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Infinitely Divisible Completely Positive Mappings

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Summary. We generalise the theory of infinitely divisible positive definite functions $f: \mathscr{G} \to \mathbb{C}$ on a group $\mathscr{G}$ to a theory of infinite divisibility for completely positive mappings $\Phi: \mathscr{G} \to \mathscr{B}(\mathscr{H})$ taking values in the algebra of bounded operators on some Hilbert space $\mathscr{H}$.

We prove a structure theorem for normalised infinitely divisible completely positive mappings Φ which shows that the mapping Φ, its Stinespring representation and its Stinespring isometry are of type S (in the sense of Guichardet [Gui]). Furthermore, we prove that a completely positive mapping is infinitely divisible if and only if it is the exponential (as defined in this paper) of a hermitian conditionally completely positive mapping.

Introduction

Let X be a real valued random variable with characteristic function $\phi_X(t) = \langle e^{itX} \rangle$, $t \in \mathbb{R}$. Then $\phi_X: \mathbb{R} \to \mathbb{C}$ has the following properties:

(1) ϕ_X is positive definite, i.e., for all functions $t \in \mathbb{R} \to \lambda_t \in \mathbb{C}$ with finite support, we have that

$$\sum_{t,t'} \bar{\lambda}_t \lambda_{t'} \phi_X(t'-t) \geqq 0 \tag{i}$$

(2) ϕ_X is normalised, i.e. $\phi_X(0) = 1$

(3) ϕ_X is continuous.

Conversely, if $\phi: \mathbb{R} \to \mathbb{C}$ is a function having the three properties mentioned above, then, by Bochner's theorem, there exists a real valued random variable X such that ϕ is the characteristic function of X.

By definition a random variable X is infinitely divisible iff its characteristic function ϕ_X satisfies the following condition: for each $n \in \mathbb{N}_0$, there exists a normalised positive definite function $\phi_n: \mathbb{R} \to \mathbb{C}$ such that

$$\phi_X(t) = \phi_n(t)^n \tag{ii}$$

for all $t \in \mathbb{R}$.

The notion of positive definiteness (i) and infinite divisibility (ii) can be naturally extended to complex valued functions ϕ: $\mathscr{G} \to \mathbb{C}$ on an arbitrary group $\mathscr{G}$. This leads to the study of infinitely divisible positive definite functions on groups and their associated representations [Str], [Pa], [Gui], [PaSch]. Infinite divisibility has also been extended to positive functionals on some particular algebras such as CCR-algebras [Ar] [CoGuHu], CAR-algebras [MaStr]-[HuWiPe] and Lie algebras [Str]. On the other hand, the structure of bi-algebras provided a natural setting to implement the notion of infinitely divisible functionals [GvW], [vW], [Sch].

A further extension consists in considering infinite divisibility for positive definite functions for which not only the space on which they are defined is non-commutative (e.g., a group, an algebra), but also the range space is allowed to be non-commutative (e.g., a*-algebra). Under this extension, the notion of positive definiteness (i) is carried over to the notion of complete positivity (cf. Definition I.1.), whereas the proper reformulation of infinite divisibility (ii) will invoke tensor products (cf. Definition I.6.). Examples of completely positive mappings satisfying this extended notion of infinite divisibility have already been constructed in [FQ], [Q], [AcBa] by means of a central limit procedure.

In this paper we study this fully non-commutative natural extension of infinite divisibility for completely positive mappings Φ: $\mathscr{G} \to \mathscr{B}(\mathscr{H})$ defined on a group $\mathscr{G}$ and taking values in the algebra of bounded operators on some Hilbert space $\mathscr{H}$. Two important results of the theory of infinitely divisible positive definite functions on groups are extended. The first result, which is known as the "Araki-Woods embedding theorem", shows essentially that the representation induced canonically by an infinitely divisible positive definite function has a continuous tensor product structure [ArWo] [Gui]. The second result characterises infinitely divisible positive definite functions as those functions having a hermitian conditionally positive definite logarithm.

The paper is divided into three main sections. In the first section we collect the basic definitions and elementary properties of complete positivity and infinite divisibility. In the second section we obtain an "Araki-Woods embedding" result for the group representation induced canonically by an infinitely divisible completely positive mapping, as well as for the mapping itself. Finally, in the last section we construct a logarithm for an infinitely divisible completely positive mapping. As in the function case, this logarithm is hermitian and conditionally completely positive. It also exhibits the new property of infinite additivity (a logarithmic version of infinite divisibility) which remained hidden for functions. Moreover using an inverse ("exponential") construction we show that there is a one to one correspondence between infinitely divisible completely mappings and hermitian infinitely additive conditionally completely positive mappings.

All new notions and constructions in the paper are illustrated by an explicit example.

I. Definition and Elementary Properties of Infinitely Divisible Completely Positive Mappings

I.1. Completely Positive Mappings

Definition I.1. (i) A mapping Φ from a group $\mathscr{G}$ into the bounded operators $\mathscr{B}(\mathscr{H})$ on a Hilbert space $\mathscr{H}$ is called *completely positive* (C.P.) if

$$\sum_{g,g'} X_g^* \Phi(g^{-1} g') X_{g'}$$

is a positive operator on $\mathscr{H}$ for all choces of $g \in \mathscr{G} \to X_g \in \mathscr{B}(\mathscr{H})$ vanishing everywhere but on a finite number of elements of $\mathscr{G}$. Any function of this type will be called in the sequel an almost zero function.

(i) Such a C.P. mapping Φ is said to be *normalised* if $\Phi(e) = \mathbb{1}$ where $e \in \mathscr{G}$, is the neutral element and $\mathbb{1} \in \mathscr{B}(\mathscr{H})$ is the identity operator on $\mathscr{H}$.

