), \tag{1.4}$$

where $\theta = \cos^{-1} \rho(W)$ and

$$\rho(W) = \sup\{|((Y, Z))|;\ Y \in M_{-\infty}^{0}(X), Z \in M_{1}^{\infty}(X) \text{ and } \|Y\| \leqq 1, \|Z\| \leqq 1\}.$$

It is clear that $0 \leqq \rho(W) \leqq 1$. The past-present and future are said to be at *positive angle* if $\rho(W) < 1$.

Our first result in this paper shows that the condition $W^{-1} \in L^1_{q \times q}$ or minimality of $\{X_n\}$ in (1.3) is dispensable for the existence of a mean convergent ARR of $\{X_n\}$. Also this result gives a more general sufficient condition for the mean convergent ARR of $\{X_n\}$ which includes both (1.3) and (1.4) as special cases (cf. Theorem 2.2).

To state this result and for later use we denote the class of densities satisfying (1.3) by M, those satisfying (1.4) by A and define a new class $A \otimes M$ by $A \otimes M = \{W;\ W = W_1^{1/2} W_2 W_1^{1/2}, W_1 \in A \text{ and } W_2 \in M\}$, where $W_1^{1/2}$ denotes the positive square root of W_1. It is evident that by choosing $W_1 = I (W_2 = I)$ this new class has $M(A)$ as its proper subset.

It is easy to check that a W in $A \otimes M$ does not necessarily have the property $W^{-1} \in L^1_{q \times q}$. (As an example when $q = 1$, one can take $W = |1 - e^{i\theta}|^{\lambda}$, $1 \leqq \lambda < 2$.) But, for $W \in A \otimes M$, W^{-1} is necessarily in $L^{1/2}_{q \times q}$. Note that the scalar density $W = |1 - e^{i\theta}|^2$ which corresponds to the univariate SSP $X_n = \varepsilon_n - \varepsilon_{n-1}$ does not belong to the univariate version of $A \otimes M$. Thus, Theorem 2.2 does not provide any information concerning the existence of a mean convergent ARR for this process. However, this $\{X_n\}$ can not have a mean convergent ARR, since in this case the infinite series in (1.1), i.e. $-\sum_{k=1}^{\infty} X_{n-k}$ does not converge in the mean. This example shows that processes $\{X_n\}$ for which W is not in $A \otimes M$ (or in other words if W has zeros of order 2 or more), can not have mean convergent ARR. In view of this it is natural to ask whether such processes can have an ARR with a weaker requirement of convergence, say summability, for the infinite series $\sum_{k=1}^{\infty} A_k X_{n-k}$ in (1.1). Theorems 2.4(c) and 2.5(b) show that this is actually possible for a large class of processes when we replace the mean convergence of the series (1.1) by its mean Abel summability, and compounded Cesáro summability.

Finally, we would like to note that Theorems 2.4(c) and 2.5(c) show the existence of a mean Abel (compounded Cesáro) summable ARR of $\{X_n\}$ only when the spectral density matrix has a finite number of zeros (of any finite orders) on $[-\pi, \pi]$. As yet, we do not have any information on the ARR of a process whose spectral density has a zero of infinite order. For $q = 1$, $W(\theta) = \exp\{-|\theta|^{-\lambda}\}$, $0 < \lambda < 1$, provides a family of such densities.

2. Mean convergence of the Fourier Series of Φ^{-1} in $L^2(W)$

As noted in Sect. 1, mean convergence of the Fourier series of Φ^{-1} in $L^2(W)$ emerges as the necessary and sufficient condition for the autoregressive representation of a purely nondeterministic full rank process $\{X_n\}$ with the spectral density W. This section is devoted to finding useful sufficient conditions on W which guarantee the mean convergence of the Fourier series of Φ^{-1} and many more functions in $L^2(W)$.

We note that for a general density W, Φ^{-1} is not necessarily in $L^1_{q\times q}$ and therefore the Fourier coefficients of Φ^{-1} are not well-defined. In the next theorem we need a condition on W such that $L^2(W)\subset L^1_{q\times q}$. Under this condition the Fourier coefficients of every function in $L^2(W)$ is well-defined. It can be shown [10, 13, 14] that $L^2(W)\subset L^1_{q\times q}$ if $W^{-1}\in L^1_{q\times q}$, and only if $(\det W)^{-1/2q}\in L^1$. Thus this assumption is weaker than $\rho(W)<1$.

