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Probability Theory and Related Fields

On the Invariance Principle for Stationary ϕ-Mixing Triangular Arrays with Infinitely Divisible Limits

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Summary. Given a stationary, ϕ-mixing triangular array of Banach space valued random vectors whose row sums converge weakly to an infinitely divisible probability measure, necessary and sufficient conditions for the validity of the corresponding invariance principle in distribution are given.

1. Introduction

Let $\{X_{nj}\}=\{X_{nj}: j=1, \ldots, j_n, n\in N\}$ (N is the set of non-zero natural numbers) be a triangular array of B-valued random vectors (r.v.'s) defined on a common probability space $(\Omega, \mathscr{A}, P)$; here and throughout the paper, B denotes a real separable Banach space with norm $\|\cdot\|$ and it will be assumed that $j_n \to \infty$ as $n \to \infty$. We shall write $S_{nk}=\sum_{j=1}^{k} X_{nj}$ if $k=1, \ldots, j_n$.

Assuming that $\{X_{nj}\}$ is stationary and ϕ-mixing (see the definitions below) we give conditions which added to the weak convergence of $\{\mathscr{L}(S_{nj_n})\}$ (if X is a random vector, $\mathscr{L}(X)$ denotes its law) to an infinitely divisible probability measure imply that the corresponding invariance principle in distribution (or functional central limit theorem) holds; we also show the necessity of those conditions. This is contained in Theorem 3.2 which gives an extension of a well-known result for the independent case due to A.V. Skorohod [12, Theorem 2.7]. For the proof of the sufficiency we adapt the method of [4] to the dependent case; this is carried out by using an appropriate version of an inequality in [4] (Lemma 2.3 below) and a certain maximal inequality ([7, (3.5)]; [5, Lemma 3.1]) together with some results of [9]; for the converse, we use an inequality of T.L. Lai [6, (3.28)]. Let us remark that Theorem 3.2 seems to be new even for the real-valued case; moreover, we give an example which together with that result contradicts a statement which appears in the literature (see Remark 3.4.4).

In the Gaussian case, Corollary 3.5 below generalizes both part of [5, Theorem 2.13] (which deals with sequences of real valued r.v.'s with finite variances) and the almost sure invariance principle given in [9]; Corollary 3.7 generalizes Theorem 20.1 of [2]. See [5] and [7] for more information about the Gaussian - even nonstationary - case; Herrndorf's work provided an important stimulus for the realization of the present paper.

At the end of the paper we indicate an application of the results given here and in [9].

Now we recall some definitions. If $\mathscr{M}$, $\mathscr{N}$ are two sub-σ-algebras of $\mathscr{A}$ we will consider the coefficients

$$\phi(\mathscr{M}, \mathscr{N}) = \sup\left\{\left|\frac{P(E\cap F)}{P(E)} - P(F)\right| : E\in\mathscr{M}, F\in\mathscr{N}, P(E)>0\right\}$$

and

$$\psi^*(\mathscr{M}, \mathscr{N}) = \sup\left\{\frac{P(E\cap F)}{P(E)\,P(F)} : E\in\mathscr{M}, F\in\mathscr{N}, P(E)\,P(F)>0\right\}.$$

Given a triangular array $\{X_{nj}: j=1, \dots, j_n, n\in N\}$ let $\mathscr{M}_{hk}^{(n)} = \sigma(\{X_{nj}: j=h, \dots, k\})$ for $n\in N$ and $1\leqq h\leqq k\leqq j_n$ (if $\mathscr{R}$ is a set of r.v.'s, $\sigma(\mathscr{R})$ is the σ-algebra generated by $\mathscr{R}$) and define

$$\phi(k) = \sup_{n\in N,\, j_n>k}\ \max_{1\leqq h\leqq j_n-k} \phi(\mathscr{M}_{1h}^{(n)}, \mathscr{M}_{h+k,\, j_n}^{(n)}) \qquad (k\in N),$$

$$\psi^* = \sup_{n\in N,\, j_n>1}\ \max_{1\leqq h\leqq j_n-1} \psi^*(\mathscr{M}_{1h}^{(n)}, \mathscr{M}_{h+1,\, j_n}^{(n)}).$$

Note that $\phi(1)\leqq 1$, $\psi^*\leqq\infty$ and $\{\phi(k)\}$ is non-increasing. It is said that $\{X_{nj}\}$ *is ϕ-mixing* if $\phi(k)\downarrow 0$ as $k\to\infty$; we will say that $\{X_{nj}\}$ is *stationary* (has *stationary sums*) if $\mathscr{L}(X_{n1}, \dots, X_{nh}) = \mathscr{L}(X_{n,k+1}, \dots, X_{n,k+h})$ $(\mathscr{L}(X_{n1}+\dots+X_{nh}) = \mathscr{L}(X_{n,k+1}+\dots+X_{n,k+h})$, respectively) for $1\leqq h\leqq j_n$, $1\leqq k\leqq j_n-h$, $n\in N$. We have similar definitions for a sequence and for a finite set $\{X_1, \dots, X_n\}$ of r.v.'s; in the last case, we shall write $S_k = \sum_{j=1}^{k} X_j$ for $k=1, \dots, n$ and $S_0=0$.

We denote by $\xrightarrow[w]{}$ the weak convergence of probability measures and by $\xrightarrow[P]{}$ the convergence in probability of random vectors. Sometimes we shall write $E(X; A)$ for the integral of the r.v. X over the event A.

The space of functions from $[0, 1]$ into B which are right-continuous on $[0, 1)$ and have left-hand limits on $(0, 1]$ equipped with the Skorohod J_1-topology will be denoted by $D=D([0, 1], B)$ (see [2, Chap. 3], [11]). Given $x\in D$ and $c>0$ we shall write

$$\Delta_{J_1}(c, x) = \sup_{\substack{t-c\leqq t_1<t<t_2\leqq t+c\\ t_1,\, t,\, t_2\in[0,\,1]}} \min\{\|x(t)-x(t_1)\|, \|x(t_2)-x(t)\|\}.$$

2. Inequalities

We quote two inequalities. The first is a version of the Ottaviani inequality for the dependent case (see [7, Lemma (3.1)], [5, Lemma 3.1]); the second was obtained by T.L. Lai ([6, (3.28)]).

2.1. Lemma. *Let $\{X_1, \ldots, X_n\}$ be a set of B-valued r.v.'s. Suppose $q \in N$, $q+1 \leqq n$ and let $a>0$. Then*

$$(1-\phi(q) - \max_{q \leqq k \leqq n} P(\|S_n - S_k\| > a))\, P(\max_{1 \leqq k \leqq n} \|S_k\| > 3a)$$
$$\leqq P(\|S_n\| > a) + P((q-1) \max_{1 \leqq j \leqq n} \|X_j\| > a).$$

2.2. Lemma. *Let $\{X_1, \ldots, X_n\}$ be a set of B-valued r.v.'s such that $\mathscr{L}(X_1) = \ldots = \mathscr{L}(X_n)$. Suppose $q \in N$, $q \leqq n$ and let $\varepsilon > 0$. Then*

$$P(\max_{1 \leqq j \leqq n} \|X_j\| > \varepsilon) \geqq (P(\max_{1 \leqq j \leqq n} \|X_j\| \leqq \varepsilon) - \phi(q))[n/q]\, P(\|X_1\| > \varepsilon)$$

($[\cdot]$ *denotes the integer part of a real number*).

We will use the following generalization of Lemma 1, p. 480 of Gihman and Skorohod [4].

