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Probability Theory and Related Fields

Stationary Min-Stable Stochastic Processes

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Summary. We consider the class of stationary stochastic processes whose margins are jointly min-stable. We show how the scalar elements can be generated by a single realization of a standard homogeneous Poisson process on the upper half-strip $[0,1]\times R_+$ and a group of L_1-isometries. We include a Dobrushin-like result for the realizations in continuous time.

1. Introduction

We say that a random vector $\tilde{Z}$, with elements Z_k, is "min-stable" if and only if its distribution is a limiting extreme value distribution with negative exponential margins.

In fact $\tilde{Z}$ is min-stable if and only if $\min\{Z_n/a_n|n=1,2,\ldots\}$ has a negative exponential distribution for any nonrandom vector $\tilde{a}$, with elements $a_n\in[0,\infty]$ at least one of which is positive. Notice that if $a_n\equiv 0$, then $Z_n/a_n=\infty$ and the term plays no role in the minimization. Multivariate extreme value distributions and their domains of attraction have been studied extensively. See de Haan and Resnick [1977]. For a general source on extreme value theory and applications see the book by Galambos [1978]. See, also, the book by Leadbetter, Lindgren and Rootzen [1983]. We are not concerned here with domains of attraction. The univariate limiting extreme value types can be transformed into one another by means of simple functional transformations ($\log x$, $1/x$, x^α etc.). The same is true for multivariate extreme value distributions. That is a distribution is determined by its margins and independently, by its dependence function. The choice of marginal type is one of convenience. We use the negative exponential family. That is we use X which is such that $-\log P\{X>x\}\equiv x/EX$. Notice that these are limiting distributions of *smallest* values.

In Sect. 2, we present a representation for any finite or infinite dimensional min-stable random vector $\tilde{Z}$. It depends upon a standard homogeneous Poisson process on the strip $[0,1]\times R_+$ and a set of nonrandom functions f_n:

$[0,1] \to R_+$ which correspond to the components Z_n of $\tilde{Z}$. This is Theorem 2 of de Haan [1984], followed by a transformation. We consider the nonuniqueness of $\{f_n\}$ and introduce the group of "pistons": a class of function transformations Γ which are such that $\int_0^1 \Gamma(f)(s)\,ds = \int_0^1 f(s)\,ds$ for all non-negative f. We discuss this group in Sect. 3.

In Sect. 4 we continue the discussion of representations and we show that a "proper" one always exists, in the sense of a definition given there. In Sect. 5 we consider the implications of strict stationarity. We consider continuous time stationary processes in Sect. 6 and we include a Dobrushin-like result for the sample paths.

For another example of representing a stochastic process as a functional of a 2-dimensional standard homogeneous Poisson process, see Pickands [1971]. The embedded process, there, is on $R_+ \times R_+$ rather than on $[0,1] \times R_+$ as here. The processes generated in that paper are extremal processes which are not stationary.

2. Representation of Min-Stable Processes

Let $\tilde{Z}$, with elements Z_n, be min-stable. By definition $\tilde{Z}$ is min-stable if and only if $1/\tilde{Z}$, with elements $1/Z_n$, is max-stable in the sense of de Haan [1984]. We begin with Theorem 2 of that paper. By that theorem we can write

$$1/Z_n = \max_{l \geqq 1} \bar{f}_n(\bar{S}_l)\,\bar{U}_l \tag{2.1}$$

where $\{\bar{S}_l, \bar{U}_l\}_{l=1}^{\infty}$ are the points of a 2-dimensional Poisson process with intensity measure $\rho(d\bar{s}) \times d\bar{u}/\bar{u}^2 = \rho(d\bar{s}) \times d(1/\bar{u})$ on $[0,1] \times R_+$ with $\rho[0,1] \in (0,\infty)$ and $\bar{f}_n$ are nonnegative functions such that $\int_0^1 \bar{f}_n(s)\,\rho(ds) < \infty$. We can transform the space and rewrite (2.1). Let $u \equiv 1/\bar{u}$.

Now

$$1/Z_n \equiv \max_{l \geqq 1} \bar{f}_n(\bar{S}_l)/U_l$$

where $\{\bar{S}_l, U_l\}$ now has intensity measure $\rho(ds) \times du$.

So

$$Z_n \equiv \min_{l \geqq 1} U_l/\bar{f}_n(\bar{S}_l). \tag{2.2}$$

Let the distribution function F be defined by

$$F(t) = \rho[0,t]/\rho[0,1].$$

Now for $l = 1, 2, \ldots$ we can exhibit $\bar{S}_l$ as

$$\bar{S}_l = F^{-1}(S_l)$$

where the S_l are from a Poisson process with unit intensity measure and $F^{-1}(s)$ is a suitably defined inverse function for $F(t)$, such as $glb\{t \mid F(t) > s\}$

$= \operatorname{lub}\{t \mid F(t) \leqq s\}$. Let

$$f_n(s) \equiv \bar{f_n}(F^{-1}(s))$$

and recall (2.2). The following theorem results:

Theorem 2.1. *$\tilde{Z}$ is min-stable if and only if*

$$Z_n = \min_{l \geqq 1} U_l / f_n(S_l), \tag{2.3}$$

where $\{S_l, U_l\}_{l=1}^{\infty}$ is a homogeneous Poisson process with unit intensity on the strip $[0,1] \times R_+$ and $f_n := [0,1] \to \mathbb{R}_+$ with $\int_0^1 f_n(s)\,ds < \infty$ for $n = 1, 2, 3, \ldots$.

