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Weakly Isomorphic Transformations That Are Not Isomorphic

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Summary. A new method for construction of transformations $T_i:(X_i, \mathscr{B}_i, \mu_i)\circlearrowleft$, $i=1,2$, that are factors of each other but that are not measure-theoretically isomorphic is provided. This method uses ergodic product cocycles of the form $\varphi \circ S^{i_1} \times \varphi \circ S^{i_2} \times \dots$, where $\varphi: X \to Z_2$ is a cocycle, S belongs to the centralizer of T and T is an ergodic translation on a compact, monothetic group X.

0. Introduction

In [20] Sinai introduced a concept of weak isomorphism between ergodic transformations on a Lebesgue space. It has been unknown for some time whether this notion is strictly weaker then the measure-theoretic isomorphism. The first construction of two ergodic transformations that are factors of each other (i.e. weakly isomorphic) but that are not isomorphic was given in [18]. Then in 1978 Rudolph developed theory of transformations having the minimal self-joining property [19]. His machinery applied to the Chacon transformation $T:(X, \mathscr{B}, \mu)\circlearrowleft$ [5] gives the following example of two nonisomorphic but weakly isomorphic transformations

$$T_1 = T \times T \times \dots : (X \times X \times \dots,\ \mu \times \mu \times \dots)\circlearrowleft$$
$$T_2 = \tau \times T \times T \times \dots : (Y \times X \times X \times \dots,\ \nu \times \mu \times \mu \times \dots)\circlearrowleft$$

where $\tau:(Y, \ell, \nu)\circlearrowleft$ is the factor of $(T \times T,\ \mathscr{B} \otimes \mathscr{B}, \mu \times \mu)$ obtained from the σ-algebra $\ell \subset \mathscr{B} \otimes \mathscr{B}$ of the flip-invariant sets (i.e. $A \in \ell$ if $fA = A$, $f(x,y) = (y,x)$). Recently Thouvenot [21] constructed new examples of nonisomorphic transformations that were weakly isomorphic. His method uses some special Gaussian processes.

There are at least three movivations for this paper. The first purpose is to introduce some new method leading to some nontrivial examples in ergodic theory. This method uses merely the notion of ergodic transformation with discrete spectrum $T:(X,\mathscr{B},\mu)\circlearrowleft$ and a Z_2-extension of it. More precisely, each $\varphi: X\to Z_2$ is called a *cocycle* whenever it is measurable. Then the automorphism

$$T_\varphi:(X\times Z_2,\tilde{\mathscr{B}},\tilde{\mu})\circlearrowleft$$
$$T_\varphi(x,i)=(Tx,\ \varphi(x)+i),$$

where $\tilde{\mu}=\mu\times\nu_2\,(\nu_2(i)=1/2, i=0,1)$, $\tilde{\mathscr{B}}$ is the corresponding product σ-algebra, is called a *Z_2-extension of T*. By the *centralizer* $C(T)$ of T we mean the set of all measure-preserving $S:(X,\mathscr{B},\mu)\circlearrowleft$ commuting with T. An ergodic cocycle $\varphi: X\to Z_2$ is said to be *strongly ergodic with respect to S*, $S\in C(T)$ (shortly *S-strongly ergodic*), whenever for every $i_1<i_2<\ldots<i_k$, $k\geqq 2$ and for every $U\in C(T)$ the cocycle $\varphi S^{i_1}+\varphi S^{i_2}+\ldots+\varphi S^{i_k}+\varphi U$ is ergodic. In Sect. 3 we show that this property is not vacuous. Assuming that φ is S-strongly ergodic the following transformations

$$T_1=T_{\varphi\times\varphi S\times\varphi S^2\times\ldots}:(X\times Z_2\times Z_2\times Z_2\times\ldots,\ \mu\times\nu_2\times\nu_2\times\nu_2\times\ldots)\circlearrowleft$$
$$T_2=T_{\varphi\times\varphi S^2\times\varphi S^3\times\ldots}:(X\times Z_2\times Z_2\times Z_2\times\ldots,\ \mu\times\nu_2\times\nu_2\times\nu_2\times\ldots)\circlearrowleft$$

are not isomorphic but that are factors of each other.

It is not hard to see that the concept of S-strongly ergodic cocycles is a new invariant of the relative isomorphism [23] in the class of all ergodic Z_2-extensions over a fixed T. The class of all ergodic Z_2-extensions of a T is especially studied in case X is an adding machine and T is an ergodic translation on X because if this is the case we achieve some automatic sequences (see [1, 11]) as examples of such extensions. It has been noticed by Rudolph that Theorem 8 [8] and Theorem 9 [11] combined with the Feldman result [3] say that it is impossible to find a countable, complete set of Borel invariants (in sense of [3]) even for the relative isomorphism. That is why we seek not real-valued new invariants.

The third reason is that we exhibit some relations between the ergodicity of the cocycles of the form $\varphi S^{i_1}\times\varphi S^{i_2}\times\ldots\times\varphi S^{i_k}$ and the structure of ergodic multi-joinings of T_φ. From this point of view this paper can be regarded as the first step to describe all ergodic n-joinings of group extensions of transformations with discrete spectrum (or more generally of simple transformations [6]).

For further discussion we refer to the last section.

The author wishes to thank M.K. Mentzen for a lot of discussions on the subject. Actually he first formulated Proposition 1 and proved it using other ideas in discrete spectrum case and noticed that any cocycle φ with the trivial centralizer was prime.

1. Notations

All automorphisms are assumed ergodic unless it states otherwise.

Let $T:(X,\mathscr{B},\mu)\circlearrowleft$ be an automorphismon a Lebesgue space. By $Sp(T)$ we denote the group of all eigenvalues of T. Then by the *centralizer* $C(T)$ of T we mean the semigroup of all endomorphisms $S:(X,\mathscr{B},\mu)\circlearrowleft$ such that $ST=TS$. The centralizer is *trivial* whenever $C(T)=\{T^i: i\in Z\}$. A T-invariant sub-σ-algebra $\ell\subset\mathscr{B}$ (i.e. $T^{-1}\ell=\ell$) is said to be a *factor of T* (more precisely $T:(X,\ell,\mu)\circlearrowleft$ is called a factor of $T:(X,\mathscr{B},\mu)\circlearrowleft$). By $J(T,T)$ we denote the space of all *2-joinings of* T, i.e. $\lambda\in J(T,T)$ if λ is a $T\times T$-invariant probability measure on $\mathscr{B}_1\otimes\mathscr{B}_2$, $\mathscr{B}_i=\mathscr{B}$, $i=1,2$ and $\lambda|\mathscr{B}_i=\mu$. A standard example of ergodic 2-joinings arises from $C(T)$. Namely μ_S defined by

$$\mu_S(A\times B)=\mu(A\cap S^{-1}B),\ S\in C(T)$$

belongs to $J(T,T)$ (μ_S is concentrated on the graph of the S). T is called *2-fold simple* [6, 22] if every ergodic 2-joining is either $\mu\times\mu$ or lies on the graph of some $S\in C(T)$. Another kind of 2-joinings (not necessarily ergodic) comes from the factors of T. Namely if $\mathscr{C}$ is a factor of $\mathscr{B}$ and λ is a 2-joining of T on $\mathscr{C}$ then the formula

(1) $$\hat{\lambda}(A\times B)=\int_{X\times X}E(A|\mathscr{C})\times E(B|\mathscr{C})\,d\lambda,\qquad A,B\in\mathscr{B}$$

defines a $T\times T$-invariant measure on $\mathscr{B}\otimes\mathscr{B}$ with right marginals called the *relatively independent extension of* λ. It is not hard to see how to define the space of all n-joinings $J(T,\dots,T)$, $n=1,2,\dots,\infty$ and also the definitions of the graph joinings and the relatively independent extensions can be easily transfered. Any transformation with discrete spectrum is 2-fold simple (in fact any ergodic n-joining is an off-diagonal measure $\mu_{S_1,\dots,S_n}$, $S_i\in C(T)$) [6]. If T is 2-fold simple and does not have discrete spectrum then it is weakly mixing [6].