2.3. Lemma. *Let $\{X_1, \ldots, X_n\}$ be a stationary set of B-valued r.v.'s. Suppose $q \in N$, $2 \leqq q \leqq n-1$ and let $\varepsilon > 0$. Then*

$$P(\max_{0 \leqq i < j < k \leqq n} \min\{\|S_j - S_i\|, \|S_k - S_j\|\} > \varepsilon)$$
$$\leqq (q-1)^3 \sum_{r=2}^{2(q-1)} n P(\|X_1\| > \varepsilon/(2(q-1)), \|X_r\| > \varepsilon/(2(q-1)))$$
$$+ (\phi(q) + P(\max_{1 \leqq k \leqq n} \|S_k\| > \varepsilon/4))\, P(\max_{1 \leqq k \leqq n} \|S_k\| > \varepsilon/4).$$

Proof. Write $Y = \max_{0 \leqq i < j < k \leqq n} \min\{\|S_j - S_i\|, \|S_k - S_j\|\}$ and

$$Z = \max_{\substack{0 \leqq i < j < k \leqq n \\ j-i \leqq q-1,\, k-j \leqq q-1}} \min\{\|S_j - S_i\|, \|S_k - S_j\|\}.$$

We have $[Y > \varepsilon] \subset [Z > \varepsilon/2] \cup [Y > \varepsilon, Z \leqq \varepsilon/2]$.

Observe that if $0 \leqq i < j < k \leqq n$, $j-i \leqq q-1$, $k-j \leqq q-1$, $\|S_j - S_i\| > \varepsilon/2$ and $\|S_k - S_j\| > \varepsilon/2$ then we have $\|X_{r_1}\| > \varepsilon/(2(q-1))$ and $\|X_{r_2}\| > \varepsilon/(2(q-1))$ for some r_1, r_2 which satisfy $i+1 \leqq r_1 \leqq j$, $j+1 \leqq r_2 \leqq k$ and $r_2 - r_1 \leqq 2q-3$. Therefore, if $0 \leqq i < j < k \leqq n$, $j-i \leqq q-1$, $k-j \leqq q-1$ it follows from stationarity that

$$P(\min\{\|S_j - S_i\|, \|S_k - S_j\|\} > \varepsilon/2)$$
$$\leqq (q-1) \sum_{r=2}^{2(q-1)} P(\|X_1\| > \varepsilon/(2(q-1)), \|X_r\| > \varepsilon/(2(q-1))).$$

Then

$$P(Z > \varepsilon/2) \leqq \sum_{j=1}^{n-1} \sum_{\substack{0 \leqq i < j < k \leqq n \\ j-i \leqq q-1,\, k-j \leqq q-1}} P(\min\{\|S_j - S_i\|, \|S_k - S_j\|\} > \varepsilon/2)$$
$$\leqq n(q-1)^2 (q-1) \sum_{r=2}^{2(q-1)} P(\|X_1\| > \varepsilon/(2(q-1)), \|X_r\| > \varepsilon/(2(q-1))).$$

Now define $E_1=[\|S_1\|>\varepsilon/4]$, $E_r=[\max_{1\leqq i\leqq r-1}\|S_i\|\leqq\varepsilon/4,\ \|S_r\|>\varepsilon/4]$ for $r=2,\dots,n-q$ and

$$F_r=[\max_{r+q\leqq k\leqq n}\|S_k-S_{r+q-1}\|>\varepsilon/4],\qquad U_r=E_r\cap F_r\qquad \text{for } r=1,\dots,n-q.$$

Suppose that $Z\leqq\varepsilon/2$ and that $0\leqq i<j<k\leqq n$, $\|S_j-S_i\|>\varepsilon$ and $\|S_k-S_j\|>\varepsilon$. By the definition of Z, three cases are possible: (1) $j-i\geqq q$ and $k-j\geqq q$, (2) $j-i\geqq q$ and $k-j\leqq q-1$, (3) $j-i\leqq q-1$ and $k-j\geqq q$. In any case, one of the events U_r occurs. In order to see this in case (1), observe that since $Z\leqq\varepsilon/2$ we have $(1-a)$ $\|S_j-S_{j-q+1}\|\leqq\varepsilon/2$ or $(1-b)$ $\|S_{j+q-1}-S_j\|\leqq\varepsilon/2$. If $\|S_j-S_{j-q+1}\|\leqq\varepsilon/2$ then $\|S_{j-q+1}-S_i\|\geqq\|S_j-S_i\|-\|S_j-S_{j-q+1}\|>\varepsilon/2$ which implies that $\|S_i\|>\varepsilon/4$ or $\|S_{j-q+1}\|>\varepsilon/4$. Then if $\bar r$ is the first r such that $\|S_r\|>\varepsilon/4$ we have $\bar r\leqq j-q+1\leqq n-q$ and $\|S_j-S_{\bar r+q-1}\|>\varepsilon/2$ or $\|S_k-S_{\bar r+q-1}\|>\varepsilon/2$ because $\|S_k-S_j\|>\varepsilon$; this shows that $U_{\bar r}$ occurs. In case $(1-b)$, first note that $\|S_i\|>\varepsilon/2$ or $\|S_j\|>\varepsilon/2$; then if $\bar r$ is defined as above we have $\bar r\leqq j$. On the other hand, $\|S_k-S_{j+q-1}\|\geqq\|S_k-S_j\|-\|S_{j+q-1}-S_j\|>\varepsilon/2$ which implies that $\|S_{j+q-1}-S_{\bar r+q-1}\|>\varepsilon/4$ or $\|S_k-S_{\bar r+q-1}\|>\varepsilon/4$ and then $U_{\bar r}$ occurs.

In the second case, since $k-j\leqq q-1$ and $\|S_k-S_j\|>\varepsilon$, from the definition of Z we conclude that $\|S_j-S_{j-q+1}\|\leqq\varepsilon/2$ and we can argue as in case $(1-a)$. The case (3) can be treated in a similar manner.

We have proved that $[Y>\varepsilon, Z\leqq\varepsilon/2]\subset\bigcup_{r=1}^{n-q}U_r$. On the other hand, using stationarity, we see that $P(F_r)\leqq P(\max_{1\leqq k\leqq n}\|S_k\|>\varepsilon/4)$ $(r=1,\dots,n-q)$. Therefore

$$\begin{aligned}P(Y>\varepsilon, Z\leqq\varepsilon/2)&\leqq\sum_{r=1}^{n-q}P(U_r)\leqq\sum_{r=1}^{n-q}P(E_r)(\phi(q)+P(F_r))\\&\leqq(\phi(q)+P(\max_{1\leqq k\leqq n}\|S_k\|>\varepsilon/4))\sum_{r=1}^{n-q}P(E_r)\\&\leqq(\phi(q)+P(\max_{1\leqq k\leqq n}\|S_k\|>\varepsilon/4))\,P(\max_{1\leqq k\leqq n}\|S_k\|>\varepsilon/4).\quad\square\end{aligned}$$

3. Results

Given an infinitely divisible probability measure ν on (the Borel σ-algebra of) B, $\{\nu^t: t\geqq 0\}$ will denote the associated weakly continuous convolution semigroup of measures and Q_ν will be the distribution on (the Borel σ-algebra of) D of a stochastic process $\xi_\nu=\{\xi_\nu(t): t\in[0,1]\}$ with stationary independent increments, $\xi(0)=0$ a.s., trajectories in D and $\mathscr{L}(\xi(1))=\nu$. On the other hand, if $a_1,\dots,a_n\in B$ we define $p_n(a_1,\dots,a_n)\in D$ by $p_n(a_1,\dots,a_n)(t)=a_{[nt]}$ if $(1/n)\leqq t\leqq 1$, $=0$ if $0\leqq t<1/n$ ($[\cdot]$ denotes the integer part of a real number).

The following remark will be useful.