(iii) A C.P. mapping Φ on a topological group $\mathscr{G}$ is *continuous* if

$$g \in \mathscr{G} \to \Phi(g)$$

is weakly continuous.

Remark I.2. (i) For $\mathscr{H} = \mathbb{C}$, Φ is a complex-valued function on $\mathscr{G}$ and the notion of complete positivity reduces to positive definiteness.

(ii) If $\mathscr{G}$ is the group of unitary elements of a unital C^*-algebra $\mathscr{A}$, and Φ is C.P. mapping from $\mathscr{G}$ into $\mathscr{B}(\mathscr{H})$, which is linear with respect to the linear structure on $\mathscr{G}$ inherited from $\mathscr{A}$, then Φ extends uniquely to a linear mapping from $\mathscr{A}$ into $\mathscr{B}(\mathscr{H})$ which is C.P. in the usual sense. [Tak].

(iii) Definition I.1 (i) is equivalent to

$$\sum_{g,g'} \langle \xi_g | \Phi(g^{-1} g') \xi_{g'} \rangle \geqq 0$$

for all choices of almost zero functions $g \in \mathscr{G} \to \xi_g \in \mathscr{H}$.

(iv) It follows immediately from complete positivity that Φ is a self-adjoint mapping, indeed:

$$\Phi(g^{-1}) = \Phi(g)^*, \qquad g \in \mathscr{G}.$$

I.2. The Stinespring Decomposition of a Completely Positive Mapping

Theorem I.3. *Let V be an isometry from a Hilbert space $\mathscr{H}$ into a Hilbert space $\mathscr{K}$ and π be a representation from a group $\mathscr{G}$ into the unitary operators $\mathscr{U}(\mathscr{K})$ on $\mathscr{K}$ then*

$$g \mapsto V^* \pi(g) V$$

is a normalised C.P. *mapping from $\mathscr{G}$ into $\mathscr{B}(\mathscr{H})$.*

Conversely if $\Phi: \mathscr{G} \to \mathscr{B}(\mathscr{H})$ *is a normalised* C.P. *mapping, there exists a triplet* $(\mathscr{K}, \pi, V)$ *as above such that*

$$\Phi(g) = V^* \pi(g) V.$$

Moreover $(\mathscr{K}, \pi, V)$ *is unique up to unitary equivalence if it satisfies the minimality condition*

$$[\{\pi(\mathscr{G})\}'' V\mathscr{H}] = \mathscr{K}.$$

Finally continuity of Φ *is equivalent to strong continuity of* π.

Proof. [Sti].

The triplet $(\mathscr{K}, \pi, V)$ is called the Stinespring triplet of Φ. If Φ is a function of positive tye on $\mathscr{G}$ the Stinespring decomposition reduces to the well known G.N.S. representation theorem.

Remark I.4. From the Stinespring decomposition $(\mathscr{K}, \pi, V)$ of a normalised C.P. mapping from Φ into $\mathscr{B}(\mathscr{H})$ a few useful properties of Φ are easily deduced.

i) For $g_1, g_2 \in \mathscr{G}$ and $\xi \in \mathscr{H}$:

$$\begin{aligned} \|(\Phi(g_1) - \Phi(g_2))\,\xi\|^2 &= \|V^*(\pi(g_1) - \pi(g_2))\,V\xi\|^2 \\ &\leqq \|(\pi(g_1) - \pi(g_2))\,V\xi\|^2 \\ &= 2\,\mathrm{Re}\langle \xi | (1 - \Phi(g_1^{-1} g_2))\,\xi\rangle. \end{aligned} \tag{1}$$

Hence it is sufficient that $g \to \Phi(g)$ is (weakly) continuous at $g = e$ in order obtain strong continuity of $g \to \Phi(g)$ everywhere.

ii) Let $X: g \in \mathscr{G} \to X_g \in \mathscr{B}(\mathscr{H})$ be an almost zero function then we have that

$$\begin{aligned} \sum_{g, g'} X_g^* \Phi(g^{-1}) \Phi(g') X_{g'} &= \sum_{g, g'} X_g^* V^* \pi(g^{-1}) V V^* \pi(g') V X_{g'} \\ &= (\sum_g \pi(g) V X_g)^* V V^* (\sum_{g'} \pi(g') V X_{g'}) \\ &\leqq \sum_{g, g'} X_g^* V^* \pi(g^{-1} g') V X_{g'} \\ &= \sum_{g, g'} X_g^* \Phi(g^{-1} g') X_{g'}. \end{aligned} \tag{2}$$

This inequality is known as the 2-positivity inequality. In particular it implies

$$\|\Phi(g)\| \leqq 1.$$

I.3. Products of Completely Positive Mappings

Let $\Phi_i: \mathscr{G}_i \to \mathscr{B}(\mathscr{H}_i)$ $i = 1, \ldots, n$, $n \in \mathbb{N}_0$ be normalised C.P. mappings, then there exists a unique normalised C.P. mapping $\Phi \equiv \Phi_1 \otimes \ldots \otimes \Phi_n$ from $\mathscr{G}_1 \times \ldots \times \mathscr{G}_n$ into $\mathscr{B}(\mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_n)$ such that

$$\Phi((g_1, \ldots, g_n)) = \Phi_1(g_1) \otimes \ldots \otimes \Phi_n(g_n).$$

Indeed let $(\mathscr{K}_i, \pi_i, V_i)$ be the Stinespring triplets for Φ_i, then

$$\begin{aligned} (g_1, \ldots, g_n) \mapsto &(V_1^* \otimes \ldots \otimes V_n^*)(\pi_1(g_1) \otimes \ldots \otimes \pi_n(g_n))(V_1 \otimes \ldots \otimes V_n) \\ &= \Phi_1(g_1) \otimes \ldots \otimes \Phi_n(g_n) \end{aligned}$$

is normalised and C.P. by Theorem I.3. By the uniqueness of the Stinespring decomposition it follows that $(\mathscr{K}_1 \otimes \ldots \otimes \mathscr{K}_n, \pi_1 \otimes \ldots \otimes \pi_n, V_1 \otimes \ldots \otimes V_n)$ is the Stinespring triplet of Φ.