Let G be a compact, abelian, metric group with the Haar measure μ_G. Let $\varphi:X\to G$ be measurable (i.e. φ is a *G-cocycle*, if $G=Z_2$ φ is simply called a *cocycle*). Then the automorphism

$$T_\varphi:(X\times G,\tilde{\mu})\circlearrowleft\qquad \tilde{\mu}=\mu\times\mu_G$$
$$T_\varphi(x,g)=(Tx,\varphi(x)+g)$$

is called a *G-extension of* T. φ is said to be *ergodic* if T_φ is ergodic. It turns out that

φ is ergodic iff whenever $\chi\in\hat{G}$ (the character group of G) *and* $f:X\to S^1$ *is measurable satisfy*

(2) $$\frac{f(Tx)}{f(x)}=\chi(\varphi(x))$$

then $\chi=1$. [17]

Let us observe that the automorphisms σ_g, $\sigma_g(x,h)=(x,h+g)$ belong to $C(T_\varphi)$. We say that φ has the *trivial centralizer* whenever $C(T_\varphi)=\{(T_\varphi)^n\sigma_g:n\in Z, g\in G\}$. Also, the sub-$\sigma$-algebra $\{A\times G:A\in\mathscr{B}\}$ is T_φ-invariant (this factor is isomorphic to T). By abuse of the notations we use the letter $\mathscr{B}$ to denote the factor. A cocycle φ is said to be *prime* if the only proper factors of T_φ are $\mathscr{B}$ and the factors of $\mathscr{B}$.

Assume that T is 2-fold simple. When consider T with discrete spectrum we claim that φ is ergodic. If T is weakly mixing we require that φ is weakly mixing as well. Assume that $\hat{S}\in C(T_\varphi)$. Then there are $f:X\to G$ measurable, v a continuous epimorphism of G and $S\in C(T)$ such that

(3) $$\hat{S}(x,g)=(Sx,f(x)+v(g))=S_{f,v}(x,g)$$

(4) $$f(x)+\varphi(Sx)=f(Tx)+v(\varphi(x))\quad\text{(see [9, 15])}.$$

If (4) holds then we say that S *can be lifted to the centralizer of* T_φ (i.e. there is $\hat{S}$ defined by (3) such that the action of $\hat{S}$ on $\mathscr{B}$ coincides with the action of S). If $\hat{S}$ is a lifting of S, then $\hat{S}\sigma_g$ so is, $g\in G$. φ is called *2-simple* if for every ergodic 2-joining $\lambda\in J(T_\varphi, T_\varphi)$ either

$$\lambda=\tilde{\mu}\times\tilde{\mu} \qquad \text{or} \qquad \lambda=\tilde{\mu}_{\hat{S}} \quad \text{or}$$

$\lambda=\hat{\mu}_S$ (i.e. λ is the relatively independent extension of an off-diagonal measure μ_S). If $\lambda\in J(T_\varphi,\dots,T_\varphi)$ then by $\bar{\lambda}$ we denote the projection of λ on $\mathscr{B}\otimes\dots\otimes\mathscr{B}$ (i.e. $\bar{\lambda}\in J(T,\dots,T)$). We call φ *simple with respect to* S, $S\in C(T)$ as soon as for every $i_1<i_2<\dots<i_k$ and for every $U\in C(T)$ such that $US^j, j\in Z$ cannot be lifted to $C(T_\varphi)$, the relatively independent extension $\hat{\mu}_{S^{i_1},\dots,S^{i_k},U}$ of the off-diagonal measure $\mu_{S^{i_1},\dots,S^{i_k},U}$ is ergodic.

2. Z_2-Cocycles, Joinings and Product Cocycles

From now on we assume that $G=Z_2$. Having T to be 2-fold simple we observe the following

Lemma 1. *Let $\varphi: X\to Z_2$ be ergodic (weakly mixing). Then the relatively independent extension $\hat{\mu}_S$ of μ_S is ergodic iff the product cocycle $\varphi\times\varphi S: X\to Z_2\times Z_2$ is ergodic (i.e. $T_{\varphi\times\varphi S}$ is ergodic).*

Proof. Consider $T_{\varphi\times\varphi S}: (X\times Z_2\times Z_2, \mu\times\nu_2\times\nu_2)\circlearrowleft$ and the measurable map f: $X\times Z_2\times Z_2\to(X\times Z_2)\times(X\times Z_2), f(x,i,j)=(x,i,Sx,j)$. Then $(T_\varphi\times T_\varphi)f=fT_{\varphi\times\varphi S}$ which implies that the dynamical systems $(T_{\varphi\times\varphi S}, \mu\times\nu_2\times\nu_2)$ and $(T_\varphi\times T_\varphi,\lambda)$ where λ is the image of $\mu\times\nu_2\times\nu_2$ via f are isomorphic. This is an immediate observation that λ is just the relatively independent extension of μ_S. □

Proposition 1. *Every ergodic (weakly mixing) Z_2-cocycle is 2-simple.*

Proof. First, assume that T is weakly mixing and let $\lambda\in J(T_\varphi, T_\varphi)$ be ergodic. If $\bar{\lambda}=\mu\times\mu$ then it is well-known that $\lambda=\tilde{\mu}\times\tilde{\mu}$ [6]. Therefore suppose that $\bar{\lambda}=\mu_S$, $S\in C(T)$. Consider the product cocycle $\varphi\times\varphi S$. If this cocycle is ergodic then by Lemma 1 we achieve that $\hat{\mu}_S$ is ergodic. Assume this is not the case. Then in view of (2) we get

$$\frac{\tilde{f}(Tx)}{\tilde{f}(x)}=\chi_1(\varphi(x))\,\chi_2(\varphi(Sx)), \tag{5}$$

$\chi_i\in\hat{Z}_2, \tilde{f}: X\to S^1$ is measurable. Hence $\tilde{f}^2(Tx)/\tilde{f}^2(x)=1$ and by the ergodicity of T $\tilde{f}^2$ is constant. Thus $\tilde{f}$ gets either two values, say ± 1, or is constant. In both cases (5) can be rewrtitten as

$$f(Tx)+f(x)=\varphi(x)+\varphi(Sx)$$

for some measurable $f: X\to Z_2$ (if $\chi_1=1$ or $\chi_2=1$ then φ is not ergodic). Hence, by (4) S can be lifted to the centralizer of T_φ. Then the following general lemma says that λ has to be on the graph of an $\hat{S}$. □

