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Subexponential Distributions and Characterizations of Related Classes

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Summary. Let $\mathscr{S}(\gamma)$, $\gamma \geqq 0$, denote the class of distributions F satisfying

(i) $\lim_{x\to\infty} \bar{F}^{2*}(x)/\bar{F}(x) = 2\int_0^\infty e^{\gamma y}\,dF(y) < \infty$

(ii) $\lim_{x\to\infty} \bar{F}(x-y)/\bar{F}(x) = e^{\gamma y} \quad \forall y \in \mathbb{R}$.

The classes $\mathscr{S}(\gamma)$, for $\gamma > 0$, are characterized by means of subexponential densities. As an application we derive a result on the asymptotic behaviour of densities of random sums. In particular for an $M/G/1$ queue, we relate the tail behaviour of the stationary waiting time density to that of the service time distribution.

§ 1. Notations, Introduction

We consider proper distribution functions concentrated on $(0, \infty)$ having unbounded support. With F and G we shall always denote those distribution functions. For F and G the convolution product $*$ is defined as

$$F * G(x) := \int_0^x F(x-y)\,dG(y) = \int_0^x G(x-y)\,dF(y).$$

For $n \in \mathbb{N}$ denote by F^{n*} the n-th convolution product of F and by $\bar{F}^{n*} := 1 - F^{n*}$ the tail of F^{n*}.

If F and G are absolutely continuous with densities f and g, then $F * G$ is absolutely continuous with density

$$f \otimes g(x) := \int_0^x f(x-y)\,g(y)\,dy = \int_0^x g(x-y) f(y)\,dy.$$

For $n \in \mathbb{N}$ denote by $f^{n\otimes}$ the density of F^{n*}.

If F is absolutely continuous with density f, then $r_F = f/\bar{F}$ is called the *hazard rate* of F.

The moment generating function of F is denoted by

$$\hat{f}(s)=\int_0^\infty e^{sy}\,dF(y).$$

If F has a finite expectation it will be denoted by $\mu(F)$.

One aim of this paper is a characterization of the following class of convolution-equivalent distributions:

Definition 1. A distribution function F belongs to the class $\mathscr{S}(\gamma)$ with $\gamma\geqq 0$ if

(i) $\lim\limits_{x\to\infty} \bar{F}^{2*}(x)/\bar{F}(x)=2d<\infty$

(ii) $\lim\limits_{x\to\infty} \bar{F}(x-y)/\bar{F}(x)=e^{\gamma y} \quad \forall y\in\mathbb{R}.$

The class $\mathscr{S}:=\mathscr{S}(0)$ is called the class of *subexponential distributions.*

Using Banach algebra methods Chover, Ney and Wainger (1973a) proved $d=\hat{f}(\gamma)$. An elementary real analytic proof was recently given by Cline (1987).

The frame of our investigation is formed by the class of distribution functions satisfying property (ii) of Definition 1.

Definition 2. A distribution function F belongs to the class $\mathscr{L}(\gamma)$ with $\gamma\geqq 0$ if

$$\lim_{x\to\infty} \bar{F}(x-y)/\bar{F}(x)=e^{\gamma y} \qquad \forall y\in\mathbb{R}.$$

In the case $\gamma=0$ we write $\mathscr{L}:=\mathscr{L}(0)$.

The class $\mathscr{L}(\gamma)$ is related to the class $\mathscr{RV}(-\gamma)$ of regularly varying functions with exponent $-\gamma$ by the fact that

$$F\in\mathscr{L}(\gamma) \quad \text{if and only if} \quad \bar{F}\circ\ln\in\mathscr{RV}(-\gamma). \tag{1.1}$$

Thus the convergence of $\bar{F}(x-y)/\bar{F}(x)$ in Definition 2 is uniform on compact y-intervals. For excellent discussions of regularly varying functions see de Haan (1970) and Bingham, Goldie and Teugels (1987).