3.1. Lemma. *Let ν be an infinitely divisible probability measure on B and let μ be its Lévy measure. Then for every $\varepsilon>0$ such that $\mu(\{x\in B: \|x\|=\varepsilon\})=0$ we have*

$$\lim_{t\downarrow 0}\frac{1}{t}Q_\nu(\{x\in D: \sup_{0<s\leqq t}\|x(s)-x(0)\|\geqq\varepsilon\})=\mu(\{x\in B: \|x\|\geqq\varepsilon\})<\infty.$$

Proof. Observe that the set $\{x \in D: \sup_{0<s\leqq t} \|x(s)-x(0)\| \geqq \varepsilon\}$ is J_1-closed. The result follows from the Ottaviani inequality and the well-known fact that $\lim_{t\downarrow 0} (1/t)\, v^t(\{x \in B: \|x\| \geqq \delta\}) = \mu(\{x \in B: \|x\| \geqq \delta\}) < \infty$ for every $\delta > 0$ such that $\mu(\{x \in B: \|x\| = \delta\}) = 0$. □

3.2. Theorem. *Let $\{X_{nj}: 1 \leqq j \leqq j_n,\ n \in N\}$ be a stationary, ϕ-mixing triangular array; write $\xi_n = p_{j_n}(S_{n1}, \ldots, S_{nj_n})$. Let v be a probability measure on B. Then assertions* (I) *to* (III) *below are equivalent.*

(I) *The following conditions hold:*

(a) $\mathscr{L}(S_{nj_n}) \xrightarrow[w]{} v$,

(b) $\{X_{nj}\}$ *satisfies*

$(*)$ $\{r_n\} \subset N,\ r_n \leqq j_n,\ r_n/j_n \to 0 \Rightarrow S_{nr_n} \xrightarrow[P]{} 0$,

(c) *for every* $\varepsilon > 0$, $\sup_n j_n P(\|X_{n1}\| > \varepsilon) < \infty$,

(d) *for each integer* $r \geqq 2$ *and every* $\varepsilon > 0$, $\lim_n j_n P(\|X_{n1}\| > \varepsilon,\ \|X_{nr}\| > \varepsilon) = 0$.

(II) *v is infinitely divisible and $\mathscr{L}(\xi_n) \xrightarrow[w]{} Q_v$ in D.*

(III) *$\{\mathscr{L}(\xi_n)\}$ is relatively compact in D and $\mathscr{L}(\xi_n(1)) \xrightarrow[w]{} v$.*

Proof. (III)$\Rightarrow$(II). First we show that $(*)$ is satisfied. Let $\{r_n\} \subset N$ such that $r_n \leqq j_n$ and $r_n/j_n \to 0$. Suppose that $\mathscr{L}(\xi_{n_k}) \xrightarrow[w]{} Q$ in D for some sequence $\{n_k\} \subset N$ and let $\varepsilon > 0$. For each $t \in (0, 1]$, since $F_{\varepsilon, t} \doteq \{x \in D: \sup_{0<s\leqq t} \|x(s)-x(0)\| \geqq \varepsilon\}$ is J_1-closed, we have $\overline{\lim}_k P(\|S_{n_k, r_{n_k}}\| > \varepsilon) \leqq \overline{\lim}_k P(\xi_{n_k} \in F_{\varepsilon, t}) \leqq Q(F_{\varepsilon, t})$; but $\lim_{t\downarrow 0} Q(F_{\varepsilon, t}) = 0$ $(x(0+) = x(0)$ for each $x \in D)$. Then $S_{n_k, r_{n_k}} \xrightarrow[P]{} 0$. Since $\{\mathscr{L}(\xi_n)\}$ is relatively compact we can deduce that $S_{nr_n} \xrightarrow[P]{} 0$.

Now observe that v is infinitely divisible by [9, Proposition 3.1]. If $0 \leqq t_1 < t_2 \leqq 1$ we have that $\mathscr{L}(\xi_n(t_2) - \xi_n(t_1)) \xrightarrow[w]{} v^{t_2 - t_1}$ ([9, Theorem 3.3]) and arguing as in the proof of [9, Proposition 3.1] we can prove that

$$\mathscr{L}(\xi_n(t_1), \xi_n(t_2) - \xi_n(t_1), \ldots, \xi_n(t_k) - \xi_n(t_{k-1})) \xrightarrow[w]{} v^{t_1} \otimes v^{t_2 - t_1} \otimes \ldots \otimes v^{t_k - t_{k-1}}$$

(product measure) if $0 \leqq t_1 < t_2 < \ldots < t_k \leqq 1$. From this we conclude that the finite-dimensional distributions of ξ_n converge weakly to that of ξ_v. This, together with the relative compactness of $\{\mathscr{L}(\xi_n)\}$, shows that $\mathscr{L}(\xi_n) \xrightarrow[w]{} Q_v$.

(II)$\Rightarrow$(I). Assume (II) holds. For each $\varepsilon > 0$ and $t \in (0, 1]$ we have

$$\overline{\lim_n}\, P(\max_{1\leqq k\leqq tj_n} \|S_{nk}\| > \varepsilon) \leqq \overline{\lim_n}\, P(\xi_n \in F_{\varepsilon, t}) \leqq Q_v(F_{\varepsilon, t}),$$

$F_{\varepsilon, t}$ being the closed subset of D defined above. Then, by the preceding lemma, we have

(3.1) for every $\varepsilon > 0$, $\overline{\lim_{t\downarrow 0}}\, \frac{1}{t}\, \overline{\lim_n}\, P(\max_{1\leqq k\leqq tj_n} \|S_{nk}\| > \varepsilon) < \infty$.

This implies that $\{X_{nj}\}$ satisfies $(*)$. To prove (c), fix $\varepsilon > 0$ and observe that from (3.1) we obtain that $\overline{\lim_{t\downarrow 0}}\, (1/t)\, \overline{\lim_n}\, P(\max_{1\leqq j\leqq tj_n} \|X_{nj}\| > \varepsilon) < \infty$. Then we can find $t_1>0$ and $n_1\in N$ such that $\sup_{n\geqq n_1} P(\max_{1\leqq j\leqq t_1 j_n} \|X_{nj}\|>\varepsilon)\leqq 1/2$; now choose $q\in N$ such that $\phi(q)\leqq 1/4$ and apply Lemma 2.2 to obtain

$$1/2\geqq P(\max_{1\leqq j\leqq t_1 j_n} \|X_{nj}\|>\varepsilon)\geqq(1/4)[[t_1 j_n]/q]\, P(\|X_{n1}\|>\varepsilon)$$

if $n\geqq n_1$. This implies (c).

Now we will prove the following claim:

(3.2) for every integer $r\geqq 2$ and each $\varepsilon>0$, $\lim_n j_n P\left(\|X_{n1}\|>\varepsilon, \left\|\sum_{j=2}^{r} X_{nj}\right\|>\varepsilon\right)=0.$

Since $\mathscr{L}(\xi_n)\xrightarrow[w]{} Q_v$ in D we have that for every $\varepsilon>0$, $\lim_{c\downarrow 0}\overline{\lim_n}\, P(\Delta_{J_1}(c,\xi_n)>\varepsilon)=0$ (argue as in [2, Chap. 3]; see [11, Theorem 3.2.2]). Fix an integer $r\geqq 2$ and $\varepsilon>0$. The previous relation implies

$$\lim_n P(\max_{1\leqq j\leqq j_n-r+1} \min\{\|X_{nj}\|, \|S_{n,j+r-1}-S_{nj}\|\}>\varepsilon)=0.$$

Write $Y_{nj}=\min\{\|X_{nj}\|, \|S_{n,j+r-1}-S_{nj}\|\}$ for $j=1,\ldots,j_n-r+1$ and observe that the triangular array $\{Y_{nj}\}$ is also ϕ-mixing and stationary. By applying Lemma 2.2 to $\{Y_{nj}\}$, we deduce that $\lim_n j_n P(Y_{n1}>\varepsilon)=0$ which proves (3.2). But (d) follows from that claim, since for every $r\geqq 3$ and every $\varepsilon>0$ we have

$$\begin{aligned}P(\|X_{n1}\|>\varepsilon, \|X_{nr}\|>\varepsilon)&\leqq P(\|X_{n1}\|>\varepsilon, \|X_{n2}+\ldots+X_{nr}\|>\varepsilon/2)\\&+P(\|X_{n1}\|>\varepsilon, \|X_{nr}\|>\varepsilon, \|X_{n2}+\ldots+X_{nr}\|\leqq\varepsilon/2)\end{aligned}$$

and the last term is bounded by $P(\|X_{n1}\|>\varepsilon, \|X_{n2}+\ldots+X_{n,r-1}\|>\varepsilon/2)$.