Lemma 2. *Let T be 2-fold simple and assume that $\varphi: X \to G$ is ergodic (weakly mixing). Assume that $\lambda \in J(T_\varphi, T_\varphi)$ is ergodic and $\bar{\lambda} = \mu_S$ and besides that S can be lifted to the centralizer of T_φ. Then $\lambda = \tilde{\mu}_{\hat{S}}$ for some lifting $\hat{S}$ of S.*

Proof. Consider $\lambda \in J(T_\varphi, T_\varphi)$ which is ergodic and $\bar{\lambda} = \mu_S$. Then λ is concentrated on the set $C = \{(x, Sx, g_1, g_2): x \in X,\ g_1, g_2 \in G\}$. Now, S can be lifted, so the formula (4) holds. Denote $\hat{S} = S_{f,v}$ and observe that the support of $\mu_{\hat{S}}$ is the set $\{(x, Sx, g_1, f(x) + v(g_1)): x \in X, g_1 \in G\} = \{((x, g_1), S_{f,v}(x, g_1)): (x, g_1) \in X \times G\}$. Consider the following measurable map $\xi: C \to G$, $\xi(x, Sx, g_1, g_2) = f(x) + v(g_1) - g_2$. It follows from (4) that $\xi(T_\varphi \times T_\varphi(x, Sx, g_1, g_2)) = f(Tx) + v(\varphi(x)) - \varphi(Sx) + v(g_1) - g_2 = f(x) + v(g_1) - g_2 = \xi(x, Sx, g_1, g_2)$. Therefore ξ is a.e. λ constant. In other words $f(x) + v(g_1) - g_2 = g_0$ λ a.e. It implies that the support of λ is $\{(x, Sx, g_1, f(x) + v(g_1) - g_0): x \in X, g_1 \in G\}$ which is the support of $\tilde{\mu}_{\hat{S}_{\sigma(-g_0)}}$. □

Proposition 1 says that the structure of 2-joinings is determined by the structure of $C(T_\varphi)$. However 2-joinings determine the structure of factors. Hence, not surprisingly, the structure of factors arises from the centralizer of T_φ. Let T be 2-fold simple. Assume that $\ell \subset \mathscr{B}$ is a factor and let

$$H(\ell) = \{S \in C(T): (\forall A \in \ell) \qquad SA = A\}.$$

Thus $H(\ell)$ is a subgroup of the centralizer of T. Conversely, if $H \subset C(T)$, then

$$\ell(H) = \{A \in \mathscr{B}: (\forall S \in H) \qquad SA = A\}$$

defines a factor. In [22] Veech proved that if ℓ is a factor of $\mathscr{B}$ then there is a compact group $H \subset C(T)$ such that

$$\ell = \ell(H) = \ell(H(\ell)) \tag{6}$$

(see also [6]).

Proposition 2 (Veech theorem). *If φ is ergodic (weakly mixing) Z_2-cocycle and if $\ell \subset \hat{\mathscr{B}}$ is a factor which is not a factor of $\mathscr{B}$ then there exists a compact subgroup $H \subset C(T_\varphi)$ such that* (6) *holds.*

Proof. Let ℓ be a factor of $\tilde{\mathscr{B}}$. Consider the relatively independent extension of the diagonal measure on ℓ (see (1))

$$\tilde{\mu} \times_\ell \tilde{\mu}(\tilde{A} \times \tilde{B}) = \int_{X \times Z_2} E(\tilde{A}|\ell) \cdot E(\tilde{B}|\ell)\, d\tilde{\mu}. \tag{7}$$

This measure need not be ergodic, so

$$\tilde{\mu} \times_\ell \tilde{\mu} = \int_{J^e} \tilde{\mu}_\gamma\, dv(\gamma). \tag{8}$$

where J^e denotes the set of all ergodic 2-joinings of T_φ and v is a probability measure on J^e. However the correspondence $C(T_\varphi) \ni \hat{S} \leftrightarrow \tilde{\mu}_{\hat{S}} \in J^e$ defines a Borel embedding (see the proof of Theorem 1.8.2 in [6]), therefore (8) can be rewritten as

$$\tilde{\mu} \times_\ell \tilde{\mu} = \int_{C(T_\varphi)} \tilde{\mu}_{\hat{S}}\, dv(\hat{S}) + \int_{J^e \setminus C(T_\varphi)} \tilde{\mu}_\gamma\, dv(\gamma)$$

If $\mu \times_{\ell} \mu = \int_{C(T_\varphi)} \tilde{\mu}_{\hat{S}} \, d\nu(S)$ then the proof of Veech result [22] (see also [6]) says that our assertion holds. Suppose that in the ergodic decomposition (8) there is an ergodic 2-joining which is not on the graph of any $\hat{S} \in C(T)$. Then by Proposition 1 this joining is $\hat{\mu}_{S_0}$ for some $S_0 \in C(T)$. In view of (7), (8) it follows that $\tilde{A} \in \ell$ iff $\mu_\gamma(\tilde{A} \times \tilde{A}^c) = 0$ ν a.e. In particular if $\tilde{A} \in \ell$ then $\hat{\mu}_{S_0}(\tilde{A} \times \tilde{A}^c) = 0$. The latter condition forces $\tilde{A}$ to belong to $\mathscr{B}$.

Corollary 1. *If the centralizer of φ is trivial then φ is prime.* □

2.1. Remark on Factors of Z_2-Extension of Adding Machines

The first intriguing question we intend to answer is whether or not each factor of a Z_2-extension is canonical (i.e. different T_φ-invariant sub-σ-algebras should lead to nonisomorphic factors). We will consider the following case $T:(X, \mathscr{B}, \mu)\circlearrowleft$, X is a group of n_t-adic numbers, (i.e. $X = \{(s_0, s_1, s_2, \ldots): 0 \leqq s_i \leqq \lambda_i - 1\}$, $\lambda_0 = n_0$, $\lambda_t = n_t/n_{t-1}$, T is the rotation on $\hat{1} = (1, 0, 0, \ldots)$). In addition we will assume that φ has partly continuous spectrum and that T_φ is a factor of $T'_{\varphi'}: (X' \times Z_2, \mu')$ where $T' = T \times T_1$. (We recall here that each factor of a Z_2-extension of a transformation with discrete spectrum is again a Z_2-extension of another transformation with discrete spectrum). We assert that

$$T'_{\varphi'} = T_\varphi \times T_1 \tag{9}$$

(in other words T_φ is a direct factor of $T'_{\varphi'}$). Indeed $T'_{\varphi'}$ is ergodic. Therefore $T \times T_1$ is ergodic, whence $T_\varphi \times T_1$ is ergodic since $Sp(T) = Sp(T_\varphi)$ (φ has partly continuous spectrum). Moreover $T'_{\varphi'}$ has both T_φ and T_1 as factors. We conclude that $T'_{\varphi'}$ has $T_\varphi \times T_1$ as a factor because in fact T_φ and T_1 are disjoint [4]. Hence (9) holds. Thus if we intend to classify all Z_2-extensions of adding machines we ought to classify all prime cocycles. These cocycles can arise from Corollary 1 (see [10]), but this is not the only reason for φ to be prime. In Sect. 3 we exhibit a rigid Z_2-extension (of an adding machine) which is prime.