In particular if for all i $\mathscr{G}_i = \mathscr{G}$ then clearly the mapping

$$g \in \mathscr{G} \mapsto \Phi_1(g) \otimes \ldots \otimes \Phi_n(g)$$

is C.P. and normalised and will by abuse of notation be denoted by $(\bigotimes_i \Phi_i)(g)$.

The Stinespring triplet of this mapping is clearly given by $(\mathscr{K}, \pi, V)$ where

$$\mathscr{K} = [\{\bigotimes_i \pi_i(g) \mid g \in \mathscr{G}\}'' \bigotimes_i V_i \mathscr{H}_i]$$
$$\pi(g) = \bigotimes_i \pi_i(g)|_{\mathscr{K}}$$
$$V = \bigotimes_i V_i|_{\mathscr{K}}.$$

Remark that in general $\mathscr{K}$ can be a proper subspace of $\bigotimes_i \mathscr{K}_i$.

I.4. Infinitely Divisible Completely Positive Mappings

As we need in the sequel a lot of cyclicity conditions we introduce the following notions:

Definition I.5. (i) We call $(\mathscr{H}, \Phi, \Omega)$ a C.P. *triplet* on a group $\mathscr{G}$ if

a) Φ is a normalised C.P. mapping from $\mathscr{G}$ into $\mathscr{B}(\mathscr{H})$

b) $\Omega \in \mathscr{H}$ is a normalised vector cyclic for $\{\Phi(\mathscr{G})\}''$

Furthermore $(\mathscr{H}, \Phi, \Omega)$ is called a *continuous* C.P. triplet whenever Φ is continuous.

(ii) Two C.P. triplets $(\mathscr{H}_i, \Phi_i, \Omega_i)$ $i = 1, 2$ on a group $\mathscr{G}$ are *equivalent* $((\mathscr{H}_1, \Phi_1, \Omega_1) \cong (\mathscr{H}_2, \Phi_2, \Omega_2))$ if there exists a unitary operator $U: \mathscr{H}_1 \to \mathscr{H}_2$ such that

$$U\Omega_1 = \Omega_2$$
$$U\Phi_1 U^* = \Phi_2.$$

(iii) If $(\mathscr{H}_i, \Phi_i, \Omega_i)$ $i = 1, 2, \ldots, n$, $n \in \mathbb{N}_0$ are n C.P. triplets on a group $\mathscr{G}$ the *product triplet* $\bigotimes_i (\mathscr{H}_i, \Phi_i, \Omega_i) = (\mathscr{H}, \Phi, \Omega)$ on $\mathscr{G}$ is defined by

$$\mathscr{H} = [\{\bigotimes_i \Phi_i(g) \mid g \in \mathscr{G}\}'' \bigotimes_i \Omega_i]$$
$$\Phi(g) = \bigotimes_i \Phi_i(g)|_{\mathscr{H}}$$
$$\Omega = \bigotimes_i \Omega_i.$$

The notion introduced in definition I.5 allows us now to introduce infinite divisibility for C.P. mappings.

Definition I.6. A C.P. triplet $(\mathscr{H}, \Phi, \Omega)$ on a group $\mathscr{G}$ is *infinitely divisible* if there exists for each $n \in \mathbb{N}_0$ a C.P. triplet $(\mathscr{H}^{1/n}, \Phi^{1/n}, \Omega^{1/n})$ on $\mathscr{G}$ satisfying

$$(\mathscr{H}, \Phi, \Omega) = \otimes^n (\mathscr{H}^{1/n}, \Phi^{1/n}, \Omega^{1/n}).$$

$(\mathscr{H}^{1/n}, \Phi^{1/n}, \Omega^{1/n})$ is called an n^{th} *root* of $(\mathscr{H}, \Phi, \Omega)$.

It is clear that our definition extends the usual notion of infinite divisibility for positive definite functions on groups [Gui], [PaSch] and infinite divisible representations [Str]. Indeed let f be an infinite divisible positive definite function on a group $\mathscr{G}$ with G.N.S. triplet $(\mathscr{H}, \pi, \Omega)$, then the triplets $(\mathbb{C}, f, 1)$ and $(\mathscr{H}, \pi, \Omega)$ are both C.P. on $\mathscr{G}$ and infinitely divisible in the sense of definition I.6.

As we will mainly be concerned with continuous C.P. mappings we briefly investigate the relation between continuity and infinite divisibility.

Definition I.7. Let $\mathscr{G}$ be a topological group. We say that $\mathscr{G}$ is *continuously divisible* if for each net $(g_\alpha)_\alpha$ in $\mathscr{G}$ converging to the neutral element e and for each $n \in \mathbb{N}_0$, there exists a net $(h_\alpha)_\alpha$ in converging to e such that eventually $g_\alpha = h_\alpha^n$.

Remark that a lot of groups are continuously divisible: Lie groups, connected locally compact groups [MZ], unitary groups of von Neumann algebras, ...

Lemma I.8. *Let Φ be a continuous positive definite function on a continuously divisible group $\mathscr{G}$.*

If for some $n \in \mathbb{N}_0$ there exists a positive definite n^{th}-root $\Phi^{1/n}$ of Φ then $\Phi^{1/n}$ continuous.