The functions $\{f_n\}$ will be called the *spectral functions* for the process $\{Z_n\}$.

Remark. One could also take e.g. a Poisson process on $\mathbb{R} \times \mathbb{R}_+$ with unit intensity as a basis for the representation.

For $z_n \in (0, \infty)$, the event

$$\begin{aligned}\{Z_n > z_n\} &= \bigcap_{l=1}^{\infty} \{U_l / f_n(S_l) > z_n\} = \bigcap_{l=1}^{\infty} \{U_l > z_n f_n(S_l)\} \\ &= \operatorname{Emp}\{(s,u) \mid u \in [0, z_n f_n(s)]\}\end{aligned}$$

where $\operatorname{Emp} A$ denotes the event that $A \subset [0,1] \times R_+$ contains no points of the homogeneous Poisson process on $[0,1] \times R_+$. But the number of points of the process in A has the Poisson distribution with mean (parameter) $\lambda_2(A)$ where λ_2 is 2-dimensional Lebesgue measure. It follows that

$$\begin{aligned}-\log P\{Z_n > z_n\} &= \lambda_2\{(s,u) \mid u \in [0, z_n f_n(s)]\} \\ &= \int_0^1 z_n f_n(s)\,ds = z_n \int_0^1 f_n(s)\,ds.\end{aligned} \tag{2.4}$$

Notice that Z_n, then, has a negative exponential distribution with mean

$$EZ_n = 1 \Big/ \int_0^1 f_n(s)\,ds.$$

Let the event $\{\tilde{Z} > \tilde{z}\} = \bigcap_{n=1}^{\infty} \{Z_n > z_n\} = \operatorname{Emp} \bigcup_{n=1}^{\infty} A_n$, where, for each n,

$$A_n = \{(s,u) \mid u \in [0, z_n f_n(s)]\}.$$

But

$$\bigcup_{n=1}^{\infty} A_n = \{(s,u) \mid u \in [0, \max_n z_n f_n(s)]\}$$

and

$$\lambda_2\left(\bigcup_{n=1}^{\infty} A_n\right) = \int_0^1 \max_n z_n f_n(s)\,ds$$

and so

$$-\log P\{\tilde{Z} > \tilde{z}\} = \int_0^1 \max_n z_n f_n(s)\,ds. \tag{2.5}$$

Thus min-stable joint distributions are determined by the values of all integrals of the form (2.5) above. For every min-stable (joint) distribution there exists a representation of the form (2.3) but the sequence $\{f_n\}$ is *not* uniquely determined. Another sequence $\{g_n\}$ yields the same distribution if and only if integrals of the form (2.5), above, are all unchanged if $\{f_n\}$ are replaced by $\{g_n\}$.

Definition 2.1. *Two sequences $\{f_n\}$ and $\{g_n\}$ of spectral functions are called **equivalent** and we write $\{f_n\} \sim \{g_n\}$, if for every sequence $\{z_n\}$ of non-negative numbers*

$$\int_0^1 \max_n z_n f_n(s)\, ds = \int_0^1 \max_n z_n g_n(s)\, ds.$$

We shall now introduce a class of mappings Γ which is such that if

$$g_n = \Gamma(f_n)$$

for all n, then $\{g_n\} \sim \{f_n\}$.

Definition 2.2. *A function mapping Γ is called a **piston** if for all $f\colon [0,1] \to \mathbb{R}_+$*

$$\int_0^1 \Gamma f(s)\, ds = \int_0^1 f(s)\, ds \tag{2.6}$$

and

$$\Gamma f(s) = r(s) f(H(s)) \tag{2.7}$$

where $r(s)$ and $H(s)$ are measurable, $r(s) > 0$ for $0 \leqq s \leqq 1$ and $H(s)$ is a one-to-one mapping of $[0,1]$ onto itself.

The set of pistons, thus defined, is just the class of linear $L_{1+}[0,1]$ isometries. See Royden [1968] Theorem 16, page 333.

Note that if $g_n := \Gamma f_n$ for all n with Γ a piston then $\{g_n\} \sim \{f_n\}$.

3. Pistons

In this section we study pistons. They are essential for the development in the subsequent sections. First note that H in (2.7) determines r. As examples we could let

$$\begin{aligned} H(t) &= 1 - t, \\ &= t + \theta \bmod 1, \qquad \theta \in (-\infty, \infty) \end{aligned}$$

or

$$= t^a, \qquad a \in (0, \infty).$$

By (2.6) and (2.7), for the first 2 examples $r(t) \equiv 1$. For the third $r(t) = a t^{a-1}$.

Lemma 3.1. *The pistons constitute a group.*

a) *If $\Gamma_i f(s) = r_i(s) f \circ H_i(s)$ $(i = 1, 2)$, then $\Gamma_1 \Gamma_2 f(s) = r_1(s) r_2(H_1(s)) f \circ H_2 \circ H_1(s)$.*

b) *If $\Gamma f(s) = r(s) f \circ H(s)$, then $\Gamma^{-1} f(s) = f \circ H^{\leftarrow}(s) / r \circ H^{\leftarrow}(s)$, where $H^{\leftarrow}$ is the inverse function of H.*

We consider two kinds of pistons.