(I)$\Rightarrow$(II). Assume (I) holds. Using (a) and (b) and arguing as in the proof of the implication (III)$\Rightarrow$(II) we can conclude that the finite-dimensional distributions of ξ_n converge weakly to that of ξ_v.

Now it is sufficient to show that

(3.3) for every $\varepsilon>0$, $\lim_{c\downarrow 0}\overline{\lim_n}\, P(\Delta_{J_1}(c,\xi_n)>\varepsilon)=0,$

by an application of [2, Theorem 15.4] since $Q_v(\{x\in D: x(1)\neq x(1-)\})=0$ (we remark that the result remains valid when we replace in the definition of D the real line by a complete separable metric space (X,d) - see [11, Theorem 3.2.1 and 2.7.3]. A slight modification of the compactness argument in its proof is needed: first we can prove that for every $\eta>0$ and every $j\in N$ there exists a compact set $K_j\subset X$ such that $\sup_n P(\xi_n(t)\notin K_j^{\eta/2^j}$ for some $t)\leqq\eta/2^j$ - here $K^\varepsilon=\{x\in X: \inf\{d(x,k): k\in K\}\leqq\varepsilon\}$ if $K\subset X$ and $\varepsilon>0$; this implies that for every $\eta>0$ there exists a compact set $K\subset X$ such that $\sup_n P(\xi_n(t)\notin K$ for some $t)\leqq\eta$ and then we use appropriate versions of Theorems 15.3, 14.4 and 14.3 of [2]).

Fix $\varepsilon>0$. Consider $c\in(0,1/2)$ and write r for the integer part of $1/c$, $I_k=[kc,(k+3)c]$ for $k=0,1,\ldots,r-3$ and $I_{r-2}=[(r-2)c,1]$. Note that if $x\in D$

$$\Delta_{J_1}(c,x) \leqq \max_{0\leqq k\leqq r-2} \sup_{\substack{t_1,t,t_2\in I_k\\ t_1<t<t_2}} \min\{\|x(t)-x(t_1)\|, \|x(t_2)-x(t)\|\}$$

and that if $j_n > 2/c$ and $k=0,1,\dots,r-2$ then, by stationarity,

$$\begin{aligned} P(&\sup_{\substack{t_1,t,t_2\in I_k\\ t_1<t<t_2}} \min\{\|\xi_n(t)-\xi_n(t_1)\|, \|\xi_n(t_2)-\xi_n(t)\|\}>\varepsilon)\\ &\leqq P(\max_{0\leqq i<j<k\leqq 4cj_n} \min\{\|S_{nj}-S_{ni}\|, \|S_{nk}-S_{nj}\|\}>\varepsilon)\end{aligned}$$

(where $S_{n0}=0$); thus, using Lemma 2.3, we conclude that if $q\in N$, $q\geqq 2$, $c\in(0,1/2)$ and n is sufficiently large

$$\begin{aligned} (3.4)\quad P(\Delta_{J_1}(c,\xi_n)>\varepsilon) &\leqq \frac{1}{c} P(\max_{0\leqq i<j<k\leqq 4cj_n} \min\{\|S_{nj}-S_{ni}\|, \|S_{nk}-S_{nj}\|\}>\varepsilon)\\ &\leqq (q-1)^3 \sum_{r=2}^{2(q-1)} 4j_n P(\|X_{n1}\|>\varepsilon/2(q-1), \|X_{nr}\|>\varepsilon/2(q-1))\\ &\quad + (\phi(q)+P(\max_{1\leqq k\leqq 4cj_n}\|S_{nk}\|>\varepsilon/4))\frac{1}{c} P(\max_{1\leqq k\leqq 4cj_n}\|S_{nk}\|>\varepsilon/4)\\ &= A_{q,n}+B_{q,c,n} \quad \text{(say).}\end{aligned}$$

Now we claim that

$$(3.5)\qquad M \doteq \overline{\lim_{c\downarrow 0}}\, \frac{1}{c}\, \overline{\lim_{n}}\, P(\max_{1\leqq k\leqq 4cj_n}\|S_{nk}\|>\varepsilon/4)<\infty.$$

To prove this, take $q\in N$ such that $\phi(q)<1$ and take $\alpha\in(\phi(q),1)$. By hypothesis (b), there exists $c_0>0$ such that for all sufficiently large n we have $\max_{1\leqq k\leqq c_0 j_n} P(\|S_{nk}\|>\varepsilon/12)\leqq 1-\alpha$ and then, by Lemma 2.1, if $c\in(0,c_0/4)$

$$\begin{aligned} P(&\max_{1\leqq k\leqq 4cj_n}\|S_{nk}\|>\varepsilon/4)\\ &\leqq (\alpha-\phi(q))^{-1}\{P(\|S_{n,[4cj_n]}\|>\varepsilon/12)+P((q-1)\max_{1\leqq j\leqq 4cj_n}\|X_{nj}\|>\varepsilon/12)\}\end{aligned}$$

for those n; hence, by [9, Theorem 3.3],

$$\begin{aligned} \overline{\lim_{n}}\, P(&\max_{1\leqq k\leqq 4cj_n}\|S_{nk}\|>\varepsilon/4)\\ &\leqq (\alpha-\phi(q))^{-1}\{v^{4c}(\{x\in B\colon \|x\|\geqq\varepsilon/12\})+4c\sup_n j_n P((q-1)\|X_{n1}\|>\varepsilon/12)\}\end{aligned}$$

for each $c\in(0,c_0/4)$. By hypothesis (c) and the fact that $\lim_{c\downarrow 0}(1/c)\,v^{4c}(\{x\in B\colon \|x\|\geqq\varepsilon/12\})<\infty$ (see the proof of Lemma 3.1) we conclude that (3.5) holds.

But (3.5) implies that $\lim_{c\downarrow 0}\overline{\lim_{n}}\, P(\max_{1\leqq k\leqq 4cj_n}\|S_{nk}\|>\varepsilon/4)=0$. Then for every $q\geqq 2$ we have $\overline{\lim_{c\downarrow 0}}\,\overline{\lim_{n}}\, B_{q,c,n}\leqq\phi(q)M$ and, by hypothesis (d), $\lim_n A_{q,n}=0$ which imply, by (3.4), that

$$\overline{\lim_{c\downarrow 0}}\ \overline{\lim_{n}}\ P(\Delta_{J_1}(c,\xi_n)>\varepsilon)\leqq\phi(q)M$$

for each integer $q\geqq 2$. Now the ϕ-mixing condition implies (3.3). □

3.3. Corollary. *Let* $\{X_{nj}\}$, ξ_n, v *be as in Theorem* 3.2.

(i) *If* $\psi^*<\infty$ *then assertion* (I) *can be replaced by*
(I′) $\{X_{nj}\}$ *satisfies* (a), (b), (c).

(ii) *If* $\phi(1)<1$ *then assertion* (I) *can be replaced by*
(I″) $\{X_{nj}\}$ *satisfies* (a), (b), (d).

(iii) *If* $\psi^*<\infty$ *and* $\phi(1)<1$ *then assertion* (I) *can be replaced by*
(I‴) $\{X_{nj}\}$ *satisfies* (a) *and* (b).