Proof. Without loss of generality we can assume Φ to be normalised. As $(\Phi^{1/n}(g))^n = \Phi(g)$ and as Φ is continuous at $g = e$ we have: for all $\varepsilon > 0$ there exists a neighborhood $\mathcal{O}_\varepsilon$ of e such that for all $g \in \mathcal{O}_\varepsilon$ there exists $k \in \{0, \ldots, n-1\}$ such that

$$|\Phi^{1/n}(g) - \delta_k| < \varepsilon$$

where $\delta_k = \exp \dfrac{2\pi i k}{n}$.

Furthermore, by continuity of the group inversion, $\mathcal{O}_\varepsilon$ can be chosen such that $\mathcal{O}_\varepsilon^{-1} = \mathcal{O}_\varepsilon$. Obviously for ε small enough $\mathcal{O}_\varepsilon$ can be written as a union of disjoint sets $\mathcal{O}_\varepsilon^k$, $k \in \{0, \ldots, n-1\}$ where

$$\mathcal{O}_\varepsilon^k = \{g \in \mathcal{O}_\varepsilon \,|\, |\Phi^{1/n}(g) - \delta_k| < \varepsilon\}$$

and as $\Phi^{1/n}$ is self-adjoint if follows immediately that $(\mathcal{O}_\varepsilon^k)^{-1} = \mathcal{O}_\varepsilon^{n-k}$.

We now prove that if $h \in \mathcal{O}_\varepsilon^k$, $h' \in \mathcal{O}_\varepsilon^{k'}$ and $hh' \in \mathcal{O}_\varepsilon$ then $hh' \in \mathcal{O}_\varepsilon^j$ where $j = (k + k') \bmod n$.

Indeed, choose $\lambda_2 = 1$, $\lambda_{h^{-1}} = -\frac{1}{2}\delta_k$, $\lambda_{h'} = -\frac{1}{2}\delta_{k'}$ and λ_g zero elsewhere, then

$$\sum_{g, g'} \bar{\lambda}_g \lambda_{g'} \Phi^{1/n}(g^{-1} g') \geqq 0$$

becomes

$$\operatorname{Re}\{\delta_{-k-k'} \Phi^{1/n}(hh')\} \leqq 1 - 4\varepsilon. \tag{3}$$

Because $hh' \in \mathcal{O}_\varepsilon$ it belongs to some $\mathcal{O}_\varepsilon^j$ and (3) can only hold if $j=(k+k') \bmod n$.

Suppose now that $\Phi^{1/n}$ is discontinuous, then, as $\Phi^{1/n}$ is positive definite, it must be discontinuous at $g=e$ (Remark I.4.i)). As $\Phi^{1/n}$ is uniformly bounded by $\Phi^{1/n}(e)=1$ we can find a net $(g_\alpha)_\alpha$ in $\mathcal{G}$ converging to e such that $\lim_\alpha \Phi^{1/n}(g_\alpha) = \delta \neq 1$. Since $\Phi=(\Phi^{1/n})^n$ and $\lim_\alpha \Phi(g_\alpha)=1$ we have that $\delta=\delta_k$ for some $k\in\{1,\ldots,n-1\}$. Therefore g_α belongs eventually to $\mathcal{O}_\varepsilon^k$. By the continuous divisibility of the group there exists a net $(h_\alpha)_\alpha$ converging to e such that eventually $g_\alpha = h_\alpha^n$. Therefore h_α^m, $m=1,2,\ldots,n$, belongs eventually to $\mathcal{O}_\varepsilon$ and h_α belongs frequently to some $\mathcal{O}_\varepsilon^j$. It then follows from the argument above that frequently

$$g_\alpha = h_\alpha^n \in \mathcal{O}_\varepsilon^{nj \bmod n} = \mathcal{O}_\varepsilon^0$$

which contradicts that eventually $g_\alpha \in \mathcal{O}_\varepsilon^k$, $k \neq 0$.

Hence $\Phi^{1/n}$ cannot be discontinuous. □

Before proving the analogous result for mappings we introduce some useful notations.

Let $\tilde{\mathcal{G}}$ be the set of all n-tuples $(g_1, \ldots, g_n)$, $g_i \in \mathcal{G}$, $i=1,\ldots,n$, $n \in \mathbb{N}$ where a 0-tuple is the empty set. $\tilde{\mathcal{G}}$ is a semigroup for the composition law:

$$(\Delta, \Delta') \in \tilde{\mathcal{G}} \times \tilde{\mathcal{G}} \mapsto \Delta \times \Delta' \in \tilde{\mathcal{G}}$$

where

$$\begin{aligned} \Delta &= \Delta(g_1, \ldots, g_n) \\ \Delta' &= (g_1', \ldots, g_m') \\ \Delta \times \Delta' &= (g_1, \ldots, g_n, g_1', \ldots, g_m') \end{aligned}$$

Furthermore $\tilde{\mathcal{G}}$ is equipped with a natural involution induced by the group inversion in $\mathcal{G}$:

$$*\colon\ \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}\colon\ \Delta=(g_1,\ldots,g_n) \mapsto \Delta^* = (g_n^{-1}, \ldots, g_1^{-1}).$$

If X is a function on $\mathcal{G}$ with values in the linear operators (possibly unbounded) on some hilbert space we will use the notation

$$X(\Delta) = X(g_1)\, X(g_2) \ldots X(g_n), \qquad \Delta = (g_1, \ldots, g_n)$$

if such a product is well defined. Finally by convention $X(\phi) = \mathbb{1}$.

Theorem I.9. *Let $(\mathcal{H}, \Phi, \Omega)$ be a continuous* C.P. *triplet on a continuously divisible group $\mathcal{G}$.*

If $(\mathcal{H}^{1/n}, \Phi^{1/n}, \Omega^{1/n})$ is a n^{th}-root of $(\mathcal{H}, \Phi, \Omega)$ (in the sense of definition I.6.), then it is continuous.