Definition 3.1. *A piston Γ_M is* ***monotone*** *if the function H in (2.7) is non-decreasing.*

Lemma 3.2. *If a piston is monotone,*

$$H(s)=\int_0^s r(u)\,du \qquad \text{for } 0\leqq s\leqq 1.$$

Proof. Define $R(s)=\int_0^s r(u)\,du$. Now by definition

$$\int_0^1 f(s)\,ds=\int_0^1 r(s)\,f\circ H(s)\,ds=\int_0^1 f\circ H\circ R^{\leftarrow}(s)\,ds.$$

This holds for any non-negative $f\in L_1$. Now take $f(s)=1$ if $s\leqq a$ and 0 elsewhere. Then

$$a=\lambda_1[0,a]=\lambda_1\{(H\circ R^{\leftarrow})^{\leftarrow}[0,a]\}=R\circ H^{\leftarrow}(a).$$

Another kind of piston is given in the following definition.

Definition 3.2. *A piston Γ_p is a* ***permutation*** *if $r\equiv 1$.*

Lemma 3.3. *If a piston is a permutation, H is measure-preserving.*

Proof. $\int_0^1 f\circ H(s)\,ds=\int_0^1 f(s)\,ds$ for all $f\in L_{1+}$.
Now take f tabe the indicator function of an arbitrary Borel set.

Remark. The permutations and the monotone transformations form subgroups of the group of pistons.

The final lemma gives some insight into the nature of pistons.

Lemma 3.4. *A piston Γ can be factored, uniquely,*

$$\Gamma=\Gamma_M\Gamma_P=\Gamma_P^*\Gamma_M^*$$

where Γ_M, Γ_M^* are monotone and Γ_P, Γ_P^* are permutations.

Proof. Write $\Gamma f(s)=r(s)\,f\circ H(s)$ for all $f\in L_H$. Define $\Gamma_M f(s)=r(s)\,f\circ R(s)$ with $R(s)=\int_0^s r(u)\,du$, then $\Gamma_M^{-1}\Gamma f(s)=f\circ H\circ R^{\leftarrow}(s)=\Gamma_p f(s)$ and Γ_p is a permutation by definition.

Apply the representation just obtained to $\tilde{\Gamma}=\Gamma^{-1}$, then we find $\tilde{\Gamma}=\tilde{\Gamma}_M\tilde{\Gamma}_P$ i.e. $\Gamma=\Gamma_P^*\Gamma_M^*$ with $\Gamma_P^*=\tilde{\Gamma}_P^{\leftarrow}$ and $\Gamma_M^*=\tilde{\Gamma}_M^{\leftarrow}$.

Remark. Using the decomposition of Lemma 3.4, the requirement (2.6) is fulfilled if one writes (2.7) as $r(s)\,f(H_0\circ R(s))$ with $R(s)=\int_0^s r(u)\,du$ and one requires that $\int_0^1 r(u)\,du=1$ and H_0 is measure preserving.

4. Proper Representations

We are now going to use the transformations discussed in Sect. 3 to transform an arbitrary sequence of spectral functions into a nice one. As we have seen, a representation of the form (2.3), involving a sequence of L_{1+}-functions $\{f_n\}$, is always possible.

Definition 4.1. *We call such a representation **proper**, or equivalently we say that the sequence $\{f_n\}$ is proper, if two conditions are met.*

First

$$\lambda_1\{s|\sup_n f_n(s)>0\}=1, \tag{4.1}$$

where λ_1 is 1-dimensional Lebesgue measure. Second, the σ-field generated by the ratios $\{f_m/f_n$ for all n, $m\}$ including the values 0 and ∞ on $[0,1]$ is, except for atoms, the Borel field.

As we will see the atoms can be taken without loss of generality to be intervals. This condition is similar to Hardin's [1982] concept of minimality.

Remark. Note that if for a sequence of positive constants $\{c_n\}$ the identity $\sup_n c_n f_n(s)=1$ holds a.e., then the spectral functions of a proper representation generate the same σ-field as their ratios.

Suppose (4.1) is not true. Then a measurable portion of the strip is "wasted" in the sense that $u/f_n(s)=\infty$ for all $u>0$ and all n if s is such that $\max_n f_n(s)=0$. So the content of this portion of the strip plays no role in the minimization of (2.3).

We include an example in which the second condition is violated. Let

$$f_1(s)\equiv 1$$

and let

$$f_n(s)=|\tfrac{1}{2}-s|$$

for $n\geqq 2$. Pairs of points of the form $\{s,1-s\}$, $s\in[0,\frac{1}{2})$ are "elementary" in that no set which is measurable $\{f_m/f_n\}$ can include one but not the other. Also such a pair has Lebesgue measure 0 and so it is not an atom.

Theorem 4.1. *Every min-stable process $\tilde{Z}$ has a proper representation.*

Proof. We proceed constructively. Let $\{f_n\}$ be a sequence of spectral functions for $\tilde{Z}$. First, we will show that there exists a nonrandom vector $\tilde{c}$ with elements $c_n>0$ such that

$$\int_0^1 \bar{f}(s)\,ds=1 \tag{4.2}$$

where

$$\bar{f}(s)=\sup_n c_n f_n(s). \tag{4.3}$$

To see that this is so, notice that $\max_n c_n f_n(s)\leqq\sum_n c_n f_n(s)$, whose integral is finite for appropriate choice of $\tilde{c}$.

Now let

$$\bar{F}(s)=\int_0^s \bar{f}(x)\,dx. \tag{4.4}$$

Notice that $\bar{F}(s)$ is an absolutely continuous probability distribution function.

Define $f_n^*(s)=f_n(\bar{F}^{\leftarrow}(s))/\bar{f}(\bar{F}^{\leftarrow}(s))$ for $n=1,2,\ldots$ and $s\in\bar{F}(A)$ where $A=\{s\,|\,\bar{f}(s)>0\}$. Let $f_n^*(s)=0$ for $s\notin\bar{F}(A)$.