Proof. (i) If $\psi^*<\infty$, (c) implies (d). (ii) See the proof of claim (3.5) or note that in this case, by [9, Theorem 3.4], (a) and (b) imply (c). □

Condition (∗) was considered in [9] (see Sect. 3 there) and [10]. We do not know if it can be omitted in Theorem 3.2; when $\{X_{nj}\}$ is obtained from a single stationary sequence by normalization, condition (∗) is related with variation properties of the sequence of norming constants (see Remark 3.4.3.1 below). We do not know if condition (c) can be omitted, but (a) and (b) together do not imply (c) (see the example in Remark 3.6.5). The example in Remark 3.4.4 shows that (d) can not be removed from (I) of Theorem 3.2.

3.4. Remarks. 3.4.1. As a consequence of Theorem 3.2, we can obtain a version for the dependent case of the arc-sine law in [1, Theorem 5.1]. On the other hand, we have not been able to obtain a version for the dependent case of the invariance principle in probability given in that article.

3.4.2. There are in [9] sufficient (and also necessary) conditions for some stationary mixing triangular arrays $\{X_{nj}\}$ of B-valued r.v.'s under which (I‴) of Corollary 3.3 holds (see, for example, [9, Corollary 6.5] and its proof). For the case where $\{X_{nj}\}$ arises from a single sequence $\{X_j\}$ by normalization, see Corollaries 5.9, 5.10 of [9] and [3], which contains sufficient conditions for convergence under weaker assumptions when the X_j's are real valued; we note that (I) is fulfilled by sequences which satisfy the hypotheses of Theorems 2 or 3 of [3] (see 3.4.3.1).

3.4.3.1[1]. *Let* $\{X_j: j\in N\}$ *be a stationary,* ϕ*-mixing sequence of* B*-valued r.v.'s,* $\{a(n)\}\subset(0,\infty)$ *such that* $a(n)\to\infty$ *and* $\{b(n)\}\subset B$. *Assume that* $\mathscr{L}(a(n)^{-1}(X_1+\ldots+X_n-nb(n)))\xrightarrow[w]{}v$, *a non degenerate probability measure, and write* $X_{nj}=a(n)^{-1}(X_j-b(n))$ *for* $j=1,\ldots,n$, $n\in N$. *Then* v *is stable and, if* $\alpha\in(0,2]$ *is the index of* v, $\{X_{nj}\}$ *satisfies* (∗) *if and only if*

(3.6) *for some slowly varying function* $L:(0,\infty)\to(0,\infty)$, *integrable over finite intervals,* $a(n)=n^{1/\alpha}L(n)$

and

(3.7) $\{r_n\}\subset N,\quad r_n\leqq n,\quad r_n/n\to 0\Rightarrow a(n)^{-1}r_n(b(r_n)-b(n))\to 0$ *in* B.

[1] This remark is related to Remark 1 in [9, p. 395], where Theorem 2 of [8] was used. We give the present statement since we are aware that there is a forthcoming correction to [8]

On the other hand, $\{X_{nj}\}$ *satisfies* (*) *if* (3.6) *holds,* $\sup_n nP(\|X_1\| > a(n)) < \infty$ *and*

(3.8) $\{na(n)^{-1}(b(n) - E(X_1; \|X_1\| \leq a(n)))\}$ *is bounded in* B.

Proof. First we show that ν is stable. Fix $p \in N$. Take $\{d_n\} \subset N$ such that $d_n \to \infty$, $d_n \leq n-1$ for all sufficiently large n and $d_n \sigma(a(n)^{-1} X_1) \to 0$, where $\sigma(X) = E(\|X\|(1+\|X\|)^{-1})$ if X is a random vector. Define $S_n^{(k)} = \sum_{(k-1)n+1 \leq j \leq kn} X_j$ and $\xi_n^{(k)} = \sum_{(k-1)n+1 \leq j \leq kn - d_n} X_j$ for $k = 1, \ldots, p$. We have

$$\sum_{k=1}^{p} a(n)^{-1}(S_n^{(k)} - nb(n)) = a'_{n,p}\, a(np)^{-1}(S_{np}^{(1)} - npb(np)) + b'_{n,p} \tag{3.9}$$

where $a'_{n,p} = a(n)^{-1} a(np)$ and $b'_{n,p} = a(n)^{-1} np(b(np) - b(n))$. By the choice of $\{d_n\}$ we have that $\lambda_n \doteq \mathscr{L}(a(n)^{-1}(\xi_n^{(1)} - nb(n))) \underset{w}{\longrightarrow} \nu$ and arguing similarly as in the proof of [9, Proposition 3.1] we can conclude that the law of the left member of (3.9) converges to ν^p. Take $f \in B'$ such that $\nu \circ f^{-1}$ is non degenerate (this is possible since ν is non degenerate). Now, applying f to both members of (3.9), by the convergence of types theorem, we deduce that there exists $\lim_n a'_{n,p} > 0$, because $\mathscr{L}(a(np)^{-1}(S_{np}^{(1)} - npb(np))) \underset{w}{\longrightarrow} \nu$, and then that $\{b'_{n,p}\}$ is relatively compact in B. Hence ν^p and ν are of the same type for each $p \in N$, which shows that ν is stable.

Suppose that $\{X_{nj}\}$ satisfies (*). We prove (3.6) in a way analogous to that of [5, proof of Remark 2.3]. Define $L(t) = t^{-1/\alpha} a([t])$ for $t > 0$ (put $a(0) = 1$). Fix $t \in (0, 1]$ and write $S_{n,[nt]} = a(n)^{-1} a([nt]) S_{[nt],[nt]} + c_{n,t}$ where $c_{n,t} \in B$. Since $\mathscr{L}(S_{n,[nt]}) \underset{w}{\longrightarrow} \nu^t$ by [9, Theorem 3.3] we conclude from the convergence of types theorem that there exists $\lim_n a(n)^{-1} a([nt]) > 0$ (ν is non degenerate); looking at the Lévy-Khintchine representation of ν we have that that limit must be $t^{1/\alpha}$. On the other hand, we have $S_{n+1,n+1} = a(n+1)^{-1} a(n) S_{n,n} + d_n + a(n+1)^{-1} X_{n+1}$ with $d_n \in B$ and the third term tends to zero in probability because $a(n) \to \infty$. Hence $\lim_n a(n+1)^{-1} a(n) = 1$ and $\lim_{s \to \infty} L(s)^{-1} L(ts) = 1$. Then (3.6) holds. Now take $\{r_n\}$ as in (3.7) and write

$$S_{n,r_n} = a(n)^{-1} a(r_n) S_{r_n,r_n} + a(n)^{-1} r_n(b(r_n) - b(n)). \tag{3.10}$$

By using the Karamata representation of slowly varying functions we obtain that $a(n)^{-1} a(r_n) \to 0$; this, together with $S_{n,r_n} \to 0$ in probability, implies that the last term in (3.10) tends to zero in B. Thus we have proved that if (*) is fulfilled then (3.6) and (3.7) hold. The proof of the converse is now clear.