Proof. Let $\Delta \in \tilde{\mathcal{G}}$. As Φ is C.P. the function

$$g \in \mathcal{G} \mapsto \langle \Phi(\Delta)\, \Omega \,|\, \Phi(g)\, \Phi(\Delta)\, \Omega \rangle \in \mathbb{C}$$

is positive definite. If $(\mathscr{H}^{1/n}, \Phi^{1/n}, \Omega^{1/n})$ is a n^{th}-root of $(\mathscr{H}, \Phi, \Omega)$, we have that

$$\langle \Phi(\Delta)\,\Omega \,|\, \Phi(g)\,\Phi(\Delta)\,\Omega \rangle = (\langle \Phi^{1/n}(\Delta)\,\Omega^{1/n} \,|\, \Phi^{1/n}(g)\,\Phi^{1/n}(\Delta)\,\Omega^{1/n} \rangle)^n$$

and

$$g \in \mathscr{G} \mapsto \langle \Phi^{1/n}(\Delta)\,\Omega^{1/n} \,|\, \Phi^{1/n}(g)\,\Phi^{1/n}(\Delta)\,\Omega^{1/n} \rangle \in \mathbb{C}$$

is again positive definite. From lemma I.8 it now follows that for $n \in \mathbb{N}_0$, $\Delta \in \tilde{\mathscr{G}}$

$$g \in \mathscr{G} \mapsto \langle \Phi^{1/n}(\Delta)\,\Omega^{1/n} \,|\, \Phi^{1/n}(g)\,\Phi^{1/n}(\Delta)\,\Omega^{1/n} \rangle \tag{4}$$

is continuous.

Because $\operatorname{span}\{\Phi^{1/n}(\Delta)\,\Omega^{1/n} \,|\, \Delta \in \tilde{\mathscr{G}}\}$ is dense in $\mathscr{H}^{1/n}$ and $\|\Phi^{1/n}(g)\| \leqq 1$ for all $g \in \mathscr{G}$, the mapping

$$g \in \mathscr{G} \mapsto \Phi^{1/n}(g) \in \mathscr{B}(\mathscr{H}^{1/n})$$

will be continuous if we show that for any $\Delta \in \tilde{\mathscr{G}}$

$$g \in \mathscr{G} \mapsto \Phi^{1/n}(g)\,\Phi^{1/n}(\Delta)\,\Omega^{1/n} \in \mathscr{H}^{1/n}$$

is continuous, but this is an immediate consequence of (4) and the inequality (1). □

We now give an example which will be followed throughout the paper in order to illustrate various constructions.

I.5. An Example

Let H be a complex hilbert space. The scalar product $\langle .|. \rangle$ defines a symplectic form

$$(\xi, \eta) \mapsto \sigma(\xi, \eta) = \operatorname{Im} \langle \xi \,|\, \eta \rangle$$

on H. The Heisenberg group H_σ is then given by $H_\sigma = \{(\xi, \theta) \,|\, \xi \in H,\ \theta \in \mathbb{R}\}$ with composition law

$$(\xi_1, \theta_1)(\xi_2, \theta_2) = (\xi_1 + \xi_2, \theta_1 + \theta_2 - \sigma(\xi_1, \xi_2)).$$

Note that (0,0) is the neutral element and that $(-\xi, -\theta)$ is the inverse of (ξ, θ). Moreover the group is non-commutative. H_σ becomes a topological group if equipped with the product topology of $H \times \mathbb{R}$.

For any complex hilbert space H one constructs the symmetric hilbert space $\mathscr{S}(H)$ (also known as Fock space, exponential space, ...) as follows [Gui].

$$\mathscr{S}(H) = \bigoplus_{n \in \mathbb{N}} H_n$$

where $H_0 = \mathbb{C}$ and H_n is the symmetric subspace of $\otimes^n H$, $n \in \mathbb{N}_0$. $\mathscr{S}(H)$ is generated by the exponential vectors $\{\operatorname{Exp} \eta \,|\, \eta \in H\}$ given by

$$\operatorname{Exp} \eta = \bigoplus_{n \geqq 0} \frac{1}{\sqrt{n!}} \otimes^n \eta$$

where

$$\otimes^0 \eta = 1 \in \mathbb{C}.$$

Notice that

$$\langle \operatorname{Exp} \eta | \operatorname{Exp} \eta' \rangle = \exp \langle \eta | \eta' \rangle.$$

Furthermore the exponential vectors are mutually linearly independent. Also if $H = H_1 \oplus H_2$ there is a natural isomorphism $\mathscr{S}(H_1) \otimes \mathscr{S}(H_2) \to \mathscr{S}(H)$ given by $\operatorname{Exp} \varphi_1 \otimes \operatorname{Exp} \varphi_2 \mapsto \operatorname{Exp} \varphi_1 \oplus \varphi_2$, $\varphi_i \in H_i$, $i = 1, 2$.

We now define a class of C.P. mappings on H_σ. Let $c \in \mathbb{R}^+$ and R, $Q \in \mathscr{B}(H)$ be such that

$$\text{(i) } Q \geqq 0$$
$$\text{(ii) } |c - R^* R| \leqq Q.$$

Define

$$\mathscr{H} = \mathscr{S}(H_R) \subset \mathscr{S}(H) \qquad \text{where } H_R = \overline{\operatorname{Ran} R}$$
$$\Omega = \operatorname{Exp} 0$$
$$\Phi_{R,Q,c}(\xi, \theta) \operatorname{Exp} \eta = \exp(ic\theta - \tfrac{1}{2} \langle \xi | (R^* R + Q) \xi \rangle - \langle R\xi | \eta \rangle)$$
$$\operatorname{Exp}(\eta + R\xi), \qquad \eta \in H_R, \qquad (\xi, \theta) \in H_\sigma.$$

One can check that $(\mathscr{H}, \Phi_{R,Q,c}, \Omega)$ is a continuous C.P. triplet on H_σ [DVV]. This triplet is infinitely divisible, indeed for any $n \in \mathbb{N}_0$ its n^{th} root is given by $(\mathscr{H}, \Phi_{R/\sqrt{n}, Q/N, c/n}, \Omega)$.