First we prove

$$\sup_n c_n f_n^*(s)=1 \quad \text{a.s.} \tag{4.5}$$

This is clear for all s with $\bar{f}\circ\bar{F}^{\leftarrow}(s)>0$. It remains to prove that $\bar{f}\circ\bar{F}^{\leftarrow}(s)>0$ a.s. Let U be a random variable with uniform distribution on $[0,1]$. Take $X=\bar{F}^{\leftarrow}(U)$, then X has density $\bar{f}$.

It follows that

$$\lambda_1\{s\,|\,\bar{f}\circ\bar{F}^{\leftarrow}(s)=0\}=P\{\bar{f}(X)=0\}=\int_{\bar{f}(s)=0}\bar{f}(s)\,ds=0.$$

Next we prove $\{f_n^*\}\sim\{f_n\}$. Take the random variable X as above. For arbitrary $z_n\geqq 0$ $(n\in\mathbb{N})$

$$\begin{aligned}\int_0^1 \sup_n z_n f_n^*(s)\,ds &= \int_{\bar{f}\circ\bar{F}^{\leftarrow}(s)>0} \sup_n z_n f_n(\bar{F}^{\leftarrow}(s))/\bar{f}\circ\bar{F}^{\leftarrow}(s)\,ds\\ &= E\sup_n z_n f_n(X)/\bar{f}(X)=\int_{\bar{f}(s)>0}\sup_n z_n f_n(s)/\bar{f}(s)\cdot\bar{f}(s)\,ds\\ &= \int_{\bar{f}(s)>0}\sup_n z_n f_n(s)\,ds=\int_0^1\sup_n z_n f_n(s)\,ds.\end{aligned}$$

Next we apply a second transformation T in order to satisfy the second part of the definition. The functions $\{f_n^*(s)\}$ generate a σ-field R of $[0,1]$. By (4.5) this is the same as the σ-field generated by the ratio's $\{f_m^*(s)/f_n^*(s)\}$. The σ-field R is included in the Borel field since the functions $f_n^*(s)$ are measurable. We will show that there exists a measure preserving point transformation T: $[0,1]\to[0,1]$, which maps intervals on the beginning of $[0,1]$ into the atoms of R and maps the Borel field off of intervals corresponding to the atoms into R, off of the atoms. Let $\{C_k\}$ be the atoms of R. For each k, let $T(x)=C_k$ for x in the interval

$$\left[\sum_{j=1}^{k-1}\lambda_1(C_j),\ \sum_{j=1}^{k}\lambda_1(C_j)\right).$$

Now from Halmos [1950] it follows that there is a measure algebra isometry (i.e. a set operations and measure preserving transformation) $T^{\leftarrow}$ mapping $R\cap\{C_1,C_2,\ldots\}^c$ onto the Borel field B restricted to $\left[\sum_{j=1}^{\infty}\lambda(C_j),1\right]$.

Such a transformation induces a canonical transformation $T^{\leftarrow}$ mapping functions measurable with respect to $R\cap\{C_1,C_2,\ldots\}^c$ into functions measurable with respect to B restricted to $\left[\sum_{j=1}^{\infty}\lambda(C_j),1\right]$.

Define $f_n^{**} = T^{\leftarrow} f_n^*$ for $n = 1, 2, \ldots$. Since T is linear isometry, for all $z_n > 0$ $(n = 1, 2, \ldots)$

$$\int_0^1 \max_n z_n f_n^{**}(s)\, ds = \int_0^1 \max_n z_n f_n^*(s)\, ds.$$

Consequently $\{f_n^*\} \sim \{f_n^{**}\}$ in the sense of Definition 2.1. Clearly the representation $\{f_n^{**}\}$ satisfies the second requirement of Definition 4.1. It is now sufficient to prove that $\max_n c_n f_n^{**}(s) = 1$ a.e. Let S be a uniformly distributed random variable. Obviously the distribution of $\max_n c_n f_n^*(S)$ is the same as that of $\max_n c_n f_n^{**}(S)$. Since the former is identically 1, it follows that $\max_n c_n f_n^{**}(S) = 1$ in distribution, hence a.s.

Remarks. If $\{f_n\}$ is already proper then there exists a piston Γ_M of monotone type such that $f_n^* = \Gamma_M(f_n)$ for all n. Furthermore, then there exists a piston Γ_p of permutation type, such that $f_n^{**} = \Gamma_p(f_n^*)$ for all n since A off the atoms is just the Borel field, itself, there. That Γ_p is pointwise one-to-one, then, follows from the Halmos construction, explained above. Relation (4.5) means that the $\{f_n^*(s)\}$ and hence also the $\{f_n^{**}(s)\}$, which are random variables, reside on the "generalized rectangle" $\sup_n c_n f_n^{**}(s) \equiv 1$. Instead of this one could also construct a proper representation satisfying the restriction

$$\sum_n c_n f_n^{**}(s) \equiv 1 \tag{4.6}$$

(a generalized simplex). We prefer to use (4.5), however, because it enables us to use Lemma 4.1, below.

Theorem 4.2. *Let $\{f_n\} \sim \{g_n\}$ in the sense of Definition 2.1. If both are proper, there exists a piston Γ such that*

$$g_n \equiv \Gamma(f_n) \tag{4.7}$$

for all n.