Assuming that (9) holds we answer the question when T_φ is a canonical factor of $T'_{\varphi'}$. Let us notice that if T_φ is a canonical factor of $T'_{\varphi'}$ then

$$C(T_\varphi \times T_1) = C(T_\varphi) \times C(T_1) \tag{10}$$

holds. First of all we divide $C(T)$ into 3 mutually disjoint sets

$$C(T) = C_1(T) \cup C_2(T) \cup C_3(T),$$

$C_1(T) = \{S: \varphi + \varphi S \text{ is not ergodic}\}$,
$C_2(T) = \{S: \varphi + \varphi S \text{ is ergodic and has discrete spectrum}\}$,
$C_3(T) = \{S: \varphi + \varphi S \text{ is ergodic and has partly continuous spectrum}\}$.

Notice that $C_1(T)$ is precisely the set of all S's which lift to $C(T_\varphi)$. We prove that (10) holds if $C_2(T) = \emptyset$. Indeed, assume that $S \times S_1 \in C(T \times T_1) = C(T) \times C(T_1)$ can be lifted to the centralizer of $T_\varphi \times T_1 = (T \times T_1)_{\tilde{\varphi}}$, where $\tilde{\varphi}(x, x_1) = \varphi(x)$. Then

$$\tilde{\varphi}(S \times S_1) + \tilde{\varphi} = f + f(T \times T_1)$$

for some cocycle $f: X\times X_1 \to Z_2$. This equality means that $(T\times T_1)_{\tilde{\varphi}(S\times S_1)+\tilde{\varphi}}$ is not ergodic or that $T_{\varphi+\varphi S}\times T_1$ is not ergodic. There are two possibilities either $T_{\varphi+\varphi S}$ is not ergodic or $T_{\varphi+\varphi S}$ is ergodic but $Sp(T_{\varphi+\varphi S})\cap Sp(T_1)\ni\alpha\neq 1$. The former condition says that $S\in C_1(T)$, so S can be lifted to $C(T_\varphi)$. The latter says that $S\in C_2(T)$ because otherwise $\mathrm{Sp}(T_{\varphi+\varphi S})=Sp(T)$.

Let us notice that Veech Theorem (combined with the analysis in [6]) shows that *a factor $\mathscr{C}$ of T_φ is canonical iff $\hat{S}^{-1}\mathscr{C}=\mathscr{C}$ for every $\hat{S}\in C(T_\varphi)$ iff every compact subgroup $H\subset C(T_\varphi)$ is normal.* Therefore there is only one reason for $\mathscr{C}$ not to be canonical. Namely if $\mathscr{C}$ has partly continuous spectrum and

$$\hat{U}H(\mathscr{C})\,\hat{U}^{-1}\ni\sigma\hat{S}\qquad\text{for some }\hat{S}\in H(\mathscr{C})$$

then $\hat{U}\mathscr{C}\neq\mathscr{C}$ although these two factors are isomorphic (we recall that $\sigma(x,i)=(x,i+1)$).

In [13] the authors raised the following question. Is the formula

$$C(U\times U')=C(U)\times C(U') \tag{11}$$

valid whenever $U\perp U'$ (U is disjoint from U')? Although this is not the case, we will deal with the problem for $U=T_\varphi$ and $U'=T'_{\varphi'}$ where T and T' have discrete spectra, φ,φ' are cocycles and $T_\varphi\times T'_{\varphi'}$ is ergodic to achieve some criterion of the validity of (11).

Proposition 3. *For U and U' as above the formula* (11) *holds as soon as the sets $C_2(T)$ and $C_2(T')$ are empty.*

Proof's sketch. We see that $T_\varphi\times T'_{\varphi'}=(T\times T')_{\varphi\times\varphi'}=W$. Then all W-invariant sub-σ-algebras $\mathscr{C}\supset\mathscr{B}\otimes\mathscr{B}$ with 2-point fibers over $\mathscr{B}\otimes\mathscr{B}$ are

$$\begin{aligned}\mathscr{A}_1&\leftrightarrow\{(x,0,x',i)\sim(x,1,x',i)\},\\ \mathscr{A}_2&\leftrightarrow\{(x,i,x',0)\sim(x,i,x',1)\},\\ \mathscr{A}_3&\leftrightarrow\{(x,i,x',j)\sim(x,i+1,x',j+1)\}\end{aligned}$$

corresponding to $T_\varphi\times T'$, $T\times T'_{\varphi'}$, $(T\times T')_{\varphi+\varphi'}$ respectively. Let $S=S_1\times S_2\in C(T\times T')$ be lifted to $C(T_\varphi\times T'_{\varphi'})$. Thus $\hat{S}$ can only permute $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$. If $\hat{S}\mathscr{A}_1=\mathscr{A}_1$, $\hat{S}\mathscr{A}_2=\mathscr{A}_2$ then $\hat{S}=\hat{S}_1\times\hat{S}_2$ since $C(T_\varphi\times T')=C(T_\varphi)\times C(T')$ as we have observed earlier. If $\hat{S}\mathscr{A}_1=\mathscr{A}_2$ then $T\times T'_{\varphi'}$ and $T_\varphi\times T'$ are isomorphic which means that $T\times T'_{\varphi'}$ has $T_\varphi\times T'_{\varphi'}$ as a factor ($T_\varphi\perp T'_{\varphi'}$) which is impossible. Finally if $\hat{S}\mathscr{A}_1=\mathscr{A}_3$, $\hat{S}\mathscr{A}_2=\mathscr{A}_2$ then $(T\times T')_\varphi$ and $(T\times T')_{\varphi\times\varphi'}$ are isomorphic via a lifting $\hat{S}$, so

$$\tilde{\varphi}(S_1\times S_2)+(\varphi+\varphi')=\psi(T\times T')+\psi$$

which implies that $(T\times T')_{\varphi S_1+\varphi}$ is isomorphic to $(T\times T')_{\varphi'}$. In other words $T_{\varphi S_1+\varphi}\times T'$ is isomorphic to $T\times T'_{\varphi'}$. Since $C_2(T)=\emptyset$, either $T_{\varphi S_1+\varphi}$ is ergodic and then $T_{\varphi S_1+\varphi}\perp T'_{\varphi'}$ and a contradiction as in case $\hat{S}_1=\mathscr{A}_2$ or $T_{\varphi S_1+\varphi}$ is not ergodic. Then S_1 can be lifted. But S_2 can be lifted as well since $\hat{S}\mathscr{A}_2=\mathscr{A}_2$. □

Remark. The partition $\{C_1(T),\ C_2(T),\ C_3(T)\}$ of $C(T)$ is an invariant of the relative isomorphism. It would be interesting to know what kind of cocycles admits $C_2(T)=\emptyset$. We state without a proof the following result as a sample.