The next theorem which is a multivariate extension of a deep theorem of Helson and Szegö [6, p. 131] provides a necessary and sufficient condition for the mean convergence of the Fourier series of every function in $L^2(W)$, cf. [10, 13, 14].

Theorem 2.1. *Let W be a $q\times q$ matrix-valued density function with* $\log\det W\in L^1$. *Then $\rho(W)<1$, if and only if $L^2(W)\subset L^1_{q\times q}$ and the Fourier series of any Ψ in $L^2(W)$ converges to Ψ in the norm of $L^2(W)$.*

Next we find a weaker sufficient condition on W which guarantees the convergence of the Fourier series of every function in a small subclass S of $L^2(W)$. For a density W, we define $S=\{\Psi\in L^2(W);\ \Psi W\Psi^*\in L^\infty_{q\times q}\}$. Note that this class has Φ^{-1} as one of its elements. In the following theorem we show that if $W\in A\otimes M$, then the Fourier series of every Ψ in S converges to Ψ in the norm of $L^2(W)$. This theorem is a matricial extension of a similar univariate result due to Bloomfield [3].

Theorem 2.2. *Let W be a $q\times q$ matrix-valued density function and $S=\{\Psi\in L^2(W);$ $\Psi W\Psi^*\in L^\infty_{q\times q}\}$. If $W\in A\otimes M$, then the Fourier series of every function $\Psi\in S$ converges to Ψ in the norm of $L^2(W)$.*

Proof. Let ω_1 and ω_q denote the smallest and largest eigenvalue of W_2. Then for $W\in A\otimes M$ we have

$$\omega_1 W_1\leqq W\leqq\omega_q W_1 \qquad \text{a.e. (Leb.)}, \tag{2.1}$$

and for $\Psi\in S$ we have

$$\int \operatorname{tr}\Psi W_1\Psi^*\,dm\leqq\int\omega_1^{-1}\operatorname{tr}\Psi W\Psi^*\,dm\leqq\|\operatorname{tr}\Psi W\Psi^*\|_\infty\int\omega_1^{-1}\,dm<\infty,$$

which proves that $\Psi\in S$ implies $\Psi\in L^2(W_1)$.

Since $\Psi\in L^2(W_1)$ and $\rho(W_1)<1$, it follows from Theorem 2.1 (and the argument preceeding it) that the Fourier series of every $\Psi\in S$ converges to Ψ in the norm of $L^2(W_1)$; i.e., with S_n^Ψ denoting the symmetric n-th partial sum of the Fourier series of Ψ we have

$$\|\Psi-S_n^\Psi\|_{W_1}\to 0 \text{ as } n\to\infty. \tag{2.2}$$

To finish the proof we need to show that $\|\Psi - S_n^{\Psi}\|_W \to 0$ as $n \to \infty$. But this is the consequence of (2.2) and the inequality

$$\|\Psi - S_n^{\Psi}\|_W^2 \leqq \|\omega_q\|_\infty \|\Psi - S_n^{\Psi}\|_{W_1}^2$$

which follows from (2.1). Q.E.D.

For a density W we define $H^2(W) = \overline{sp}\{e^{in\theta} I; n \geqq 0\}$ in $L^2(W)$. Note that $\Phi^{-1} \in H^2(W)$. Also we would like to note that for a general W, Φ^{-1} and other elements of $H^2(W)$ are not necessarily in $L^1_{q \times q}$ and therefore their Fourier coefficients are not well-defined. By an argument similar to that of Rosenblum [15, p. 41] one can identify the elements of $H^2(W)$ with analytic matrix-valued functions, and thus here by the Fourier coefficients of elements of $H^2(W)$ we actually mean the Taylor coefficients of these analytic functions.