Assume that $M \doteq \sup_n nP(\|X_1\| > a(n)) < \infty$ and that (3.6), (3.8) both hold. We will prove (3.7). Note that it is sufficient to prove that statement with $b'(n) = E(X_1; \|X_1\| \leq a(n))$ in place of $b(n)$ (if $\{c(n)\}$ is the sequence in (3.8), write

$$a(n)^{-1} r_n(b(r_n) - b(n)) = a(n)^{-1} a(r_n)\, c(r_n) - n^{-1} r_n\, c(n) + a(n)^{-1} r_n(b'(r_n) - b'(n))).$$

For this, take $\{r_n\}$ as in (3.7) and observe that for all sufficiently large n

$$\|a(n)^{-1} r_n(b'(r_n)-b'(n))\| \leqq a(n)^{-1} r_n \int_{(a(r_n),\, a(n)]} x \mathscr{L}(\|X_1\|)(dx)$$

$$\leqq a(n)^{-1} a(r_n) r_n P(\|X_1\| > a(r_n)) + a(n)^{-1} r_n \int_{a(r_n)}^{a(n)} P(\|X_1\| > x)\, dx$$

$$= u_n + v_n \quad \text{(say)}.$$

Given $s \in (0, 1)$ take $\{r_n'\} \subset N$ such that $r_n' \leqq n$ and $r_n'/n \to s$; again by the theorem of Karamata, we have $a(n)^{-1} a(r_n) \to 0$ and $a(n)^{-1} a(r_n') \to s^{1/\alpha}$. Then, breaking the integral involved in v_n at $a(r_n')$ we see that, for all sufficiently large n,

$$v_n \leqq (a(n)^{-1} a(r_n') - a(n)^{-1} a(r_n))\, r_n P(\|X_1\| > a(r_n))$$
$$+ (1 - a(n)^{-1} a(r_n'))(r_n/r_n')\, r_n' P(\|X_1\| > a(r_n'))$$

which shows that $\overline{\lim_n}\, v_n \leqq s^{1/\alpha} M$. This implies $v_n \to 0$; also $u_n \to 0$. Then (3.7) is proved and (*) holds. □

3.4.3.2. Let $\{X_{nj}\}$ be a ϕ-mixing triangular array with stationary sums such that $\mathscr{L}(S_{nj_n}) \xrightarrow[w]{} v$. Then $\{X_{nj}\}$ satisfies (*) if and only if $X_{n1} \xrightarrow[P]{} 0$ and for every $\varepsilon > 0$ there exists $a > 0$ such that

$$\overline{\lim_n} \max_{1 \leqq k \leqq a j_n} P(\|S_{nk}\| > \varepsilon) < 1.$$

Proof. We prove the "if" part. Let $\{r_n\}$ be as in (*) and fix $\varepsilon > 0$. By hypothesis, we can find $q \in N$, $\alpha \in (\phi(q), 1)$, $p_0 \in N$ and $n_0 \in N$ such that $\max_{1 \leqq k \leqq j_n/p_0} P(\|S_{nk}\| > \varepsilon) \leqq 1 - \alpha$ if $n \geqq n_0$. Given $p \in N$, $p \geqq p_0$ let $n_1 \in N$, $n_1 \geqq n_0$ such that $r_n + q < j_n/p$ if $n \geqq n_1$; for such n's write $U_n^{(p)} = \sum_{r_n + q \leqq j \leqq j_n/p} X_{nj}$, $V_n^{(p)} = S_{nr_n} + U_n^{(p)}$. By [9, Proposition 3.1] and since $\sum_{r_n < j < r_n + q} X_{nj} \to 0$ in probability (because $X_{n1} \to 0$ in probability), we have $\mathscr{L}(V_n^{(p)}) \xrightarrow[w]{} v^{1/p}$. On the other hand, if $n \geqq n_1$ it holds that $(\alpha - \phi(q)) P(\|S_{nr_n}\| > 2\varepsilon) \leqq P(\|S_{nr_n}\| > 2\varepsilon, \|U_n^{(p)}\| \leqq \varepsilon) \leqq P(\|V_n^{(p)}\| > \varepsilon)$. Then $\overline{\lim_n}\, P(\|S_{nr_n}\| > 2\varepsilon) \leqq (\alpha - \phi(q))^{-1} v^{1/p}(\{x\colon \|x\| \geqq \varepsilon\})$ for every $p \geqq p_0$. This implies $\lim_n P(\|S_{nr_n}\| > 2\varepsilon) = 0$. □

3.4.4. Fix $\alpha \in (0, 2)$. Let Y_j, $j \in N$, η_j, $j \geqq 0$ be independent identically distributed real random variables with $\mathscr{L}(Y_1)(dx) = I_{\{y:\, |y| \geqq 1\}}(x)(\alpha/2)|x|^{-1-\alpha} dx$. Then $P(|Y_1| > x) = x^{-\alpha}$ if $x \geqq 1$. Define $X_j = Y_j + \eta_j - \eta_{j-1}$ for $j \in N$ and $X_{nj} = n^{-1/\alpha} X_j$ for $j = 1, \ldots, n$, $n \in N$; then $\{X_j\}$ is stationary and 1-dependent. Since $\mathscr{L}(n^{-1/\alpha}(Y_1 + \ldots + Y_n)) \xrightarrow[w]{} v$, a stable measure of index α, (a) of (I) of Theorem 3.2 holds with this v. Remark 3.4.3.1 (or, by symmetry, Remark 2 on p. 395 of [9]) shows that $\{X_{nj}\}$ satisfies (*). Also (c) is fulfilled: given $\varepsilon > 0$, $nP(|X_{n1}| > \varepsilon) \leqq 3nP(|Y_1| > (\varepsilon/3) n^{1/\alpha}) = 3(\varepsilon/3)^{-\alpha}$. But $\{X_{nj}\}$ does not satisfy (d): given $\varepsilon > 0$, for all sufficiently large n

$$\begin{aligned}&nP(|X_{n1}|>\varepsilon, |X_{n2}|>\varepsilon)\\&\geqq nP(|\eta_1|>2\varepsilon n^{1/\alpha}, |Y_1|\leqq(\varepsilon/2)\, n^{1/\alpha}, |\eta_0|\leqq(\varepsilon/2)\, n^{1/\alpha}, |Y_2|\leqq(\varepsilon/2)\, n^{1/\alpha}, |\eta_2|\leqq(\varepsilon/2)\, n^{1/\alpha})\\&=nP(|\eta_1|>2\varepsilon n^{1/\alpha})(P(|Y_1|\leqq(\varepsilon/2)\, n^{1/\alpha}))^4\\&=(2\varepsilon)^{-\alpha}(1-(\varepsilon/2)^{-\alpha} n^{-1})^4\end{aligned}$$

which goes to $(2\varepsilon)^{-\alpha}$ as $n\to\infty$. Hence (d) can not be omitted in Theorem 3.2.

Now we point out another feature of this example. Let $r\in N$, $r\geqq 2$ and consider the subsequence $\{X_{jr}: j\in N\}$; we have $\phi(r)=0$ and $\mathscr{L}(n^{-1/\alpha}(X_r+X_{2r}+\ldots+X_{nr}))\xrightarrow[w]{} v^3$ which has the same index α (note that $Y_r+Y_{2r}+\ldots+Y_{nr}$, $\eta_r+\eta_{2r}+\ldots+\eta_{nr}$ and $\eta_{r-1}+\eta_{2r-1}+\ldots+\eta_{nr-1}$ are independent for each n). Here (4.2) of Theorem 4 of [8] is satisfied, even with the same norming constants than those for the whole sequence, but the conclusion there can not hold because it would imply assertion (II) of Theorem 3.2. (For the Gaussian case, see Remark 3.6.5.)

Now we turn to the Gaussian case. Our next result generalizes Theorem 4.8 and Corollaries 4.9 and 4.10 of [9] (see Remark 3.6.1 and, concerning to hypotheses (1) and (2) of [9, Theorem 4.8], Remark 3.4.3.2) and part of Theorem 2.13 of [5] (see Remark 3.4.3.1 and [5, Lemma 3.3]).

Recall that if γ is a Gaussian probability measure on B then the process ξ_γ can be taken with trajectories in C, the Banach space of continuous functions from $[0, 1]$ into B endowed with the supremum norm. We will denote by Q'_γ the distribution of ξ_γ on (the Borel σ-algebra of) C. If $a_1, \ldots, a_n\in B$ we define $p'_n(a_1, \ldots, a_n)\in C$ by $p'_n(a_1, \ldots, a_n)(t)=a_{[nt]}+(nt-[nt])(a_{[nt]+1}-a_{[nt]})$ for $t\in[0, 1]$ (with the convention $a_0=0$).