Proposition 4. *If $\varphi: X \to Z_2$ is a cocycle generated by a continuous substitution on two symbols of constant length* [2] *then* $C_2(T) = \emptyset$. □

3. An Example of an S-Strongly Ergodic Cocycle

We start with $T:(X, \mathscr{B}, \mu)\circlearrowleft$, where X is the group of n_t-adic numbers,

$$\lambda_t = 2^{t+1} + 1,\ t \geqq 0.$$

In other words $X = \{\bar{v} = (v_0, v_1, \ldots, v_t, \ldots): 0 \leqq v_t \leqq \lambda_t - 1\}$ and $T = \sigma_{(1,0,0,\ldots)}$.
Denoting

$$D_0^{n_t} = \{\bar{v} \in X: v_i = 0 \quad \text{for} \quad i = 0, 1, \ldots, t\}$$

we get a T-tower of height n_t

$$D^{n_t} = \{D_0^{n_t}, D_1^{n_t}, \ldots, D_{n_t-1}^{n_t}\},\ T^i D_0^{n_t} = D_i^{n_t} \bmod n_t.$$

Since X is a compact, monothetic group and T is an ergodic translation

$$C(T) = \{\sigma_{\bar{v}}: \bar{v} \in X\} \cong X.$$

Let us consider the action of $S = \sigma_{\bar{v}}$ on D^{n_t} and $D^{n_{t+1}}$

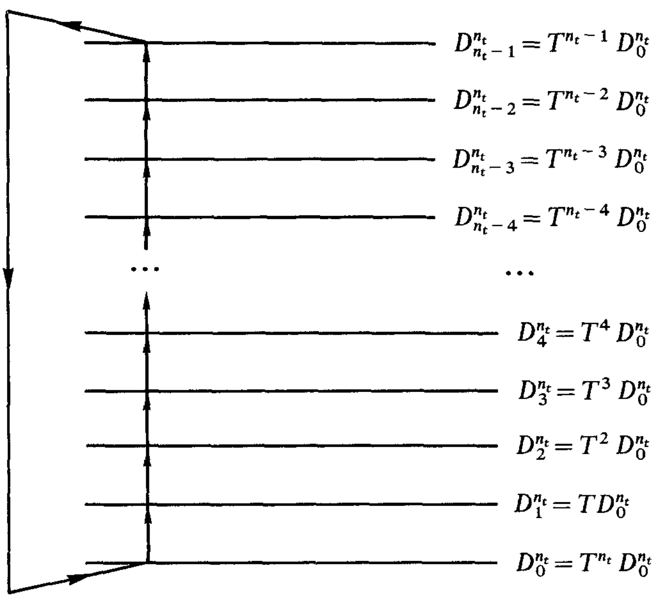


Fig. 1

Definition of φ. The definition will be inductive. At the t-th step our cocycle will be defined on $D_0^{n_t}, \ldots, D_{n_t-2}^{n_t}$ and cannot be defined on $D_{n_t-1}^{n_t}$. Moreover $\varphi | D_i^{n_t} = \text{const} = a_i^t \in \{0, 1\}$, $i = 0, \ldots, n_t - 2$. Now we define the passage into the $(t+1)$-th step.

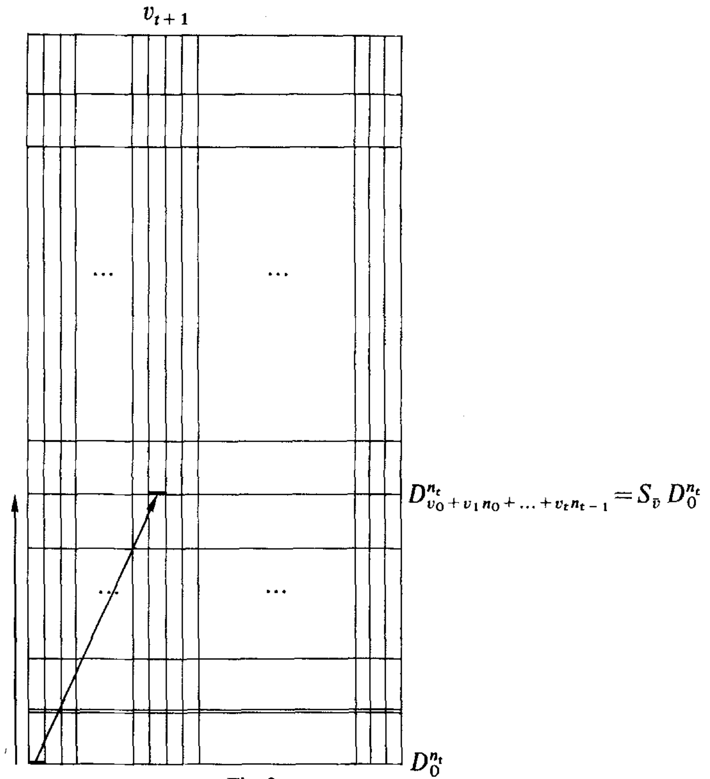

v_{t+1}
...
...
$D^{n_t}_{v_0+v_1n_0+\dots+v_tn_{t-1}} = S_{\bar v}\, D^{n_t}_0$
...
...
$D^{n_t}_0$

Fig. 2

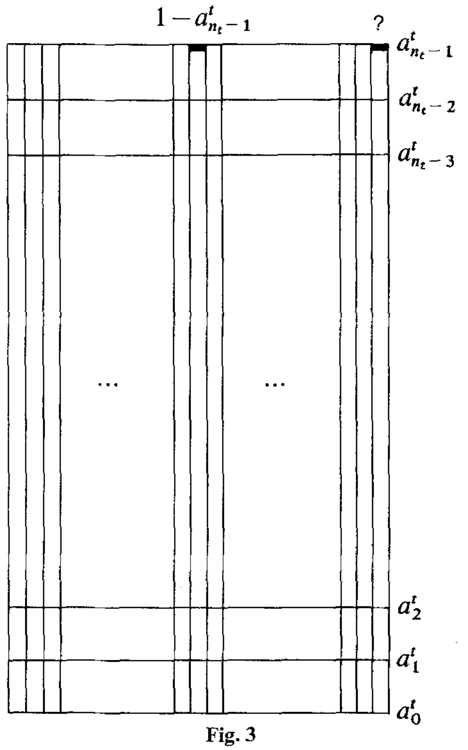

$1-a^t_{n_t-1}$
?
$a^t_{n_t-1}$
$a^t_{n_t-2}$
$a^t_{n_t-3}$
...
...
a^t_2
a^t_1
a^t_0

Fig. 3

First of all we define $a^t_{n_t-1}$ (=0 or 1) so that $\sum_{i=0}^{n_t-1} a^t_i = 1$. Then we put

$$a^{t+1}_{sn_t-1} = \varphi \mid D^{n_{t+1}}_{sn_t-1} = \begin{cases} a^t_{n_t-1} & s \neq 2^{t+1}+1,\ 2^{t+2}+1 \\ 1 - a^t_{n_t-1} & s = 2^{t+1}+1 \\ \text{undefined} & s = 2^{t+2}+1 \end{cases}$$

$(s = 1, 2, \ldots, 2^{t+2}+1)$.

Agree to call this fat level in Fig. 3 an *error* (because the value of φ on $D^{n_t}_{n_t-1}$ is near constant and equal to $a^t_{n_t-1}$ in spite of the value of φ on $D^{n_{t+1}}_{(2^{t+1}+1)n_t-1}$). This is a correct definition of a cocycle φ. Let us observe that our cocycle is "constant" on each level $D^{n_t}_i$ because for $i < n_t - 1$ it is constant indeed and for $i = n_t - 1$ the (relative) measure of the errors is less than $2/\lambda^{t+1} \underset{t}{\rightarrow} 0$. Hence $\varphi \mid D^{n_{t+1}}_{n_{t+1}-1}$ is also almost constant. This is an immediate computation that in fact $\varphi \mid D^{n_{t+1}}_{n_{t+1}-1}$ defines another error for the t-th step.