Applying Karamata's representation theorem for regularly varying functions to the class $\mathscr{L}(\gamma)$ we obtain for every $F\in\mathscr{L}(\gamma)$, $\gamma\geqq 0$, the representation

$$\bar{F}(x)=c(x)\exp\left\{-\int_0^x b(y)\,dy\right\}, \qquad x\geqq 0,$$

where $c:\mathbb{R}_+\to(0,\infty)$, $b:\mathbb{R}_+\to\mathbb{R}_+$ are such that $\lim\limits_{x\to\infty} c(x)=c>0$ and $\lim\limits_{x\to\infty} b(x)=\gamma$.

The classes $\mathscr{L}(\gamma)$ are closed with respect to tail-equivalence, where F and G are called *tail-equivalent* if there exists some $c\in(0,\infty)$ such that $\bar{F}\sim c\bar{G}$, i.e. $\lim\limits_{x\to\infty} \bar{F}(x)/\bar{G}(x)=c$. The representation above guarantees for each $F\in\mathscr{L}(\gamma)$ the existence of a tail-equivalent $G\in\mathscr{L}(\gamma)$ such that G is absolutely continuous and has a hazard rate r_G satisfying $\lim_{x\to\infty} r_G(x)=\gamma$. Thus considering distribution functions in $\mathscr{S}(\gamma)$, we can always assume that they are absolutely continuous.

Densities of distributions of $\mathscr{S}(\gamma)$ have already been considered by Chover, Ney and Wainger (1973a, b).

Definition 3. A function $f\colon \mathbb{R}\to\mathbb{R}_+$ such that $f(x)>0$ on $[A,\infty)$ for some $A\in\mathbb{R}_+$ belongs to the class $\mathscr{S}d(\gamma)$ with $\gamma\geqq 0$ if

(i) $\lim_{x\to\infty} f^{2\otimes}(x)/f(x)=2d<\infty$

(ii) $\lim_{x\to\infty} f(x-y)/f(x)=e^{\gamma y} \qquad \forall y\in\mathbb{R}.$

The class $\mathscr{S}d:=\mathscr{S}d(0)$ is called the class of *subexponential densities.*

If we define $g(x)=e^{-\gamma x}f(x)$ for some $\gamma>0$ and a function $f\colon \mathbb{R}\to\mathbb{R}_+$ such that $f(x)>0$ on $[A,\infty)$ for some $A\in\mathbb{R}_+$, then immediately by definition, $f\in\mathscr{S}d$ if and only if $g\in\mathscr{S}d(\gamma)$.

The following result is known for continuous functions f [Chover, Ney and Wainger (1973a, b)].

Theorem 1.1. *For $f\in\mathscr{S}d(\gamma)$ define a distribution function concentrated on* $(0,\infty)$ *by*

$$F(x):=\left(\int_0^\infty f(y)\,dy\right)^{-1}\int_0^x f(y)\,dy.$$

Then $F\in\mathscr{S}(\gamma)$.

Proof. W.l.o.g. assume $\int_0^\infty f(y)\,dy=1$.

For $\varepsilon>0$ there exists some $v\in\mathbb{R}_+$ such that for all $t\geqq v$

$$(2d-\varepsilon)f(t)\leqq\int_0^t f(t-y)f(y)\,dy\leqq(2d+\varepsilon)f(t).$$

and hence by integration

$$\lim_{x\to\infty}\frac{\bar{F}^{2*}(x)}{\bar{F}(x)}=2d.$$

Furthermore, $F\in\mathscr{L}(\gamma)$ holds by Karamata's theorem. □

Thus, also in Definition 3 $\quad d=\int_0^\infty e^{\gamma y}f(y)\,dy$ must hold.