II. Structure of Infinitely Divisible Completely Positive Triplets

II.1. Mappings of Type S

The aim of this subsection is to give some general properties of mappings of type S. Such mappings generalise those considered in [Gui] and arise naturally in the study of infinite divisibility.

Definition II.1. Let $\mathscr{H}$ and $\mathscr{K}$ be complex hilbert spaces and $\mathscr{S}(\mathscr{H})$ and $\mathscr{S}(\mathscr{K})$ their corresponding symmetric hilbert spaces. A linear operator C: $\mathscr{S}(\mathscr{H}) \to \mathscr{S}(\mathscr{K})$ with $\operatorname{Span}\{\operatorname{Exp} \eta | \eta \in \mathscr{H}\} \subset \operatorname{Dom} C$ is said to be of type S if it maps exponential vectors into multiples of exponential vectors, i.e., if there exist mappings

$$\gamma\colon \mathscr{H} \to \mathbb{C}\colon \eta \mapsto \gamma(\eta)$$
$$\Gamma\colon \mathscr{H} \to \mathscr{K}\colon \eta \mapsto \Gamma(\eta)$$

such that

$$C \operatorname{Exp} \eta = \gamma(\eta) \operatorname{Exp} \Gamma(\eta).$$

Proposition II.2. *Let* C: $\mathscr{S}(\mathscr{H}) \to \mathscr{S}(\mathscr{K})$ *and its adjoint* C^*: $\mathscr{S}(\mathscr{K}) \to \mathscr{S}(\mathscr{H})$ *be operators of type S. There exist*

(i) $\gamma \in \mathbb{C}$

(ii) $\beta \in \mathscr{K}$, $\beta^* \in \mathscr{H}$

(iii) *a bounded linear operator*, $A: \mathscr{H} \to \mathscr{K}$

such that

$$C \operatorname{Exp} \eta = \gamma \exp\langle \beta^* | \eta \rangle \operatorname{Exp}(\beta + A\eta), \qquad \eta \in \mathscr{H}$$
$$C^* \operatorname{Exp} \zeta = \bar{\gamma} \exp\langle \beta | \zeta \rangle \operatorname{Exp}(\beta^* + A^* \zeta), \qquad \zeta \in \mathscr{K}.$$

Proof. As C and C^* are of type S there exist mappings

$$\eta \in \mathscr{H} \mapsto \gamma(\eta) \in \mathbb{C}, \qquad \zeta \in \mathscr{K} \mapsto \gamma^*(\zeta) \in \mathbb{C},$$
$$\eta \in \mathscr{H} \mapsto \Gamma(\eta) \in \mathscr{K} \quad \text{and} \quad \zeta \in \mathscr{K} \mapsto \Gamma^*(\zeta) \in \mathscr{H}$$

such that

$$C \operatorname{Exp} \eta = \gamma(\eta) \operatorname{Exp} \Gamma(\eta), \qquad \eta \in \mathscr{H}$$
$$C^* \operatorname{Exp} \zeta = \gamma^*(\zeta) \operatorname{Exp} \Gamma^*(\zeta), \qquad \zeta \in \mathscr{K}.$$

One has then for $\eta \in \mathscr{H}$, $\zeta \in \mathscr{K}$

$$\begin{aligned} \gamma(\eta) \exp\langle \zeta | \Gamma(\eta) \rangle &= \langle \operatorname{Exp} \zeta | C \operatorname{Exp} \eta \rangle \\ &= \langle C^* \operatorname{Exp} \zeta | \operatorname{Exp} \eta \rangle \\ &= \overline{\gamma^*(\zeta)} \exp\langle \Gamma^*(\zeta) | \eta \rangle. \end{aligned} \tag{5}$$

Putting in (5) successively $\zeta = 0$ and $\eta = 0$ it follows that

$$\begin{aligned} \gamma(\eta) &= \overline{\gamma^*(0)} \exp\langle \Gamma^*(0) | \eta \rangle, \qquad \eta \in \mathscr{H} \\ \gamma^*(\zeta) &= \overline{\gamma(0)} \exp\langle \Gamma(0) | \zeta \rangle, \qquad \zeta \in \mathscr{K}. \end{aligned} \tag{6}$$

If $\gamma(0) = 0$ then one has $C = C^* = 0$ and the proposition follows trivially we can therefore assume without loss of generality that $0 \neq \gamma(0) = \overline{\gamma^*(0)}$. Inserting (6) in (5) leads to

$$\exp(\langle \Gamma^*(0) | \eta \rangle + \langle \zeta | \Gamma(\eta) \rangle) = \operatorname{Exp}(\langle \zeta | \Gamma(0) \rangle + \langle \Gamma^*(\zeta) | \eta \rangle)$$

and so there exists a mapping

$$N: \mathscr{H} \times \mathscr{K} \to \mathbb{Z}: (\eta, \zeta) \mapsto N(\eta, \zeta)$$

such that

$$\langle \Gamma^*(0) | \eta \rangle + \langle \zeta | \Gamma(\eta) \rangle = \langle \zeta | \Gamma(0) \rangle + \langle \Gamma^*(\zeta) | \eta \rangle + 2\pi i \, N(\eta, \zeta). \tag{7}$$