Remark. If $\sup_n c_n f_n(s) = 1$ a.s. for some sequence of positive constants $\{c_n\}$, then Γ is essentially unique. For details and proof see Sect. 6.

Before proceeding to prove the theorem we state and prove a lemma which is due to A.A. Balkema.

Lemma 4.1 (A.A. Balkema, personal communication.). *Let $X_1, X_2, \ldots, X_k$ be the non-negative elements of a random vector with finite means. Define*

$$\phi(a_1, a_2, \ldots, a_k) = E(\max\{1, X_1/a_1, X_2/a_2, \ldots, X_k/a_k\}) \tag{4.8}$$

for all $a_i \in (0, \infty]$. Then

$$\begin{aligned} &P\{X_1 \leqq a_1, \ldots, X_k \leqq a_k\} \\ &\quad = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} [(1+\varepsilon)\, \phi(a_1(1+\varepsilon), \ldots, a_k(1+\varepsilon)) - \phi(a_1, \ldots, a_k)]. \end{aligned} \tag{4.9}$$

Remark. ϕ, given by (4.8), is a "sup-characteristic function". The result (4.9) is an inversion formula.

Proof. For $a_i \in (0, \infty]$, $v_i \in [0, \infty)$ and $\varepsilon > 0$, let

$$\psi_\varepsilon(v_1, v_2, \ldots, v_k) = \max\{(1+\varepsilon), v_1/a_1, \ldots, v_k/a_k\} - \max\{1, v_1/a_1, \ldots, v_k/a_k\}.$$

Notice that

$$\psi_\varepsilon = \varepsilon$$

if $v_i \leqq a_i$ for $i = 1, 2, \ldots, k$, and that

$$\psi_\varepsilon = 0$$

if $v_i \geqq a_i(1+\varepsilon)$ for some i. Consequently $\varepsilon^{-1}\psi_\varepsilon$ converges monotonically, as $\varepsilon \downarrow 0$, to the indicator function of the set $[0, a_1] \times [0, a_2] \times \ldots \times [0, a_k]$, whose expectation is the probability on the left hand side of (4.8).

Proof of Theorem 4.2. This proof depends upon the steps in the proof of Theorem 4.1, above. Let $c_n > 0$ be so chosen that (4.2) holds where $\bar{f}(s)$ is given by (4.3). Define the monotone piston F by letting

$$F^{-1} h(s) = \bar{f}(s) \cdot h\left(\int_0^s \bar{f}(x)\, dx\right).$$

Similarly define the monotone piston G by letting

$$G^{-1} h(s) = \bar{g}(s) \cdot h\left(\int_0^s \bar{g}(x)\, dx\right)$$

with $\bar{g}(s) = \sup_n c_n g_n(s)$.

It follows that $\{F f_n\}$ and $\{G g_n\}$ are proper and that $\{F f_n\} \sim \{G g_n\}$. Notice that $\sup_n c_n F f_n \equiv \sup_n c_n G g_n \equiv 1$. Therefore for every sequence of non-negative constants $\{a_n\}$

$$\begin{aligned}
&\int_0^1 \max\{1, \sup_{n\geqq 1} a_n F f_n(s)\}\, ds \\
&= \int_0^1 \max\{(\sup_{n\geqq 1} c_n F f_n(s))(\sup_{n\geqq 1} a_n F f_n(s))\}\, ds \\
&= \int_0^1 \sup_{n\geqq 1}\{\max(c_n, a_n) \cdot F f_n(s)\}\, ds = \int_0^1 \sup_{n\geqq 1}\{\max(c_n, a_n) \cdot G g_n(s)\}\, ds \\
&= \int_0^1 \max\{1, \sup_{n\geqq 1} a_n G g_n(s)\}\, ds.
\end{aligned}$$

By Lemma 4.1, above, the sequences $\{F f_n\}$ and $\{G g_n\}$ have the same probability distribution, i.e.

$$\lambda((\max_{n\geqq 1} a_n F f_n)^{-1}(B)) = \lambda((\max_{n\geqq 1} a_n G g_n)^{-1}(B))$$

for every Borel set $B \subset [0,1]$ and all $a_n \geqq 0$ $(n=1,2,\ldots)$. This leads to an isometry between the σ-fields $\sigma\{Ff_n\}$ and $\sigma\{Gg_n\}$ induced by the two sequences of functions (identify $(\max_{n\geqq 1} a_n Ff_n)^{-1}(B)$ with $(\max_{n\geqq 1} a_n Gg_n)^{-1}(B)\}$. Since the two representations are proper, the isometry can be realized by a one-to-one measure preserving transformation T^{-1}: $[0,1]\to[0,1]$ (Rohlin, cf. Parry and Tuncel [1982], p. 22). Now $Gg_n(s)=Ff_n(T(s))$ for $n=1,2,\ldots$ a.e. The statement of the theorem follows.

Remark. The piston Γ referred to in Eq. (4.7) can be written $\Gamma=G^{-1}TF$ where T is the permutation from the last part of the proof and G and F are of monotone type.

5. Stationarity in Discrete Time

Suppose that $\tilde{Z}$, with elements Z_n, is strictly stationary. That is $\{Z_n\}$ and $\{Z_{n+1}\}$ have the same joint distribution. It follows that $\{Z_{n+k}\}$ have the same joint distribution for all $k=0, \pm 1, \pm 2, \ldots$. Now $\{Z_n\}$ has a representation of the form (2.3). Recall Definition 2.1 of equivalence. By Theorem 4.1 there exists a proper sequence $\{f_n\}$ for the representation (2.3). It follows that $\{f_{n+k}\}$ is a proper sequence for any k.