We define a larger class than $\mathscr{S}d(\gamma)$, similar to the class $\mathscr{L}(\gamma)$ for distribution functions before:

Definition 4. A function for $f\colon \mathbb{R}\to\mathbb{R}_+$ such that $f(x)>0$ on $[A,\infty)$ for some $A\in\mathbb{R}_+$ belongs to the class $\mathscr{L}d(\gamma)$ with $\gamma\geqq 0$ if

$$\lim_{x\to\infty} f(x-y)/f(x)=e^{\gamma y} \qquad \forall\, y\in\mathbb{R}.$$

In the case $\gamma=0$ we write $\mathscr{L}d:=\mathscr{L}d(0)$.

The convergence of $f(x-y)/f(x)$ is uniform on compact y-intervals.

For $f\in\mathscr{L}d(\gamma)$ with $\gamma>0$ such that $f\in L^1(\mathbb{R}_+)$

$$F(x):=\left(\int_0^\infty f(y)\,dy\right)^{-1}\int_0^x f(y)\,dy$$

defines a distribution function concentrated on $(0,\infty)$. By Karamata's theorem

$$\lim_{x\to\infty} f(x)/\bar{F}(x)=\gamma\int_0^\infty f(y)\,dy$$

holds and thus f is asymptotic equivalent to $\bar{F}$ which is decreasing on $\mathbb{R}_+$.

$f\in\mathscr{L}d$ is equivalent to slow variation of $f\circ\ln$ and thus f is not necessarily asymptotic equivalent to some monotone function [for an example see Cline (1986), p. 538].

Using Karamata's representation theorem again we obtain for every function $f\in\mathscr{L}d(\gamma)$, $\gamma\geqq 0$, the representation

$$f(x)=c(x)\exp\left\{-\int_A^x b(y)\,dy\right\}, \qquad x\geqq A, \tag{1.2}$$

where $c\colon [A,\infty)\to(0,\infty)$, $b\colon [A,\infty)\to\mathbb{R}$ are such that $\lim_{x\to\infty} c(x)=c>0$ and $\lim_{x\to\infty} b(x)=\gamma$. In the case $\gamma>0$ b may be chosen as positive.

The classes $\mathscr{L}d(\gamma)$ are closed with respect to asymptotic equivalence and the representation (1.2) guarantees for each $f\in\mathscr{L}d(\gamma)$ the existence of an asymptotic equivalent $g\in\mathscr{L}d(\gamma)$ such that g is absolutely continuous. We weaken the asymptotic equivalence as the tail-equivalence of distribution functions in Klüppelberg (1988) and call two functions f and g *weakly asymptotic equivalent* if $f \asymp g$, i.e.

$$0<\liminf_{x\to\infty} f(x)/g(x)\leqq\limsup_{x\to\infty} f(x)/g(x)<\infty.$$

The following result shows that – restricted on $\mathscr{L}d(\gamma)$ – the classes $\mathscr{S}d(\gamma)$ are closed with respect to weak asymptotic equivalence. It can be proved analogous to Theorem 2.1 of Klüppelberg (1988).

Lemma 1.2. *Suppose $f, g\in\mathscr{L}d(\gamma)$, $f, g\in L^1(0,x_0)$ for all $x_0\in\mathbb{R}_+$ and $f\asymp g$, then*

$$f\in\mathscr{S}d(\gamma) \quad \textit{if and only if} \quad g\in\mathscr{S}d(\gamma).$$

§ 2. A Characterization of $\mathscr{S}(\gamma)$

A well-known characterization of regularly varying functions says that f belongs to $\mathscr{RV}(\rho)$ if and only if there exists some slowly varying function l such that

$$f(x)=x^{\rho}\,l(x).$$

Applying this to distribution functions of $\mathscr{L}(\gamma)$ we are led to a characterization of the classes $\mathscr{S}(\gamma)$ for $\gamma>0$.