Writing out (7) for $\eta = \eta_1 + \eta_2$ and subtracting the same expressions for $\eta = \eta_1$ and $\eta = \eta_2$ one obtains

$$\begin{aligned} &\langle \zeta | \Gamma(\eta_1 + \eta_2) - \Gamma(\eta_1) - \Gamma(\eta_2) + \Gamma(0) \rangle \\ &= 2\pi i (N(\eta_1 + \eta_2, \zeta) - N(\eta_1, \zeta) - N(\eta_2, \zeta)). \end{aligned} \tag{8}$$

As N takes values in $\mathbb{Z}$ this can only hold if

$$N(\eta_1 + \eta_2, \zeta) - N(\eta_1, \zeta) - N(\eta_2, \zeta) = 0.$$

Fix now $\eta\in\mathscr{H}$, $\zeta\in\mathscr{K}$ one has then for any $k\in\mathbb{N}$

$$N(\eta,\zeta)=2^{2^k}N(2^{-2^k}\eta,\zeta)$$

and therefore $N(\eta,\zeta)=0$.

From equation (8) it now follows that

$$\Gamma(\eta_1+\eta_2)-\Gamma(\eta_1)-\Gamma(\eta_2)+\Gamma(0)=0 \qquad \eta_1,\eta_2\in\mathscr{H}$$

or equivalently that

$$\eta\in\mathscr{H}\mapsto A\eta\equiv\Gamma(\eta)-\Gamma(0)\in\mathscr{K}$$

is a linear operator.

Finally using (7) one concludes that also

$$\zeta\in\mathscr{K}\mapsto B\zeta\equiv\Gamma^*(\zeta)+\Gamma^*(0)\in\mathscr{H}$$

is linear and that

$$\langle\zeta|A\eta\rangle=\langle B\zeta|\eta\rangle, \qquad \eta\in\mathscr{H}, \qquad \zeta\in\mathscr{K}$$
$$\langle\zeta|A\eta\rangle=\langle B\zeta|\eta\rangle, \qquad \eta\in\mathscr{H}, \qquad \zeta\in\mathscr{K}.$$

Therefore A is closed and, as it is everywhere defined, bounded. Moreover $B=A^*$.

Putting now

we find that $\gamma=\gamma(0)$, $\quad\beta=\Gamma(0)\quad$ and $\quad B^*=\Gamma^*(0)$

$$C\operatorname{Exp}\eta=\gamma\exp\langle\beta^*|\eta\rangle\operatorname{Exp}(A\eta+\beta), \qquad \eta\in\mathscr{H}$$
$$C^*\operatorname{Exp}\zeta=\bar{\gamma}\exp\langle\beta|\zeta\rangle\operatorname{Exp}(A^*\zeta+\beta^*), \qquad \zeta\in\mathscr{K}. \quad \square$$

Proposition II.3. *Let $C\colon\mathscr{S}(\mathscr{H})\to\mathscr{S}(\mathscr{K})$ and $C^*\colon\mathscr{S}(\mathscr{K})\to\mathscr{S}(\mathscr{H})$ be linear operators of type S. C is bounded if and only if*

$$C\operatorname{Exp}\eta=\gamma\exp\langle\beta^*|\eta\rangle\operatorname{Exp}(A\eta+\beta), \qquad \eta\in\mathscr{H}$$

where

(i) $\gamma\in\mathbb{C}$
(ii) $A\colon\mathscr{H}\to\mathscr{K}$ *is a contraction*
(iii) $\beta\in\mathscr{K}$, $\beta^*\in\mathscr{H}$ *are such that* $\beta^*+A^*\beta\in\operatorname{Ran}(\mathbb{1}-A^*A)^{1/2}$.

Proof. By Proposition II.2 there exist $\gamma\in\mathbb{C}$, $\beta\in\mathscr{K}$, $\beta^*\in\mathscr{H}$ and a bounded linear operator $A\colon\mathscr{H}\to\mathscr{K}$ such that

$$C\operatorname{Exp}\eta=\gamma\exp\langle\beta^*|\eta\rangle\operatorname{Exp}(\beta+A\eta), \qquad \eta\in\mathscr{H}$$
$$C^*\operatorname{Exp}\zeta=\bar{\gamma}\exp\langle\beta|\zeta\rangle\operatorname{Exp}(\beta^*+A^*\zeta), \qquad \zeta\in\mathscr{K}$$

(i) Suppose that C is bounded then for $\eta\in\mathscr{H}$

$$\begin{aligned}\|C\operatorname{Exp}\eta\|^2&=|\gamma|^2\exp\{\langle\beta^*|\eta\rangle+\langle\eta|\beta^*\rangle+\|A\eta+\beta\|^2\}\\ &\leqq\|C\|^2\|\operatorname{Exp}\eta\|^2=\|C\|^2\exp\|\eta\|^2\end{aligned}$$

and therefore there exists a constant M such that

$$\langle\eta|A^*A-\mathbb{1})\eta\rangle+2\operatorname{Re}\langle\eta|\beta^*+A^*\beta\rangle\leqq M, \qquad \eta\in\mathscr{H}. \tag{9}$$

Condition (9) is equivalent to

$$\begin{aligned} &\text{i)}\ \|A\| \leqq 1 \\ &\text{ii)}\ |\langle \eta | \beta^* + A^* \beta \rangle|^2 \leqq M \langle \eta | (\mathbb{1} - A^* A)\eta \rangle, \qquad \eta \in \mathscr{H} \end{aligned} \tag{10}$$