By stationarity

$$\{f_n\} \sim \{f_{n+1}\}.$$

By Theorems 4.1 and 4.2 we have the following:

Theorem 5.1. *The elements of $\tilde{Z}$ are representable by* (2.3) *with proper sequence $\{f_n\}$. There exists a piston Γ such that*

$$f_{n+1} \equiv \Gamma(f_n)$$

and so

$$f_n \equiv \Gamma^n(f_0).$$

Example. Take $f_0(s)=s$ and $\Gamma f(s)=f(1-s)$. The process $\{Z_n\}$ is periodic with period 2 and

$$P\{Z_1>z_1, Z_2>z_2\}=\exp -\frac{z_1^2+z_2^2+z_1 z_2}{2(z_1+z_2)}.$$

Remark. Suppose that $\lambda_1\{s \mid f_0(s)>0\}=1$. Then one can take $f_0(s)$ to be constant on $[0,1]$ without loss of generality. If $\lambda_1\{s \mid f_0(s)>0\} \in (0,1)$, we can take $f_0(s)$ to be constant on $[0,a]$ and 0 thereafter for any $a \in (0,1)$. It is not necessary that $a \equiv \lambda_1\{s \mid f_0(s)>0\}$.

As we saw in the second remark following Theorem 4.1, we can choose the sequence $\{f_n(s)\}$ so that it resides on a generalized rectangle. There exists a nonrandom vector $\tilde{c}$, with elements $c_n \geqq 0$, such that for almost all $s \in [0,1]$,

$$\max_n c_n f_n(s) \equiv 1.$$

For each k, $\{f_{n+k}(s)\}$ resides on a similar but different generalized rectangle with $\{c_n\}$ replaced by $\{c_{n+k}\}$.

In discrete time, a strictly stationary stochastic process of our type is completely described, in distribution, by $\{f_0, F\}$ where

$$f_n = F^n(f_0),$$

$n=0, \pm 1, \pm 2, \ldots$. Hence F^n is a power group of pistons. The representation $\{f_0, F\}$ is not unique. Let $\{g_0, G\}$ be another, equivalent, representation. That is

$$\{f_n\} \sim \{g_n\}$$

where

$$g_n = G^n(g_0),$$

$n=0, \pm 1, \pm 2, \ldots$. Then we write

$$\{f_0, F\} \sim \{g_0, G\}. \tag{5.1}$$

Theorem 5.2. *Equivalence in the sense of* (5.1) *holds if and only if there exists a piston B such that*

$$g_0 = B(f_0) \tag{5.2}$$

and

$$G^n = BF^nB^{-1}, \tag{5.3}$$

$n=0, \pm 1, \pm 2, \ldots$.

Proof. First assume that (5.1) is true. By Theorem 4.2 there exists a piston B such that, for $n=0, \pm 1, \pm 2, \ldots$,

$$g_n = G^n(g_0) = B(f_n) = BF^n(f_0) = BF^nB^{-1}(g_0). \tag{5.4}$$

By Theorem 4.1 we can take both $\{f_n\}$ and $\{g_n\}$ to be proper. So for $n=0, \pm 1, \pm 2, \ldots$

$$G^n = BF^nB^{-1}.$$

Conversely suppose that (5.2) and (5.3) are true. For any integers m, $n \in (-\infty, \infty)$,

$$G^mG^n = BF^mB^{-1}BF^nB^{-1} = BF^mF^nB^{-1} = BF^{m+n}B^{-1} = G^{m+n}.$$

Thus (5.1) is true, verified by (5.4) from right to left.

6. Stationarity in Continuous Time

In this section we assume that $\tilde{Z}$ is a min-stable stationary random function with values $Z(t)$, $-\infty < t < \infty$. A standard assumption is continuity in probability. See e.g. de Haan [1984]. We make that assumption about $\{Z(t)\}$. A representation for such a process may be defined as follows: First we construct a representation for $\{Z(r_n)\}$ where $\{r_n\}$ is a countable sequence dense in $(-\infty, \infty)$. Since $\tilde{Z}$ is continuous in probability, a unique extension exists from the representation for $\{Z(r_n)\}$ to one for $\{Z(t)\}$. This follows by Theorem 3 of de Haan [1984] before the reciprocal transformation of Sect. 2. The latter does not invalidate it.

In this section we consider equivalence classes of pistons rather than individual pistons.

Two pistons Γ_1 and Γ_2 are equivalent (are considered the same) with respect to a family $\{f_t\}$ of L_1-functions if $\Gamma_1\Gamma_2^{-1}$ is of permutation type (i.e. the function r from Definition 2.2 equals 1 a.e.), furthermore $\Gamma_1\Gamma_2^{-1}(\chi_c)=\chi_c$ for the indicator function of any atom c of the σ-field induced by $\{f_t\}$ and $\Gamma_1\Gamma_2^{-1}$ is the identity a.e. outside the atoms of that σ-field.

Theorem 6.1. *For all* $t\in(-\infty,\infty)$,

$$f_t=\Phi^t(f_0) \tag{6.1}$$

for some nonnegative L_1 *function* $f_0(s)$ *and some power group* $\{\Phi^t\}$ *of pistons, that is* $\Phi^{t+s}\equiv\Phi^t\Phi^s$, *for all* $s, t\in(-\infty,\infty)$.

Before proving the Theorem we have the following:

Lemma 6.1. *The functions* $f_t(s)$ *from the representation for* $\tilde{Z}$ *are* L_1 *continuous in* t.