Theorem 2.1. *Let F be a distribution function, $\gamma>0$, and define*

$$h(x):=e^{\gamma x}\bar{F}(x).$$

Then

$$F\in\mathscr{S}(\gamma) \quad \textit{if and only if} \quad h\in\mathscr{S}\mathscr{d}.$$

Proof. Immediately from the definition $h\in\mathscr{S}\mathscr{d}$ if and only if $\bar{F}\in\mathscr{S}\mathscr{d}(\gamma)$. Thus we have to prove $F\in\mathscr{S}(\gamma)$ if and only if $\bar{F}\in\mathscr{S}\mathscr{d}(\gamma)$. Obviously, $F\in\mathscr{L}(\gamma)$ if and only if $h\in\mathscr{L}\mathscr{d}$. $\mathscr{S}(\gamma)$ is closed with respect to tail-equivalence [Embrechts and Goldie (1982)] and $\mathscr{S}\mathscr{d}(\gamma)$ with respect to asymptotic equivalence by Lemma 1.2. Hence, whichever implication is being considered, we can assume that F is absolutely continuous and its hazard rate $r_F(x)$ tends to γ as $x\to\infty$. Thus for $\varepsilon\in(0,\gamma)$ there exists some $x_0\in\mathbb{R}_+$ such that $\gamma-\varepsilon\leqq r_F(x)\leqq\gamma+\varepsilon$ for all $x\geqq x_0$. This can be used to show that finiteness of either side of the identity

$$\hat{f}(\gamma)=1+\gamma\int_0^{\infty} e^{\gamma y}\bar{F}(y)\,dy$$

implies that of the other and hence

$$\hat{f}(\gamma)<\infty \quad \text{if and only if} \quad h\in L^1(\mathbb{R}_+).$$

Now consider for $v>0$ and $x>2v$ the decompositions

$$\frac{\bar{F}^{2*}(x)}{\bar{F}(x)}=2\int_0^{v}\frac{\bar{F}(x-y)}{\bar{F}(x)}\,dF(y)+\int_v^{x-v}\frac{\bar{F}(x-y)}{\bar{F}(x)}\,dF(y)+\frac{\bar{F}(x-v)}{\bar{F}(x)}\,\bar{F}(v)$$

and

$$\frac{\bar{F}^{2\otimes}(x)}{\bar{F}(x)}=2\int_0^{v}\frac{\bar{F}(x-y)}{\bar{F}(x)}\,\bar{F}(y)\,dy+\int_v^{x-v}\frac{\bar{F}(x-y)}{\bar{F}(x)}\,\bar{F}(y)\,dy.$$

For $F\in\mathscr{L}(\gamma)$ and $\hat{f}(\gamma)<\infty$ we have

$$\lim_{v\to\infty}\lim_{x\to\infty}\int_0^{v}\frac{\bar{F}(x-y)}{\bar{F}(x)}\,dF(y)=\lim_{v\to\infty}\int_0^{v} e^{\gamma y}\,dF(y)=\hat{f}(\gamma),$$

$$\lim_{v\to\infty}\lim_{x\to\infty}\frac{\bar{F}(x-v)}{\bar{F}(x)}\,\bar{F}(v)=0,$$

$$\lim_{v\to\infty}\lim_{x\to\infty}\int_0^{v}\frac{\bar{F}(x-y)}{\bar{F}(x)}\,\bar{F}(y)\,dy=\lim_{v\to\infty}\int_0^{v} e^{\gamma y}\bar{F}(y)\,dy=\frac{\hat{f}(\gamma)-1}{\gamma}.$$

For $v>x_0$ we obtain

$$(\gamma-\varepsilon)\int_v^{x-v}\frac{\bar F(x-y)}{\bar F(x)}\bar F(y)\,dy\leqq\int_v^{x-v}\frac{\bar F(x-y)}{\bar F(x)}\,dF(y)\leqq(\gamma+\varepsilon)\int_v^{x-v}\frac{\bar F(x-y)}{\bar F(x)}\bar F(y)\,dy.$$