Let $P_r(\cdot)$ denote the Poisson kernel, i.e., $P_r(\theta) = 1 - r^2/1 - 2r \cos\theta + r^2$, $0 < r < 1$ and $-\pi < \theta \leqq \pi$. For a function $\Psi \in H^2(W)$ with Fourier (Taylor) coefficients $\{\Psi_k\}_{k=0}^{\infty}$ the convolution of P_r and Ψ is defined (and denoted) by

$$\Psi_r(e^{i\theta}) = \Psi(r e^{i\theta}) = (P_r * \Psi)(e^{i\theta}) = \sum_{k=0}^{\infty} \Psi_k r^k e^{ik\theta}. \tag{2.3}$$

We say that the Fourier (Taylor) series of a function $\Psi \in H^2(W)$ is Abel summable to Ψ in the norm of $L^2(W)$, if and only if

$$\lim_{r \to 1^-} \|\Psi_r - \Psi\|_W = 0. \tag{2.4}$$

It follows from the isomorphism between the time and spectral domains that the autoregressive representation of $\{X_n\}$ is mean Abel summable, if and only if the Fourier (Taylor) series of Φ^{-1} is Abel summable to Φ^{-1} in the norm of $L^2(W)$.

Next, following the pattern of Theorems 2.1 and 2.2, we find a necessary and sufficient condition on W for the Abel summability of the Fourier (Taylor) series of every function in $H^2(W)$. For this the matrix-valued function Q defined (in terms of Φ) by

$$Q(\theta) = Q(r, \theta, \Phi) = \Phi^{-1}(r e^{i\theta})\, \Phi(e^{i\theta}), \qquad -\pi < \theta \leqq \pi, \tag{2.5}$$

plays an important role. The following theorem is actually a matricial extension of some of the (univariate) results due to Rosenblum [15, Theorem 1 (ii)].

Theorem 2.3. *Let $W = \Phi\Phi^*$ be a $q \times q$ matrix-valued density function with $\log\det W \in L^1$. Then the following statements are equivalent.*

(a) *The Fourier (Taylor) series of every function Ψ in $H^2(W)$ is Abel summable to Ψ in the norm of $L^2(W)$.*

(b) *There exists a constant K_1, $0 < K_1 < \infty$, such that for all functions $\Psi \in H^2(W)$ we have*

$$\|P_r * \Psi\|_W \leqq K_1 \|\Psi\|_W, \qquad 0 < r < 1. \tag{2.6}$$

(c) *There exists a constant K_2, $0 < K_2 < \infty$, such that*

$$(P_r * \operatorname{tr} QQ^*)(\theta) \leqq K_2, 0 < r < 1 \quad \text{and} \quad -\pi < \theta \leqq \pi. \tag{2.7}$$

Proof. (a)⇒(b) follows from the uniform boundedness principle. (a)⇒(b) follows from an argument similar to that given in Rosenblum [15, pp. 32–33].

To prove that (b) implies (c), we note that for each $0<r<1$ and $-\pi<x\leqq\pi$ the function $\Psi(\theta)=\Psi(\theta,r,x)=(1-re^{i(\theta-x)})^{-1}\Phi^{-1}(\theta)$ is in $H^2(W)$. This is a consequence of the closure theorem for $H^2_{q\times q}$, cf. [8, p. 288]. By using the simple inequality $4(1-r^2e^{i\theta}|^{-2}\geqq P_r(\theta)$, and applying (2.6) to this function Ψ we get (2.7):

$$\begin{aligned}
&\int_{-\pi}^{\pi} P_r(\theta-x)\operatorname{tr}\Phi^{-1}(re^{i\theta})\,\Phi(\theta)\,\Phi^*(\theta)\,\Phi^*(re^{i\theta})\,\mathrm{dm}(\theta)\\
&\leqq 4(1-r^2)\int_{-\pi}^{\pi}|1-r^2e^{i(\theta-x)}|^{-2}\operatorname{tr}\Phi^{-1}(re^{i\theta})\,W(\theta)\,\Phi^*(re^{i\theta})\,\mathrm{dm}(\theta)\\
&\leqq 4(1-\mathrm{r}^2)\,K_1\int_{-\pi}^{\pi}|1-re^{i(\theta-x)}|^{-2}\operatorname{tr}\Phi^{-1}(\theta)\,W(\theta)\,\Phi^{*-1}(re^{i\theta})\,\mathrm{dm}(\theta)\\
&=4qK_1\int_{-\pi}^{\pi}P_r(\theta-x)\,\mathrm{dm}(\theta)=4qK_1.
\end{aligned}$$