Definition of S. We take $S = \sigma_{\bar{v}}$, where $\bar{v} = (v_i)_{i \geq 0}$, $v_i = [\lambda_i / i]$, $i > 0$. We observe that the following holds: given k

(12) $$S^k = (v^{(k)}_i)_{i \geq 0}, \qquad v^{(k)}_i = k v_i \qquad \text{for } i \geq i_0 .$$

Keane's criterion. We will need some criterion concerning ergodicity of some cocycles. Let $T: (X, \mathscr{B}, \mu)\circlearrowleft$ be an n_t-adic machine and let $\varphi: X \rightarrow Z_2$ be a cocycle such that $\varphi \mid D^{n_t}_i = a^t_i = \text{const}$ $i = 0, \ldots, n_t - 2$, $t \geq 0$. Assume that the number of errors on $D^{n_t}_{n_t-1}$ divided by λ_t tends to zero. Look at the top of D^{n_t} and pass to the $(t+1)$-th tower.

We group the errors into pairs (see Fig. 4). Then A_t is the number of places between errors including errors.

Proposition 5. (Keane's criterion). *φ is ergodic iff*

$$\sum_{t \geq 0} A_t / \lambda_t = +\infty .$$

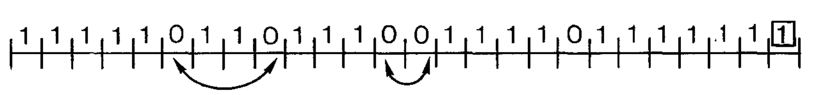
Proof. The proof follows from the observation that such a cocycle determines an almost periodic point $\omega \in \{0,1\}^Z$ and, besides, that φ is ergodic iff ω is strictly transitive (for details see [11]). Then we use Lemma 3 from [7]. □

We intend to argue that φ is S-strongly ergodic, i.e. that

$$\varphi S^{i_1} + \ldots + \varphi S^{i_k} + \varphi U$$

$(i_1 < \ldots < i_k, k \geq 2, U \in C(T))$ is ergodic. First we prove that $\psi = \varphi S^{i_1} + \ldots + \varphi S^{i_k}$ is ergodic whenever $0 \leq i_1 < \ldots < i_k$. Let us look at the passage from the t-th step into the $(t+1)$-th step for ψ.

We see that $\psi \mid D^{n_t}_i$ is constant except for k levels, say $j_1, \ldots, j_k$. At each $D^{n_t}_{j_r}$ there are two errors. The "distance" (i.e. the number of columns) between the errors in $D^n_{j_r}$



$A_t = 6$

Fig. 4

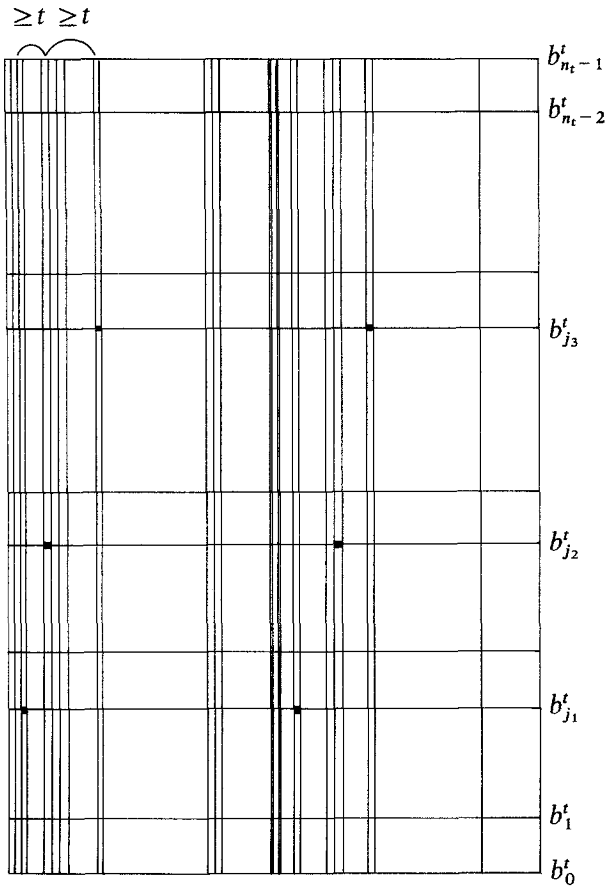


Fig. 5

and $D^n_{j_{r'}}$ is at least t and moreover the distance between these errors and the $2^{t+1}+1$-th (or $2^{t+2}+1$-th) column is at most It (these facts are a consequence of (12)), where $I=\max i_s$.

Now we intend to define a new cocycle $\tilde{\psi}$ satisfying the assumptions of Keane's Criterion and, besides, $\psi+\tilde{\psi}$ is not ergodic (i.e. T_ψ and $T_{\tilde{\psi}}$ are relatively isomorphic).

Definition of $\tilde{\psi}$. The definition is inductive. At each step t $\tilde{\psi}\,|\,D^n_i=c^t_i$, $i=0,\dots,n_t-2$ and $\tilde{\psi}$ is not defined on $D^{n_t}_{n_t-1}$. First we define $c^t_{n_t-1}$ so that

$$\sum_{i=0}^{n_t-1} c^t_i=\sum_{i=0}^{n_t-1} b^t_i. \tag{13}$$

Then we put

$$c^{t+1}_{sn_t-1}=\begin{cases} c^t_{n_t-1} \text{ if there is no error in the column} & s<\lambda_{t+1}\\ 1-c_{n_t-1} \text{ otherwise} & s<\lambda_{t+1}\\ \text{undefined} & s=\lambda_{t+1}\end{cases}$$

ψ and $\tilde{\psi}$ are relatively isomorphic. Consider the passage from the t-th step into the $(t+1)$-th step for $\psi+\tilde{\psi}$.

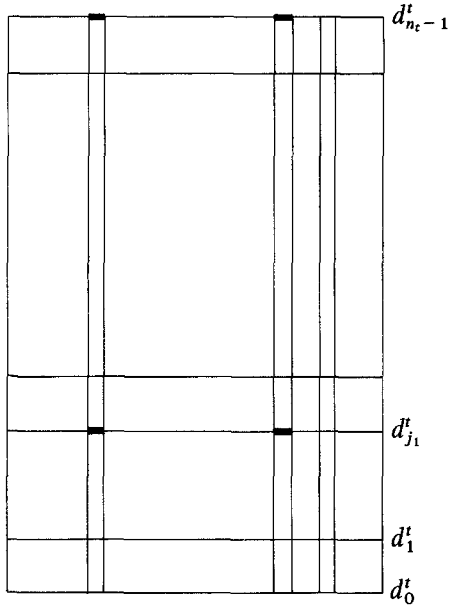


Fig. 6

We see that $\psi + \tilde{\psi}$ is almost constant on each level $D_i^{n_t}$. Moreover

$$\sum_{i=0}^{n_t-1} d_i^t = 0 \tag{14}$$

since (13) holds. Let u_t denote the number of columns with errors. Then $u_t = 2k$ and moreover in any such a column there are even numbers of errors (either no errors or precisely two errors). Consider $T_{\psi+\tilde{\psi}}:(X\times Z_2, \tilde{\mu})\circlearrowleft$ and the following sequence of sets

$$C_t = \bigcup_{i=0}^{n_t-1} D_i^{n_t} \times (d_0^t + \ldots + d_{i-1}^t)\,.$$

Then an immediate computation shows that

(i) $\tilde{\mu}(C_t) = 1/2$,

(ii) $\tilde{\mu}(T_{\psi+\tilde{\psi}}\, C_t \triangle C_t) \underset{t}{\to} 0$,

(iii) $\sum_{t\geqq 0} \tilde{\mu}(C_{t+1} \triangle C_t) < +\infty$.