This implies

$$\lim_{v\to\infty}\limsup_{x\to\infty}\int_v^{x-v}\frac{\bar F(x-y)}{\bar F(x)}\,dF(y)=0$$

if and only if

$$\lim_{v\to\infty}\limsup_{x\to\infty}\int_v^{x-v}\frac{\bar F(x-y)}{\bar F(x)}\bar F(y)\,dy=0.$$

Hence,

$$\lim_{x\to\infty}\frac{\bar F^{2*}(x)}{\bar F(x)}=2\hat f(\gamma)\quad\text{if and only if}\quad\lim_{x\to\infty}\frac{\bar F^{2\otimes}(x)}{\bar F(x)}=2\frac{\hat f(\gamma)-1}{\gamma}.\qquad\square$$

Corollary 2.2. *Suppose* $\gamma>0$.
(*a*) $F\in\mathscr{S}(\gamma)$ *if and only if* $\bar F\in\mathscr{S}d(\gamma)$
(*b*) *If* F *has a density* $f\in\mathscr{L}d(\gamma)$, *then*

$$F\in\mathscr{S}(\gamma)\Leftrightarrow\bar F\in\mathscr{S}d(\gamma)\Leftrightarrow f\in\mathscr{S}d(\gamma).$$

Example. Generalized inverse Gaussian distribution (GIGD). The treatment of the GIGD as in Embrechts (1983) can considerably be shortened. The density of the GIGD is given by

$$f(x)=(b/a)^{c/2}(2K_c(\sqrt{ab}))^{-1}x^{c-1}\exp\{-\tfrac{1}{2}(ax^{-1}+bx)\},\qquad x\in\mathbb{R}_+,$$

where K_c is the modified Besselfunction of the third kind with index c, and the following parameter set is possible:

$$\Theta_c=\begin{cases}\{(a,b);\,a\geqq 0,\,b>0\} & \text{if } c>0\\ \{(a,b);\,a>0,\,b>0\} & \text{if } c=0\\ \{(a,b);\,a>0,\,b\geqq 0\} & \text{if } c<0.\end{cases}$$

In the case of $a=0$ or $b=0$ the norming constant is interpreted as the respective limit. Set

$$h(x):=(b/a)^{c/2}(2K_c(\sqrt{ab}))^{-1}x^{c-1}\exp\left\{-\frac{a}{2x}\right\},$$

then

$$f(x)=\exp\left\{-\frac{b}{2}x\right\}h(x).$$

Since $h\in\mathscr{RV}(c-1)$ and $h\in L^1(\mathbb{R}_+)$ if and only if $c<0$, $f\in\mathscr{S}d(b/2)$ if and only if $c<0$. Thus by Cor. 2.2 $F\in\mathscr{S}(b/2)$ if and only if $c<0$.

Remark. Using the so-called γ-transform F_γ defined by

$$F_\gamma(x)=\hat{f}(\gamma)^{-1}\int_0^x e^{\gamma y}\,dF(y)$$

a class of distribution functions which provides the possibility to embed the class $\mathscr{S}(\gamma)$ into the class $\mathscr{S}$ was introduced by Teugels (1975). He defined $\mathscr{T}(\gamma)$ for $\gamma\geqq 0$ as the class of distribution functions such that $F_\gamma\in\mathscr{S}$. Obviously, $\mathscr{T}(0)=\mathscr{S}$. If $F\in\mathscr{L}(\gamma)$ for some $\gamma>0$, then by Karamata's theorem

$$\bar{F}_\gamma(x)\sim\hat{f}(\gamma)^{-1}\gamma\int_x^\infty e^{\gamma y}\bar{F}(y)\,dy.$$

If we define $h(x)=e^{\gamma x}\bar{F}(x)$ as in Theorem 2.1, then $F_\gamma\in\mathscr{S}$ if and only if $H\in\mathscr{S}$ where H is the distribution function with density h properly normalized.