To prove (c) implies (b), we note that for any $\Psi\in H^2(W)$ [using the Cauchy-Schwartz inequality, Fubini's theorem and (2.7)] that

$$\begin{aligned}
\|P_r*\Psi\|_W&=\int_{-\pi}^{\pi}\operatorname{tr}\Psi(re^{i\theta})\,\Phi(\theta)\,\Phi^*(\theta)\,\Psi^*(re^{i\theta})\,\mathrm{dm}(\theta)\\
&=\int_{-\pi}^{\pi}\operatorname{tr}\Psi(re^{i\theta})\,\Phi(re^{i\theta})\,Q(\theta)\,Q^*(\theta)\,[\Psi(re^{i\theta})\,\Phi(re^{i\theta})]^*\,\mathrm{dm}(\theta)\\
&\leqq\int_{-\pi}^{\pi}\operatorname{tr}QQ^*\cdot\operatorname{tr}\Psi(re^{i\theta})\,\Phi(re^{i\theta})\,[\Psi(re^{i\theta})\,\Phi(re^{i\theta})]^*\,\mathrm{dm}(\theta)\\
&=\sum_{k,l=1}^{q}\int_{-\pi}^{\pi}\operatorname{tr}QQ^*\left|\sum_{j=1}^{q}\Psi_{kj}(re^{i\theta})\,\Phi_{jk}(re^{i\theta})\right|^2\mathrm{dm}(\theta)\\
&=\sum_{k,l=1}^{q}\int_{-\pi}^{\pi}\operatorname{tr}QQ^*\left|\int_{-\pi}^{\pi}P_r(\theta-x)\sum_{j=1}^{q}(\Psi_{kj}\,\Phi_{jl})(x)\,\mathrm{dm}(x)\right|^2\mathrm{dm}(\theta)\\
&\leqq\sum_{k,l=1}^{q}\int_{-\pi}^{\pi}\operatorname{tr}QQ^*\left(\int_{-\pi}^{\pi}P_r(\theta-x)\left|\sum_{j=1}^{q}\Psi_{kj}\,\Phi_{jl}(x)\right|^2\mathrm{dm}(x)\right)\mathrm{dm}(\theta)\\
&=\int_{-\pi}^{\pi}\operatorname{tr}\Psi W\Psi^*\left(\int_{-\pi}^{\pi}P_r(\theta-x)\operatorname{tr}QQ^*(\theta)\,\mathrm{dm}(\theta)\right)\mathrm{dm}(x)\\
&\leqq K_2\|\Psi\|_W.\quad\text{Q.E.D.}
\end{aligned}$$

Although Theorem 2.3 provides two equivalent necessary and sufficient conditions for the Abel summability of the Fourier (Taylor) series of every function in $H^2(W)$, these conditions are hard to apply and are not explicit in terms of the components of W. In the next theorem by using Theorems 2.1, 2.2 and

2.3 we provide sufficient conditions for the Abel summability of the Fourier (Taylor) series of all functions in $H^2(W)$ and in particular Φ^{-1}, which are easy to apply and more explicit in terms of the components of W.

Theorem 2.4. *Let W be a $q\times q$ matrix-valued density function with* $\log\det W\in L^1$ *and P be a complex-valued trigonometric polynomial of some degree n.*

(a) *If $W\in A$, then the Fourier (Taylor) series of every function $\Psi\in H^2$ is Abel summable to Ψ in the norm of $L^2(W)$.*

(b) *Let $W'\in A$. If $W=|P|^2 W'$, then the Fourier (Taylor) series of every function $\Psi\in H^2(W)$ is Abel summable to Ψ in the norm of $L^2(W)$.*

(c) *Let $S=\{\Psi\in H^2(W);\ \Psi W\Psi^*\in L^\infty_{q\times q}\}$ and $W'\in A\otimes M$. If $W=|P|^2 W$, then the Fourier (Taylor) series of every function $\Psi\in S$ is Abel summable to Ψ in the norm of $L^2(W)$.*

Proof. In view of Theorems 2.1 and 2.3, proofs of (a) and (b) are the same as the proof of Lemma 6 in [12]. (c) follows from (b) by using the method of proof of Theorem 2.2 and replacing S_n^Ψ by Ψ_r. Q.E.D.