Hence $\{C_t\}$ is a Cauchy sequence and $C = \lim C_t$ is $T_{\psi+\tilde{\psi}}$ invariant, whence $T_{\psi+\tilde{\psi}}$ cannot be ergodic.

Cocycles $\varphi S^{i_1} + \ldots + \varphi S^{i_k}$ *are ergodic.* Denote $\psi = \varphi S^{i_1} + \ldots + \varphi^{\chi i_k}$ and consider $\tilde{\psi}$. It is enough to show that $\tilde{\psi}$ is ergodic. We divide $D^{n_t}_{n_t-1}$ first into λ_{t+1} pieces. Then these pieces we group into t groups $A_1, \ldots, A_t$ of consecutive pieces (the last group need not have $[\lambda_{t+1}/t]$ pieces). We observe that there is no possibility for two different errors to be in the same group A_i since $i_1 < \ldots < i_k$ and (12) holds. This means that

$$A_t \geqq \frac{\lambda_t/t}{2\lambda_t} = 1/2t.$$

Since $\sum_{t \geqq 1} 1/2t$ is divergent and Proposition 5 holds, $\tilde{\psi}$ is ergodic and consequently ψ is ergodic.

Cocycles $\varphi S^{i_1} + \ldots + \varphi^{\chi^{i_k}} + \varphi U$, $k \geqq 2$ *are ergodic.* It is enough to apply the foregoing arguments because there is no possibility to destroy the divergency $\sum_{t \geqq 0} A_t/\lambda_t$ using only two errors more (φU gives merely two new errors).

The proof that φ is S-strongly ergodic is now complete.

Because any cocycle φ is simple with respect to S as soon as it is S-strongly ergodic, we get

Proposition 6. *For the φ we have defined the relatively independent extensions $\hat{\mu}_{S^{i_1}, \ldots, S^{i_k}, U}$ are ergodic whenever $i_1 < \ldots < i_k$ and US^j, $j \in Z$ cannot be lifted to the centralizer of T_φ.* □

The cocycle φ enjoys an additional property. Namely

Proposition 7. *φ is rigid (i.e. $T_\varphi^{m_t}$ weakly converges to the identity for some sequence m_t) and is prime.*

The idea of the proof. The first part is obvious. To prove the second it is enough to show that there is no nontrivial compact subgroup $H \subset C(T)$ such that for every $U \in H$, U can be lifted to the centralizer of T_φ. However the arguments we just used to prove that φ is S-strongly ergodic show that if $U = \sigma_{\bar{w}}$, $\bar{w} = (w_i)_{i \geqq 0}$ can be lifted, then w_i is "near" 0, λ_i or $\lambda_i/2$ (in the sense made precise by Keane's Criterion). □

4. A few Applications

4.1. Weakly Isomorphic Transformations that Are not Isomorphic

In this section we assume that $T: (X, \mathscr{B}, \mu) \circlearrowleft$ has discrete spectrum and $\varphi: X \to Z_2$ is an S-strongly ergodic cocycle.

We will consider (ergodic) transformations of the form

$$(15) \qquad T_{\varphi S^{i_1} \times \varphi S^{i_2} \times \ldots}: (X \times Z_2 \times Z_2 \times \ldots, \mu \times \nu_2 \times \nu_2 \times \ldots) \circlearrowleft$$

$i_j \neq i_k$. Denote $T_{\varphi S^{i_1} \times \varphi S^{i_2} \times \ldots}$ by $T_{i_1, i_2, \ldots}$. Let us notice that $T_{j_1, j_2, \ldots}$ is a factor of (15) whenever $\{j_1, j_2, \ldots\} \subset \{i_1, i_2, \ldots\}$. Indeed, the inclusion is equivalent to say that there is an $\sigma: N \circlearrowleft$ one-to-one such that $j_k = i_{\sigma(k)}$. We define $\theta: (X \times Z_2 \times Z_2 \times \ldots, \mu \times \nu_2 \times \nu_2 \times \ldots) \circlearrowleft$ putting

$$\theta(x, r_1, r_2, \ldots) = (x, r_{\sigma(1)}, r_{\sigma(2)}, \ldots).$$

Then θ preserves the measure and $T_{j_1, j_1, \dots}\theta = \theta T_{i_1, i_2, \dots}$. If there is an $c \in Z$ such that

$$-i_k + j_k = c, \; k \geqq 1 \tag{16}$$

then $T_{i_1, i_2, \dots}$ and $T_{j_1, j_2, \dots}$ are isomorphic.

The point is that the composition of these two possibilities exhaustes all cases of isomorphisms. Namely

Proposition 7. *If $T_{i_1, i_2, \dots}$ and $T_{j_1, j_2, \dots}$ are isomorphic then there exist a permutation $\sigma: N \circlearrowleft$ and an integer c such that*

$$j_{\sigma(k)} - i_k = c \textit{ for every } k \geqq 1 .$$

Proof. Assume that $T_{i_1, i_2, \dots}$ and $T_{j_1, j_2, \dots}$ are isomorphic. These automorphisms are ergodic $Z_2 \times Z_2 \times \dots$-extensions of T, so this isomorphism must be of the form

$$U_{f, v}: (X \times Z_2 \times Z_2 \times \dots, \;\; \mu \times \nu_2 \times \nu_2 \times \dots) \circlearrowleft$$

where $U \in C(T)$, $f: X \to Z_2 \times Z_2 \times \dots$ is measurable and v is a continuous automorphism of $Z_2 \times Z_2 \times \dots$ (see [15]). Let us notice that $f = (f_1, f_2, \dots)$, $f_i: X \to Z_2$ is a cocycle and

$$fT = (f_1 T, f_2 T, \dots). \tag{17}$$

If v is a continuous automorphism of $Z_2 \times Z_2 \times \dots$ then v acts as an infinite matrix $[a_{ij}]_{ij}$, $a_{ij} = 0, 1$, where the the i-th column $[a_{ki}]_k = v(e_i) = (0, \dots, 0, \underset{i}{1}, 0, \dots)$. Moreover in any row the number of 1's is finite and

$$v(r_1, r_2, \dots)|_i = \sum_j a_{ij} r_j . \tag{18}$$

In view of (4)

$$f(x) + \varphi S^{i_i} \times \varphi S^{i_2} \times \dots (Ux) = f(Tx) + v(\varphi S^{j_1} \times \varphi S^{j_2} \times \dots)(x).$$