This is essentially Theorem 3 of de Haan [1984]. It is easily seen that it remains true after the transformation of Sect. 2.

Proof of Theorem 6.1. By Theorem 3 of de Haan [1984] and Theorem 4.1 above there is a family $\{f_t\}_{t\in\mathbb{R}}$ of non-negative L_1-functions such that for all real t

$$Z_t=\min_{k\geq 1} U_k/f_t(S_k)$$

with $\{S_k, U_k\}$ as in Sect. 1 and the representation $\{f_{r_n}\}_{r_n\in Q}$ is proper for $\{Z_{r_n}\}_{r_n\in Q}$.

We may further suppose that for some sequence $c_n>0$ ($n=0, \pm 1, \pm 2, \ldots$) we have $\sup_n c_n f_{r_n}\equiv 1$.

From Lemma 6.1 it follows that f_t is measurable with respect to $\sigma\{f_{r_n}\}_{r_n\in Q}$ (the σ-field induced by this sequence of random variables) for all $t\in\mathbb{R}$. Furthermore for any $s\in\mathbb{R}$ the family $\{f_{s+r_n}\}_{r_n\in Q}$ is L_1-dense in $\{f_t\}_{t\in\mathbb{R}}$. Now fix $s\in\mathbb{R}$.

We claim that $\sup_n c_n f_{s+r_n}(u)>0$ a.e. Suppose not, then the set A defined by

$$A=\{u\,|\,c_n f_{s+r_n}(u)=0 \text{ for all } r_n\in Q\}$$

has positive Lebesgue measure. Since $\sup_n c_n f_{r_n}(u)\equiv 1$ there exists a subset A_1 of A with positive Lebesgue measure, $\varepsilon>0$ and an index n_0 such that

$$f_{r_{n_0}}(u)>\varepsilon \quad \text{for } u\in A_1.$$

Now the sequence $\{f_{s+r_n}\}_{r_n\in Q}$ is dense in $\{f_t\}_{t\in\mathbb{R}}$ i.e. there is a subsequence such that $f_{s+r_{n'}}\to f_{r_{n_0}}$ in L_1. Since on the one hand $f_{s+r_{n'}}(u)=0$ for $u\in A_1$ and all $r_{n'}$ and on the other hand $f_{r_{n_0}}(u)>\varepsilon$ for $u\in A_1$, a contradiction is obtained.

The proper representations $\{f_{r_n}\}$ and $\{f_{s+r_n}\}$ are equivalent. Hence by Theorem 4.2 for all s there exists a Γ_s such that for all n

$$f_{r_n+s}(u)=\Gamma_s f_{r_n}(u)=r_s(u)\,f_{r_n}(T_s\circ R_s(u)) \tag{6.2}$$

with $R_s(u)=\int_0^u r_s(v)\,dv$ and T_s measure preserving. Since $\sup_n c_n f_{r_n}\equiv 1$,

$$r_s(u)=\sup_n c_n f_{r_n+s}(u).$$

Since both $\{f_{r_n}\}$ and $\{f_{r_n+s}\}$ induce the Borel field off the atoms in $[0,1]$, the mappings $\mathbf{f}$ and $\mathbf{f}_s$: $[0,1]\to\mathbb{R}^{\mathbb{N}}$ with coordinates $\{f_{r_n}\}$ and $\{f_{r_n+s}\}$ respectively, are one-to-one a.e. outside the atoms. It follows that for all s the piston Γ_s is determined up to an equivalence (as defined in the beginning of this section).

From (6.2) it follows by L_1-continuity

$$|f_{s+t}-\Gamma_s f_t|\leqq|f_{s+t}-f_{s+r_n}|+|f_{s+r_n}-\Gamma_s f_{r_n}|+|\Gamma_s f_{r_n}-\Gamma_s f_t|\to 0 \qquad (n\to\infty).$$

Hence for all $s,\ t\in\mathbb{R}$

$$f_{s+t}=\Gamma_s f_t$$

and $\{\Gamma_t\}_{t\in\mathbb{R}}$ forms a power group of pistons since all of them are determined up to an equivalence.

Theorem 6.2. *The realizations are of either of two kinds.*

1) *They are bounded away from* 0 *on every finite interval with probability* 1.

or

2) *They are arbitrarily close to* 0 *on every finite interval with probability* 1.

These are true, respectively, according as

$$\int_0^1 [\max_{0\leqq t\leqq\lambda} f_t(s)]\,ds < \text{or} = \infty. \tag{6.3}$$

The integral is finite for all $\lambda\in(0,\infty)$ if it is for any such λ.

Remark. A similar result is known to hold for stationary Gaussian processes. In fact in case 1, above, a stationary Gaussian process is continuous everywhere with probability 1. This result is due to Dobrushin [1960]. See also Cramér and Leadbetter [1967], Chap. 9.

Proof of Theorem 6.2. First suppose that $\lambda\in(0,\infty)$. By (2.5), for $z\in(0,\infty)$

$$-\log P\{Z_\lambda>z\}=z\int_0^1[\max_{0\leqq t\leqq\lambda} f_t(s)]\,ds$$

where

$$Z_\lambda=\min_{0\leqq t\leqq\lambda} Z(t)=\min_{0\leqq t\leqq\lambda}\min_{l\geqq 1} U_l/f_t(S_l)=\min_{l\geqq 1} U_l/[\max_{0\leqq t\leqq\lambda} f_t(S_l)].$$

If the integral is finite, then Z_λ has a negative exponential distribution with $EZ_\lambda=1\Big/\int_0^1 \max_{0\leqq t\leqq\lambda} f_t(s)\,ds.$

Thus 1, above, is satisfied by the sample functions. Clearly by stationarity and separability the same is true for *all* $\lambda \in (0, \infty)$. Suppose on the other hand that the integral diverges. Then $-\log P\{Z_\lambda > z\} = \infty$, $P\{Z_\lambda > z\} = 0$ and so $Z_\lambda = 0$ with probability 1. So 2, above, holds.