§ 3. Densities of Random Sums

The classes $\mathscr{S}(\gamma)$, $\gamma\geqq 0$, have received attention with applications to branching processes [Chistyakov (1964), Chover, Ney and Wainger (1973 a, b)], to queueing theory [Pakes (1975), Veraverbeke (1977)], and to risk theory [Embrechts and Veraverbeke (1982)]. All these applications are based on distributions of random sums, i.e.

$$G(x)=\sum_{n=0}^{\infty}\lambda_n F^{n*}(x),$$

where $\{\lambda_n\}$ is a sequence satisfying appropriate conditions. The basic result was provided by Embrechts, Goldie and Veraverbeke (1979). They proved for the compound Poisson distribution, i.e. $\lambda_n=e^{-\lambda}\frac{\lambda^n}{n!}$, that $\bar{G}\sim\lambda\bar{F}$ if and only if $F\in\mathscr{S}$. Embrechts and Goldie (1982) generalized this result to the classes $\mathscr{S}(\gamma)$ for $\gamma>0$. A general form of all these results can be found in Cline (1987).

If F is absolutely continuous with density f, then G is absolutely continuous with density g satisfying

$$g(x)=\sum_{n=1}^{\infty}\lambda_n f^{n\otimes}(x).$$

A result analogous to the abovementioned one is also valid for densities. Instead of repeating the whole procedure of the proof for densities in place of distributions we use the representation of Theorem 2.1.

We start with some results for densities analogous to wellknown results for distribution functions [Athreya and Ney (1972), Chover, Ney and Wainger (1973 a)]. The proof of the following Lemma can be obtained by appropriate variations. Similar results can also be found in Willekens (1986).

Lemma 3.1.

(a) Suppose $f \in \mathscr{L}d(\gamma)$, *then*

$$\liminf_{x \to \infty} \frac{f^{n\otimes}(x)}{f(x)} \geqq n \left(\int_0^\infty e^{\gamma y} f(y)\, dy \right)^{n-1} \qquad \forall n \in \mathbb{N}.$$

(b) Suppose $f \in \mathscr{S}d(\gamma)$ *is bounded. Then for any* $\varepsilon > 0$ *there exists some* $k_\varepsilon \in \mathbb{R}_+$ *such that*

$$f^{n\otimes}(x) \leqq k_\varepsilon \left(\int_0^\infty e^{\gamma y} f(y)\, dy + \varepsilon \right)^n f(x) \qquad \forall x \in \mathbb{R}_+, n \in \mathbb{N}.$$

(c) $f \in \mathscr{S}d(\gamma)$ *if and only if*

$$\lim_{x \to \infty} \frac{f^{n\otimes}(x)}{f(x)} = n \left(\int_0^\infty e^{\gamma y} f(y)\, dy \right)^{n-1} \qquad \forall n \in \mathbb{N}.$$

The next theorem is a best possible result to relate the densities of a random sum.

Theorem 3.2. *Suppose* $f \in \mathscr{L}d(\gamma)$ *is bounded,* $f(x) > 0$ *on* $[A, \infty)$ *for some* $A \in \mathbb{R}_+$, *and* $\int_0^\infty e^{\gamma y} f(y)\, dy < \infty$. *Let* $\{\lambda_n\}$ *be a sequence in* $\mathbb{R}_+$ *with* $\lambda_j > 0$ *for some* $j > 1$ *and* $\sum_{n=1}^\infty \lambda_n \left(\int_0^\infty e^{\gamma y} f(y)\, dy + \varepsilon \right)^n < \infty$ *for some* $\varepsilon > 0$. *Denote*

$$g(x) := \sum_{n=1}^\infty \lambda_n f^{n\otimes}(x).$$

Then the following assertions are equivalent:

(a) $f \in \mathscr{S}d(\gamma)$.

(b) $g \in \mathscr{S}d(\gamma)$ *and* $\limsup_{x \to \infty} g(x)/f(x) < \infty$.

(c) $g \sim cf$ *for* $c = \sum_{n=1}^\infty n\lambda_n \left(\int_0^\infty e^{\gamma y} f(y)\, dy \right)^{n-1}$.