Combining (17) and (18) we get

$$f_1(x) + \varphi(S^{i_1} Ux) = f_1(Tx) + [\varphi(S^{j_{k_1^{(1)}}} x) + \dots + \varphi(S^{j_{k_{s_1}^{(1)}}} x)]$$
$$f_2(x) + \varphi(S^{i_2} Ux) = f_2(Tx) + [\varphi(S^{j_{k_1^{(2)}}} x) + \dots + \varphi(S^{j_{k_{s_2}^{(2)}}} x)]$$
$$\dots .$$

But these conditions mean that

$$\varphi U + \varphi S^{j_{k_1^{(1)}} - i_1} + \dots + \varphi S^{j_{k_{s_1}^{(1)}} - i_1}$$
$$\varphi U + \varphi S^{j_{k_1^{(2)}} - i_2} + \dots + \varphi S^{j_{k_{s_2}^{(2)}} - i_2}$$
$$\dots$$

are not ergodic. Since our cocycle is S-strongly ergodic, $s_1 = s_2 = \dots = 1$. Then

$$\varphi U + \varphi S^{j_{k_1^{(1)}} - i_1}, \; \varphi U + \varphi S^{j_{k_1^{(2)}} - i_2}, \dots$$

are not ergodic which means that $S^{j_{k_1^{(1)}} - i_1} U^{-1}$, $S^{j_{k_1^{(2)}} - i_2} U^{-1}, \dots$ can be lifted to the centralizer of T_φ. But $C(T_\varphi)$ is a group (every ergodic Z_2-extension of a 2-fold simple map enjoys this property). Therefore $S^{j_{k_1^{(1)}}-i_1}U^{-1}\ (S^{j_{k_1^{(2)}}-i_2}U^{-1})^{-1} = S^{j_{k_1^{(1)}}-i_1-(j_{k_1^{(2)}}-i_2)}$ can be lifted to the centralizer of T. We have achieved that

$$j_{k_1^{(1)}}-i_1=j_{k_1^{(2)}}-i_2=\ldots=c$$

because S^k can be lifted iff $k=0$. Hence the proof is complete. □

Corollary 2. *$T_{0,1,2,3,\ldots}$ and $T_{0,2,3,\ldots}$ are weakly isomorphic but they are not isomorphic.* □

Corollary 3. *There exists an ergodic $Z_2\times Z_2\times\ldots$-extension T_φ of a transformation with discrete spectrum T such that there is an $\hat{S}\in C(T_\varphi)$ which is not invertible (in particular $h(\hat{S})>0$).*

Proof. Take $T_{0,1,2,\ldots}$. We see that $\hat{S}(x,r_1,r_2,\ldots)=(Sx,r_2,r_3\ldots)$ (not invertible) is in $C(T_{0,1,2,\ldots})$ and $h(\hat{S})=\log 2$. □

From the proof of Proposition 7 we deduce the following

Corollary 4. *If $U\in C(T)$ then U can be lifted to the centralizer of $T_{i_1,i_2,\ldots}$ iff*
(i) *there is $f\colon N\circlearrowleft$ one-to-one such that $i_{f(n)}-i_n=c$,*
(ii) *US^{-c} can be lifted to the centralizer of T_φ. In particular all elements $U\in C(T)$ which can be lifted to $C(T_\varphi)$ can also be lifted to $C(T_{i_1,i_2,\ldots})$ and only such elements are invertible in $C(T_{i_1,i_2,\ldots})$.* □

4.2. A Transformation with the Centralizer to Be a Group but with a Factor Whose Centralizer Is not a Group

We answer Newton question [14]. Consider "two-sided" version of the construction given in 4.1. If φ is S-strongly ergodic then $T_{\ldots,-1,0,1,2,\ldots}$ is still ergodic and the centralizer of it can be computed from Corollary 4. But condition (i) of this corollary says that f is in fact a permutation and therefore any element from $C(T_{\ldots,-1,0,1,2,\ldots})$ is invertible. However $T_{0,1,2,\ldots}$ is a factor of $T_{\ldots,-1,0,1,2,\ldots}$ and the centralizer of the former automorphism is not a group.

4.3. Compact Rank Need not Imply that the Centralizer Is a Group

(For the definition of the rank we refer to [16], a transformation has the compact rank if it is a $\bar{d}$-limit of finite rank transformations). We answer Thouvenot's question stated in a conversation. Observe that the φ we constructed in Sect. 3 satisfies: T_φ has rank 1, $T_{\varphi\times\varphi S}$ has rank at most $2^2\cdot 2$, $T_{\varphi\times\varphi S\times\varphi S^2}$ has rank at most $2^3\cdot 3,\ldots$. Therefore $T_{\varphi\times\varphi S\times\varphi S^2\ldots}$ has compact rank because it is an inverse limit of $T_{\varphi\times\varphi S\times\ldots\times\varphi S^k}$, $k\geqq 1$.

Remark. Although some of these constructions can be done using only Rudolph's machinery (see [19, 6, 12]), there is at least one advantage of our approach. All our examples are loosely Bernoulli [LB] as ergodic group extensions of transformations with discrete spectra [16], (we recall here that it is still unknown whether

$T\times T$ is LB for Chacon transformation and whether T is LB when T is a Gaussian automorphism with spectral measure concentrated on a Kronecker set). Moreover using our special φ these examples enjoy even compact rank property.

5. Final Remarks

We raise some open question. Having T and $\varphi: X\to Z_2$ for any $i_1<i_2<\ldots<i_k$, $k\geqq 2$ we define

$$C^{(1)}_{i_1,\ldots,i_k}(T)=\{S\in C(T):\varphi S^{i_1}+\ldots+\varphi S^{i_k} \text{ is not ergodic}\},$$
$$C^{(2)}_{i_1,\ldots,i_2}(T)=\{S\in C(T):\varphi S^{i_1}+\ldots+\varphi S^{i_k} \text{ is ergodic with discrete spectrum}\},$$
$$C^{(3)}_{i_1,\ldots,i_k}(T)=\{S\in C(T):\varphi S^{i_1}+\ldots+\varphi S^{i_k} \text{ is ergodic with partly continuous spectrum}\}.$$

We get a partition $C^{i_1,\ldots,i_k}=\{C^{(1)}_{i_1,\ldots,i_k}(T), C^{(2)}_{i_1,\ldots,i_k}(T), C^{(3)}_{i_1,\ldots,i_2}(T)\}$ of $C(T)$. This is an invariant of the relative isomorphism and φ and $\varphi+1$ have this invariant the same. Is the sequence $\{C^{i_1,\ldots,i_k}\}$ a complete set of invariants of the relative isomorphism up to $\varphi+1$?

We have been unable to decide whether for any T and ergodic there is $S\in C(T)$ such that φ is S-strongly ergodic. In order to get such an S, first of all we need an S such that for any k, S^k cannot be lifted. Fortunately the set of such S's has Haar measure 1 (see [9]).

Another question is the following. Let $\lambda\in J(T_\varphi, T_\varphi, T_\varphi)$ be such that $\hat{\mu}_{S_i,S_j}$, $i\neq j$ is ergodic (i.e. $S_iS_j^{-1}$ cannot be lifted). Is then $\lambda=\hat{\mu}_{S_1,S_2,S_3}$ ergodic?

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