Example. The process $\{Z(t)\}$ defined by $Z(t) = \max_{l > 1} U_l / f_t(S_l)$ where

$$f_t(s) = \Gamma_t f_0(s) \qquad \text{for } s \in \mathbb{R}$$

with

$$\Gamma_t h(s) = e^t \cdot s^{e^t - 1} \cdot h(s^{e^t})$$

is a strictly stationary min-stable process for any non-negative $f_0 \in L_1$.

We examine in detail a broad but not exhaustive class of min-stable stationary processes in continuous time. It is analogous to the class of moving average processes. In fact ours are moving minimum processes. For some properties of moving minima in discrete time see Deheuvels [1983].

Let $H(x)$ be an absolutely continuous distribution function with support on $(-\infty, \infty)$. Notice that H: $(-\infty, \infty) \to (0, 1)$. Let S_l, V_l be, as before, a homogeneous 2-dimensional Poisson process on the strip $[0, 1] \times R_+$. Let

$$(X_l, Y_l) = (H^{-1}(S_l), U_l/(H^{-1})'(S_l)). \tag{6.4}$$

Now $\{X_l, Y_l\}$ are the points of a homogeneous Poisson process on $R \times R_+$, the upper half of the plane. Inverting,

$$(S_l, U_l) = (H(X_l), Y_l/H'(X_l)).$$

Let the function ϕ: $R \to R_+$. Assume that

$$\int_{-\infty}^{\infty} \phi(s)\, ds \in (0, \infty). \tag{6.5}$$

For each $t \in (-\infty, \infty)$ let

$$Z(t) = \min_l Y_l / \phi(X_l - t). \tag{6.6}$$

The marginal distribution of $Z(t)$ is negative exponential with mean $1 \Big/ \int_{-\infty}^{\infty} \phi(s)\, ds$.

By (6.4) and (6.6)

$$Z(t) = \min_l U_l / f_t(S_l)$$

where

$$f_t(s) = (H^{-1})'(s)\, \phi(H^{-1}(s) - t).$$

Clearly

$$f_0(s) = (H^{-1})'(s)\, \phi(H^{-1}(s))$$

and we can let

$$f_t(s) = \Phi^t f_0(s)$$

where $\{\Phi^t\}$ is a power group of pistons with

$$\Phi^t \equiv A^{-1} B^t A$$

where

$$A f(s) = f(H^{-1}(s))(H^{-1})'(s)$$

$$A^{-1} f(s) = f(H(s))\, H'(s)$$

and

$$B^t f(s) = f(s-t).$$

By Theorem 6.2, the sample paths are of types 1 or 2 of that theorem according as

$$\int_{-\infty}^{\infty} [\max_{0 \leqq t \leqq \lambda} \phi(s+t)]\, ds < or = \infty. \tag{6.7}$$

For divergence of the integral it is clearly sufficient that ϕ be unbounded. For convergence it is not sufficient that ϕ be bounded as we show by the following example.

Let $\{b_n\}$ be such that $b_n > 0$, $n = 0, \pm 1, \pm 2, \ldots$. Assume that $\sum_{n=-\infty}^{\infty} b_n < \infty$. Let

$$\begin{aligned} \phi(s) &= 1 \qquad n \leqq s \leqq n + b_n \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Notice that the integral (6.5) converges but that the integral (6.7) diverges.

In order that both integrals converge, it is sufficient that ϕ be proportional to a unimodal probability density function.

For example, let

$$\phi(s) = e^{-|s|}.$$

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References

Cramér, H., Leadbetter, M.R.: Stationary and related stochastic processes. New York: J. Wiley 1967

Deheuvels, P.: Point processes and multivariate extreme values. J. Multivariate Anal. **13**, 257–272 (1983)

Dobrushin, R.L.: Properties of sample functions of a stationary gaussian process. Teor. Veroyatn. Primen. **5**, 132–134 (1960)

Galambos, J.: The asymptotic theory of extreme order statistics. New York: Wiley, 1978

Haan, L. de, Resnick, S.I.: Limit theory for multivariate sample extremes. Z. Wahrscheinlichkeitstheor. Verw. Geb. **40**, 317–337 (1977)

Haan, L. de: A spectral representation for max-stable processes. Ann. Probab. **12**, 1194–1204 (1984)

Halmos, P.R.: Measure theory. Princeton: D. van Nostrand 1950

Hardin, C.D.: On the spectral representation of symmetric stable processes. J. Multivariate Anal. **12**, 385–401 (1982)

Leadbetter, M.R., Lindgren, G., Rootzén, H.: Extremes and related properties of random sequences and processes. Berlin Heidelberg New York: Springer 1983

Parry, W., Tuncel, S.: Classification problems in ergodic theory. Cambridge: University Press 1982

Pickands, J. III: The two-dimensional Poisson process and extremal processes. J. Appl. Probab. **8**, 745–756 (1971)

Royden, H.L.: Real analysis. 2nd ed. New York: Macmillan 1968

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