Proof. (a) implies (b) and (c) by Lemma 3.1 (b, c) and the dominated convergence theorem. By Lemma 3.1 (a)

$$\liminf_{x \to \infty} g(x)/f(x) \geqq \sum_{n=1}^\infty n\lambda_n \left(\int_0^\infty e^{\gamma y} f(y)\, dy \right)^{n-1} > 0$$

and thus (b) implies $g \asymp f$. Since $f \in \mathscr{L}d(\gamma)$ and f, $g \in L^1(0, x_0)$ for all $x_0 \in \mathbb{R}_+$ by the boundedness condition above (b) implies (a) by Lemma 1.2. If (c) holds, then for $\varepsilon \in (0, c)$ there exists some $x_0 \in \mathbb{R}_+$ such that

$$(c-\varepsilon) f(x) \leqq g(x) \leqq (c+\varepsilon) f(x) \qquad \forall x \geqq x_0.$$

This implies for $x \geqq x_0$ by integration

$$(c-\varepsilon)\, \bar{F}(x) \leqq \bar{G}(x) \leqq (c+\varepsilon)\, \bar{F}(x)$$

and thus

$$\lim_{x \to \infty} \bar{G}(x)/\bar{F}(x) = c.$$

Consider first the case $\gamma > 0$. Using Theorem 2.13 of Cline (1987) and Cor. 2.2 (b) this yields $f \in \mathscr{S}d(\gamma)$. Now suppose $\gamma = 0$. Multiply $g(x)$ by $e^{-\gamma x}$ for some $\gamma > 0$, then by definition

$$g \in \mathscr{S}d \Leftrightarrow e^{-\gamma x} g(x) \in \mathscr{S}d(\gamma).$$

Since $(e^{-\gamma x} f(x))^{n\otimes} = e^{-\gamma x} f^{n\otimes}(x)$ holds for all $n \in \mathbb{N}$ we obtain

$$e^{-\gamma x} g(x) = \sum_{n=1}^{\infty} \lambda_n (e^{-\gamma x} f(x))^{n\otimes}.$$

Now (c) holds for g and f replaced by $e^{-\gamma x} g(x)$ and $e^{-\gamma x} f(x)$, respectively, and by the above $e^{-\gamma x} f(x) \in \mathscr{S}d(\gamma)$ and hence $f \in \mathscr{S}d$. Thus (c) implies (a). □

Since f and g are not necessarily monotone we had to strengthen the condition $\bar{F}(x) \neq o(\bar{G}(x))$ of Theorem 2.13 of Cline (1987) to $\limsup_{x \to \infty} g(x)/f(x) < \infty$.

Under certain conditions on the weights λ_n the additional condition $\limsup_{x \to \infty} g(x)/f(x) < \infty$ in assertion (b) is not necessary. This is valid e.g. in the important cases of Poisson and geometric weights. For distributions of random sums this was mentioned in Cor. 2.14 of Cline (1987) and we use his result for the density version.

Corollary 3.3. *Suppose that additionally to the conditions of Theorem* 3.2 *for* $n \in \mathbb{N}$

$$\lambda_n = (1-\lambda)\, \lambda^n \qquad \textit{for some } \lambda \in (0, 1)$$

or

$$\lambda_n = e^{-\lambda} \frac{\lambda^n}{n!} \qquad \textit{for some } \lambda \in (0, \infty)$$

holds. If $g \in \mathscr{S}d(\gamma)$ *holds then* $f \in \mathscr{S}d(\gamma)$.