3.5. Corollary. *Let $\{X_{nj}: 1\leqq j\leqq j_n, n\in N\}$ be a stationary, ϕ-mixing triangular array; write $\xi_n=p_{j_n}(S_{n1}, \ldots, S_{nj_n})$ and $\xi'_n=p'_{j_n}(S_{n1}, \ldots, S_{nj_n})$. Let γ be a probability measure on B. Then the following statements are equivalent:*

(1) *$\mathscr{L}(S_{nj_n})\xrightarrow[w]{}\gamma$, $\{X_{nj}\}$ satisfies $(*)$ and for every $\varepsilon>0$, $\lim_n j_n P(\|X_{n1}\|>\varepsilon)=0$.*

(2) *γ is Gaussian and $\mathscr{L}(\xi_n)\xrightarrow[w]{} Q_\gamma$ in D.*

(3) *γ is Gaussian and $\mathscr{L}(\xi'_n)\xrightarrow[w]{} Q'_\gamma$ in C.*

(4) *$\{\mathscr{L}(\xi'_n)\}$ is relatively compact in C and $\mathscr{L}(\xi'_n(1))\xrightarrow[w]{}\gamma$.*

(5) *γ is Gaussian and there exist a triangular array $\{X'_{nj}\}$ and a stochastic process $\xi=\{\xi(t): t\in[0, 1]\}$ with trajectories in C defined on a common probability space which satisfy*

(a) *$\mathscr{L}(X'_{n1}, \ldots, X'_{nj_n})=\mathscr{L}(X_{n1}, \ldots, X_{nj_n})$ for each $n\in N$,*

(b) *$\mathscr{L}(\xi)=Q'_\gamma$,*

(c) *$\max_{1\leqq k\leqq j_n}\|S'_{nk}-\xi(k/j_n)\|\to 0$ a.s. as $n\to\infty$,*

where $S'_{nk}=\sum_{j=1}^{k} X'_{nj}$.

Proof. That (1) implies (2) follows from [10, Proposition 2.4] and Theorem 3.2. In order to prove the equivalence of (2) and (3) see [2, § 18] and [5, Proof of Remark 2.11]. It it easy to deduce (3) from (5).

(4) ⇒ (1). As in the proof of the implication (III) ⇒ (II) of Theorem 3.2 we can show that $\{X_{nj}\}$ satisfies (*) ($F_{\varepsilon,t} \cap C$ is a closed subset of C). Arguing as in [5, Proof of Remark 2.3] (given $\delta > 0$ and $\varepsilon > 0$, $\{x \in C: \sup\{\|x(s) - x(t)\|: s, t \in [0,1], |s-t| \leqq \delta\} \geqq \varepsilon\}$ is a closed subset of C) we can prove that if $\{\mathscr{L}(\xi'_{n_k})\}$ converges weakly in C for some sequence $\{n_k\} \subset N$ then $\max_{j \leqq j_{n_k}} \|X_{n_k,j}\| \xrightarrow[P]{} 0$; this implies that $\max_{j \leqq j_n} \|X_{nj}\| \xrightarrow[P]{} 0$. By Lemma 2.2, $j_n P(\|X_{n1}\| > \varepsilon) \to 0$ for every $\varepsilon > 0$.

(3) ⇒ (5). Suppose that (3) holds. By applying a well-known result of Skorohod [11, 3.1.1] we obtain C-valued r.v.'s $\tilde{\xi}_0, \tilde{\xi}_1, \ldots$ (defined on some probability space) such that $\mathscr{L}(\tilde{\xi}_0) = Q'_\gamma$, $\mathscr{L}(\tilde{\xi}_n) = \mathscr{L}(\xi'_n)$ if $n \in N$ and $\tilde{\xi}_n \to \tilde{\xi}_0$ a.s.

Define $S_0 = C$, $S_n = B^{j_n}$ for $n = 1, 2, \ldots$, $T_n = C$ for $n = 0, 1, \ldots$ and let $S = \prod_{n \geqq 0} S_n$, $T = \prod_{n \geqq 0} T_n$ (with the product σ-algebras). Define $\kappa_n: S_n \to T_n$ by $\kappa_0 = \mathrm{id}_C$, $\kappa_n = p'_{j_n} \circ h_{j_n}$ if $n = 1, 2, \ldots$ where $h_{j_n}: B^{j_n} \to B^{j_n}$ is defined by $h_{j_n}(x_1, \ldots, x_{j_n}) = (x_1, x_1 + x_2, \ldots, x_1 + \ldots + x_{j_n})$. Consider the probability measures $\mu_0 = Q'_\gamma$ on S_0, $\mu_n = \mathscr{L}(X_{n1}, \ldots, X_{nj_n})$ on S_n $(n = 1, 2, \ldots)$ and $\lambda = \mathscr{L}(\{\tilde{\xi}_n\}_{n \geqq 0})$ on T. Now, letting ξ be the canonical projection from S onto S_0 and $(X'_{n1}, \ldots, X'_{nj_n})$ that of S onto S_n we have, by an application of [1, Theorem A.1], that there exists a probability measure σ on S such that $\mathscr{L}_\sigma(X'_{n1}, \ldots, X'_{nj_n}) = \mu_n$ if $n = 1, 2, \ldots$ and $\mathscr{L}_\sigma(\xi, \{\kappa_n(X'_{n1}, \ldots, X'_{nj_n})\}_{n \geqq 1}) = \lambda$. Then (a) and (b) of (5) are satisfied and (c) also holds since $\mathscr{L}_\sigma(\{\max_{k \leqq j_n} \|S'_{nk} - \xi(k/j_n)\|\}_{n \geqq 1}) = \mathscr{L}(\{\max_{k \leqq j_n} \|\tilde{\xi}(k/j_n) - \tilde{\xi}_0(k/j_n)\|\}_{n \geqq 1})$ and $\tilde{\xi}_n \to \tilde{\xi}_0$ (in C) a.s. □

3.6. Remarks. 3.6.1. The hypothesis of stationarity in Corollary 3.5 can be replaced by the assumption that $\{X_{nj}\}$ has stationary sums; we now sketch a direct proof of (1) ⇒ (2) for this case. Assume (1) holds. By [10, Proposition 2.4], γ is Gaussian. The convergence of the finite-dimensional distributions can be treated as in the proof of Theorem 3.2; by [2, Theorem 15.5] it is sufficient to show that

$$\lim_{c \downarrow 0} \overline{\lim_n} P\Big(\sup_{|s-t| \leqq c;\, s,t \in [0,1]} \|\xi_n(s) - \xi_n(t)\| > \varepsilon\Big) = 0$$

for every $\varepsilon > 0$. Fix $\varepsilon > 0$. Let $c > 0$, write r for the integer part of $1/c$, $I_k = [(k-1)c, kc)$ for $k = 1, \ldots, r$ and $I_{r+1} = [rc, 1]$. Arguing as in [2, p. 56] and by stationarity we have

$$\begin{aligned} P\Big(&\sup_{|s-t| \leqq c} \|\xi_n(s) - \xi_n(t)\| > \varepsilon\Big) \\ &\leqq \sum_{k=1}^{r+1} P\Big(\sup_{t \in I_k} \|\xi_n(t) - \xi_n((k-1)c)\| > \varepsilon/3\Big) \\ &\leqq ((1/c) + 1) P\Big(\max_{1 \leqq k < cj_n + 1} \|S_{nk}\| > \varepsilon/3\Big). \end{aligned}$$

Now the proof is concluded through an argument similar to that which led to (3.5), using Lemma 2.1 and the fact that $\lim_{c \downarrow 0} (1/c) \gamma^c(\{x: \|x\| \geqq \varepsilon/9\}) = 0$ since γ is Gaussian.

3.6.2. A. de Acosta called our attention to the fact that in the Gaussian case, in contrast to the general one, the invariance principle in probability is a direct consequence of the invariance principle in distribution; the above proof of (3)$\Rightarrow$(5) follows his suggestion.