Now, we turn to the problem of Cesaro summability of the Fourier (Taylor) series of functions in $H^2(W)$. Let $\Psi\in H^2(W)$ with Fourier (Taylor) coefficients $\{\Psi_k\}_0^\infty$ and partial sums $S_n(\theta)=\sum_{k=0}^{n}\Psi_k e^{ik\theta}$. For $\alpha>0$, we say that the Fourier (Taylor) series of Ψ is (C,α) summable to Ψ in the norm of $L^2(W)$ if

$$\lim_{n\to\infty}\left\|\sum_{k=0}^{n}\binom{n-k+\alpha-1}{n-k}\binom{n+\alpha}{n}^{-1}S_k-\Psi\right\|_W=0. \tag{2.8}$$

For scalar sequences it is well-known that the strength of (C,α) methods increases with α. However, there are series which are Abel summable but not (C,α)-summable for any $\alpha>0$, cf. [5, p. 108]. Because of this and in view of the importance of relations like (2.8) in prediction of $\{X_n\}$, cf. Sect. 1, we consider the stronger method of compounded Cesáro summability method: Let $\{\alpha_n\}_0^\infty$ be a (fixed) monotone increasing sequence of positive numbers. We say that the Fourier (Taylor) series of $\Psi\in H^2(W)$ is compounded Cesáro summable to Ψ if

$$\lim_{n\to\infty}\left\|\sum_{k=0}^{n}\binom{n-k+\alpha_n-1}{n-k}\binom{n+\alpha_n}{n}^{-1}S_k-\Psi\right\|_W=0.$$

It is known [1] that the compounded Cesáro summability method is regular, if and only if $\lim_{n\to\infty}\frac{\alpha_n}{n}=0$. For more information on the subject of summability and the definition of undefined terms the reader may refer to [5].

The next theorem establishes analogue of Theorem 2.4 for the compounded Cesáro summability.

Theorem 2.5. *Let W be a $q\times q$ matrix-valued density function with* $\log\det W\in L^1$, *P be a complex-valued trigonometric polynomial of some degree n and $\{\alpha_n\}$ a (fixed) monotone increasing sequence of positive numbers with* $\lim_{n\to\infty}\frac{\alpha_n}{n}=0$.

(a) *Let $W' \in A$. If $W = |P|^2 W'$, then the Fourier (Taylor) series of every function $\Psi \in H^2(W)$ is compounded Cesáro summable to Ψ in the norm of $L^2(W)$.*

(b) *Let $W' \in A \otimes M$ and S be as in part* (c) *of Theorem 2.4. If $W = |P|^2 W'$, then the Fourier (Taylor) series of every $\Psi \in S$ is compounded Cesáro summable to Ψ in the norm of $L^2(W)$.*

Proof. We prove only part (a) since the proof of part (b) is the same as the proof of Theorem 2.4(c).

To prove (a) first we show that every $\Psi \in H^2(W)$ has radial limits a.e. (Leb.). Let Φ and Φ_1 be the optimal factors of $|P|^2$ and W', respectively. Then Φ the optimal factor of $W = |P|^2 W'$ is given by $\phi \Phi_1$. Since $W' \in A$ it follows that $\Phi_1^{-1} \in H^{1/2}_{q \times q}$, and since ϕ is an analytic polynomial of finite degree with no zeros in the open unit disc we have that $\Phi_1^{-1} \in H^\delta$, $0 < \delta < 1/2$. Thus, $\Phi^{-1} \in H^{p_1}_{q \times q}$, $1/p_1 = 2 + 1/\delta$. Now, we note that every $\Psi \in H^2(W)$ has a representation of the form $\Psi = h\Phi^{-1}$, where $h \in H^2_{q \times q}$, and this entails that $\Psi \in H^p_{q \times q}$ with $1/p = 1/2 + 1/p_1$. Therefore, every $\Psi \in H^2(W)$ belongs to $H^p_{q \times q}$, for some $p > 0$, which implies that Ψ has radial limits a.e. The rest of the proof follows from Theorem 2.4(b) and adopting the method of proof of Theorems 6.1 and 6.3 [1]. Q.E.D.

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Received January 30, 1985; in revised form March 9, 1988