Proof. $g \in \mathscr{S}d(\gamma)$ implies $G \in \mathscr{S}(\gamma)$ by Theorem 1.1, which yields $F \in \mathscr{S}(\gamma)$ by Cor. 2.14 of Cline (1987). Since $f \in \mathscr{L}d(\gamma)$ this is for $\gamma > 0$ equivalent to $f \in \mathscr{S}d(\gamma)$ by Cor. 2.2 (b). For $\gamma = 0$ multiply $f(x)$ by $e^{-\gamma x}$ for some $\gamma > 0$. As in the proof of Theorem 3.2 this yields the assertion. □

§ 4. The $M/G/1$ Queue

Random sums occur in many stochastic models as solutions in the form of a Neumann series of a linear Volterra integral equation of the second kind. In all these situations there is a well-known input function and an unknown output function. If the input function belongs to some class $\mathcal{S}d(\gamma)$ the asymptotic behaviour of the output function is determined by Theorem 3.2. We shall concentrate on an easy example in queueing theory where our preceding results are especially useful. For more examples which have been treated or can be treated similarly see e.g. Chistyakov (1964), Teugels (1975), Pakes (1975), Veraverbeke (1977), Embrechts and Veraverbeke (1982), Murphree and Smith (1986).

Denote by η the arrival rate and by F the service-time distribution having finite mean $\mu(F)$. If $\rho := \eta\,\mu(F) < 1$ then the stationary distribution G of the virtual waiting-time can be written as

$$G(x) = \sum_{n=0}^{\infty} (1-\rho)\,\rho^n F_I^{n*}(x)$$

where $F_I(x) = \mu(F)^{-1} \int_0^x \bar{F}(y)\,dy$ is the integrated tail distribution of F.

Since Karamata's theorem implies for $F \in \mathcal{L}(\gamma)$

$$\lim_{x \to \infty} \bar{F}(x)/\bar{F}_I(x) = \gamma\,\mu(F),$$

for $\gamma > 0$ the result of Embrechts and Goldie (1982) relates the tails of F and G. Unfortunately, the answer is for $\gamma = 0$ not satisfactory.

The answer for $\gamma \geqq 0$ is given by considering that, since F_I is absolutely continuous, G is absolutely continuous with density

$$g(x) = \sum_{n=1}^{\infty} (1-\rho)\,\eta^n \bar{F}^{n\otimes}(x).$$

Thus we can apply Theorem 3.2 and Cor. 3.3 to obtain the following asymptotic relationship between the service-time distribution and the density of the stationary waiting-time distribution:

Theorem 4.1. *Suppose $F \in \mathcal{L}(\gamma)$ and $\eta \int_0^\infty e^{\gamma y} \bar{F}(y)\,dy < 1$. Then the following assertions are equivalent:*

(*a*) $\bar{F} \in \mathcal{S}d(\gamma)$
(*b*) $g \in \mathcal{S}d(\gamma)$
(*c*) $g \sim c\bar{F}$

where

$$c=\begin{cases}(1-\rho)\,\eta\left(1-\dfrac{\eta}{\gamma}(\hat{f}(\gamma)-1)\right)^{-2} & \text{if } \gamma>0\\ \eta(1-\rho)^{-1} & \text{if } \gamma=0.\end{cases}$$

For $\gamma>0$ we use Cor. 2.2 to formulate the result in terms of distribution functions.

Cor. 3.2. *Suppose $F\in\mathscr{L}(\gamma)$ for $\gamma>0$ and $\hat{f}(\gamma)<1+\gamma/\eta$. Then the following assertions are equivalent:*

(*a*) $F\in\mathscr{S}(\gamma)$

(*b*) $G\in\mathscr{S}(\gamma)$

(*c*) $\bar{G}\sim\left\{(1-\rho)\dfrac{\eta}{\gamma}\left(1-\dfrac{\eta}{\gamma}(\hat{f}(\gamma)-1)\right)^{-2}\right\}\bar{F}.$

The model of an $M/G/1$-queue is equivalent to the Sparre-Anderson-model in ruin theory when the arrival process of the claims is Poisson and the claim amounts have distribution F. The results of this section should therefore be compared with Sect. 6 of Embrechts and Veraverbeke (1982).

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