3.6.3. In (1) *of Corollary* 3.5 *the condition* $\lim_n j_n P(\|X_{n1}\|>\varepsilon)=0$ *for every* $\varepsilon>0$ *can be replaced by* $\max_{1\leqq j\leqq j_n} \|X_{nj}\| \xrightarrow[P]{} 0$ (Lemma 2.2).

3.6.4. If $\phi(1)<1$ *and* γ *is Gaussian then statement* (1) *in Corollary* 3.5 *can be replaced by*

$$(1') \qquad \mathscr{L}(S_{nj_n}) \xrightarrow[w]{} \gamma \text{ and } \{X_{nj}\} \text{ satisfies } (*).$$

(See the proof above or use [9, Theorem 4.1]).

3.6.5. We do not know if condition (*) can be removed from (1) of Corollary 3.5. The following example shows that the condition $\lim_n j_n P(\|X_{n1}\|>\varepsilon)=0$ for every $\varepsilon>0$ can not be omitted. Let Y_j, $j\in N$, η_j, $j\geqq 0$, be independent symmetric real random variables such that Y_j, $j\in N$, are identically distributed with $E(Y_1^2)=1$ and η_j, $j\geqq 0$, are identically distributed with $\mathscr{L}(\eta_0)(dx)=I_{\{y:\,|y|\geqq 1\}}(x)\,2|x|^{-3}\log|x|\,dx$. Then $P(|\eta_0|>x)=x^{-2}(1+2\log x)$ if $x\geqq 1$. Define $X_j=Y_j+\eta_j-\eta_{j-1}$ for $j\in N$ and $X_{nj}=n^{-1/2}X_j$ for $j=1,\dots,n$, $n\in N$; then $\{X_j\}$ is stationary and 1-dependent. We have that $\mathscr{L}(S_{nn}) \xrightarrow[w]{} N(0,1)$ and $\{X_{nj}\}$ satisfies (*) but, given $\varepsilon>0$, for all sufficiently large n

$$nP(|X_{n1}|>\varepsilon)\geqq(2\varepsilon)^{-2}(1+2\log(2\varepsilon n^{1/2}))\,P(|Y_1|\leqq(\varepsilon/2)\,n^{1/2})\,P(|\eta_0|\leqq(\varepsilon/2)\,n^{1/2})$$

which goes to ∞ as $n\to\infty$ (incidentally, this example shows that the hypothesis $\phi(1)<1$ can not be omitted in two results of [9] - Theorem 3.4 and "only if" part of Theorem 4.1; see [10]). Now let $r\in N$, $r\geqq 2$ and consider the subsequence $\{X_{jr}: j\in N\}$; we have $\phi(r)=0$ and $\mathscr{L}(a(n)^{-1}(X_r+X_{2r}+\dots+X_{nr})) \xrightarrow[w]{} N(0,1)$ where $a(n)=n^{1/2}\log n$. Here the norming constants for such subsequences are larger than those for the whole sequence. This example shows that condition (4.2) in Theorem 4 of [8] must be modified (in the Gaussian case); the following holds: *let* $\{X_j: j\in N\}$ *be a stationary,* ϕ*-mixing sequence of* B*-valued r.v.'s,* $\{a(n)\}\subset(0,\infty)$, $\{b(n)\}\subset B$ *such that* $\{X_{nj}\}\doteq\{a(n)^{-1}(X_j-b(n))\}$ *satisfies* (*), $\{\mathscr{L}(a(n)^{-1}(X_1+\dots+X_n-nb(n)))\}$ *converges weakly to a probability measure* γ *and, for some integer* $r\geqq 1$ *with* $\phi(r)<1$, $\mathscr{L}(a(n)^{-1}(X_r+X_{2r}+\dots+X_{nr}-nb(n))) \xrightarrow[w]{} \gamma_r$ *for some Gaussian law* γ_r*; then* (5) *of Corollary* 3.5 *holds* (since (1) is satisfied as an application of [9, Theorem 4.1] shows).

As proved in [10], the hypothesis $\phi(1)<1$ can be removed from Theorem 4.4 of [9] and its corollaries and, moreover, (1) of Corollary 3.5 above is satisfied (look at the proofs in [9]); the same holds for the results in [10]. Thus we have a functional central limit theorem in such cases. We write out the one obtained from [10, Corollary 3.4].

We will assume that B is a separable Hilbert space and we shall write $d_k(x)=\inf\{\|x-y\|: y\in F_k\}$, F_k being the subspace spanned by $\{e_1,\dots,e_k\}$, where $\{e_i: i\in N\}$ is a fixed (but arbitrary) orthonormal basis of B, when B is infinitedimensional; if the dimension of B is finite we have an orthonormal basis $\{e_1, \ldots, e_n\}$ $(n \in N)$ and we put $d_k = 0$ for $k \geqq n$. If γ is a centered Gaussian measure on B, Φ_γ denotes its covariance. For a r.v. X and $f \in B'$ (the topological dual of B) we write

$$m_X^2(f) = \lim_{x \to \infty} \frac{(E(f(X); \|X\| \leqq x))^2}{E(\|X\|^2; \|X\| \leqq x)} = \begin{cases} \dfrac{(Ef(X))^2}{E\|X\|^2} & \text{if } 0 < E\|X\|^2 < \infty \\ 0 & \text{if } E\|X\|^2 = \infty. \end{cases}$$

3.7. Corollary. *Suppose that B is a Hilbert space. Let $\{X_j : j \in N\}$ be a stationary, ϕ-mixing sequence with $\sum_{j=1}^{\infty} \phi^{1/2}(j) < \infty$. Assume that $E(\|X_1\|^2) \in (0, \infty]$ and*

(i) $\displaystyle\lim_{x \to \infty} \frac{x^2 P(\|X_1\| > x)}{E(\|X_1\|^2; \|X_1\| \leqq x)} = 0,$

(ii) *there exists a sequentially w^*-dense subset W of B' such that for every $f \in W$ and each $j \in N$ the limit*

$$\Phi_j^{(0)}(f) = \lim_{x \to \infty} \frac{E(f(X_1) f(X_j); \|X_1\| \leqq x, \|X_j\| \leqq x)}{E(\|X_1\|^2; \|X_1\| \leqq x)} \text{ exists,}$$

(iii) $\displaystyle\lim_k \overline{\lim_{x \to \infty}} \frac{E(d_k^2(X_1); \|X_1\| \leqq x)}{E(\|X_1\|^2; \|X_1\| \leqq x)} = 0.$

Then $E\|X_1\| < \infty$, the sum

$$\Phi(f) = (\Phi_1^{(0)}(f) - m_{X_1}^2(f)) + 2 \sum_{j=2}^{\infty} (\Phi_j^{(0)}(f) - m_{X_1}^2(f))$$

converges for every $f \in W$ and there exist a sequence $\{a_n\}$ with $a_n > 0$, $a_n \to \infty$ and a centered Gaussian measure γ such that $\Phi_\gamma(f, f) = \Phi(f)$ for each $f \in W$ and $\mathscr{L}(\xi_n) \xrightarrow[w]{} Q_\gamma$ in D where $\xi_n(t) = a_n^{-1} \sum_{1 \leqq j \leqq [nt]} (X_j - EX_1)$ if $0 \leqq t \leqq 1$.

Note. Let Ω = irrational numbers in (0, 1), $\mathscr{A}$ = Borel subsets of Ω, P defined by $P(d\omega) = (\log 2)^{-1}(1+\omega)^{-1}\, d\omega$ (Gauss' measure); given $\omega \in \Omega$, let $(a_1(\omega), a_2(\omega), \ldots)$ be the sequence of partial quotients of the continued fraction expansion of ω. For each $n \in N$ let h_n be a function from N into B and define $X_{nj} = h_n(a_j)$ if $j = 1, \ldots, n$, $n \in N$. Then $\{X_{nj}\}$ is stationary and satisfies all the dependence assumptions here (in particular, those of Corollary 3.3(iii)) and in [9]. Something about this will appear elsewhere.

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