Let η now be an arbitrary element of $\operatorname{Ran}\{(\mathbb{1}-A^*A)^{1/2}|_{\mathscr{H}\ominus \operatorname{Ker}(\mathbb{1}-A^*A)}\}$ then by (10)

$$|\langle (\mathbb{1}-A^*A)^{-1/2}\eta | \beta^* + A^*\beta\rangle|^2 \leqq M\langle \eta|\eta\rangle$$

and so

$$\beta^* + A^*\beta \in \operatorname{Dom}\{(\mathbb{1}-A^*A)^{-1/2}|_{\mathscr{H}\ominus \operatorname{Ker}(\mathbb{1}-A^*A)}\}$$

or

$$\beta^* + A^*\beta \in \operatorname{Ran}(\mathbb{1}-A^*A)^{1/2}.$$

(ii) Suppose conversely that $\|A\| \leqq 1$ and that there exists a $\eta_0 \in \mathscr{H}$ such that

$$\beta^* + A^*\beta = (\mathbb{1}-A^*A)^{1/2}\eta_0.$$

Define for $0 \leqq \lambda \leqq 1$

$$C_\lambda \operatorname{Exp}\eta = \gamma \exp\langle \beta^* | \eta\rangle \operatorname{Exp}(\beta + \lambda A\eta), \qquad \eta \in \mathscr{H}.$$

By a simple computation

$$\begin{aligned} &C_\lambda^* C_\lambda \operatorname{Exp}\eta \\ &\quad = |\gamma|^2 \exp(\langle \beta^* + \lambda A^*\beta|\eta\rangle + \|\beta\|^2) \operatorname{Exp}(\beta^* + \lambda A^*\beta + \lambda^2 A^*A\eta), \qquad \eta \in \mathscr{H}. \end{aligned}$$

It is straightforward to check that for $\eta_1 \in \mathscr{H}$

$$U(\eta_1)\operatorname{Exp}\eta = \exp(-\tfrac{1}{2}\|\eta_1\|^2 - \langle \eta_1|\eta\rangle)\operatorname{Exp}(\eta_1 + \eta) \qquad \eta \in \mathscr{H}$$

extends to a unitary operator on $\mathscr{S}(\mathscr{H})$ with adjoint $U^*(\eta_1) = U(-\eta_1)$. As $\|A\| \leqq 1$, $\mathbb{1} - \lambda^2 A^*A$ has for $0 \leqq \lambda < 1$ a bounded inverse. Choosing now

$$\eta_1 = -(\mathbb{1} - \lambda^2 A^*A)^{-1}(\beta^* + \lambda A^*\beta)$$

one verifies that for $\eta \in \mathscr{H}$

$$\begin{aligned} &U(\eta_1) C_\lambda^* C_\lambda U^*(\eta_1) \operatorname{Exp}\eta \\ &\quad = |\gamma|^2 \exp(\|\beta\|^2 + \|(\mathbb{1}-\lambda^2 A^*A)^{-1/2}(\beta^* + \lambda A^*\beta)\|^2) \operatorname{Exp}\lambda^2 A^*A\eta. \end{aligned} \tag{11}$$

The operator

$$\operatorname{Exp}\eta \mapsto \operatorname{Exp}\lambda^2 A^*A\eta, \qquad \eta \in \mathscr{H}$$

is easily seen to be equal to

$$\bigoplus_{n=0}^{\infty} \Big(\bigotimes^{n} \lambda^2 A^*A\Big)\Big|_{\mathscr{H}_n}$$

(where $\mathscr{H}_n$ is the symmetric subspace of $\bigotimes^n \mathscr{H}$) and has therefore norm one. It follows then from (11) that

$$\|C_\lambda\|^2 = |\gamma|^2 \exp(\|\beta\|^2 + \|(1-\lambda^2 A^* A)^{-1/2}(\beta^* + \lambda A^* \beta)\|^2).$$

As for $\|A\| \leqq 1$ and $0 \leqq \lambda < 1$ one has

$$(1-\lambda)^2(\mathbb{1} - \lambda^2 A^* A)^{-1} \leqq \mathbb{1}$$

and

$$(\mathbb{1} - \lambda^2 A^* A)^{-1}(\mathbb{1} - A^* A) \leqq \mathbb{1}$$

it follows that

$$\begin{aligned}
&\|(\mathbb{1} - \lambda^2 A^* A)^{-1/2}(\beta^* + \lambda A^* \beta)\|^2 \\
&\quad = \|(\mathbb{1} - \lambda^2 A^* A)^{-1/2}((\mathbb{1} - A^* A)^{1/2} \eta_0 - (\mathbb{1} - \lambda) A^* \beta)\|^2 \\
&\quad \leqq 2\|(\mathbb{1} - \lambda^2 A^* A)^{-1/2}(\mathbb{1} - A^* A)^{1/2} \eta_0\|^2 + 2\|(1-\lambda)(\mathbb{1} - \lambda^2 A^* A)^{-1/2} A^* \beta\|^2 \\
&\quad \leqq 2\|\eta_0\|^2 + 2\|\beta\|^2.
\end{aligned}$$

Therefore

$$\|C_\lambda\|^2 \leqq |\gamma|^2 \exp(2\|\eta_0\|^2 + 3\|\beta\|^2). \tag{12}$$

For $\eta \in \mathscr{H}$, and $\zeta \in \mathscr{K}$, we have now

$$\lim_{\lambda \uparrow 1} \langle \operatorname{Exp} \zeta | C_\lambda \operatorname{Exp} \eta \rangle = \langle \operatorname{Exp} \zeta | C \operatorname{Exp} \eta \rangle$$

and as $\operatorname{Span}\{\operatorname{Exp} \eta | \eta \in \mathscr{H}\}$, $\operatorname{Span}\{\operatorname{Exp} \zeta | \zeta \in \mathscr{K}\}$ are dense in $\mathscr{S}(\mathscr{H})$ respectively $\mathscr{S}(\mathscr{K})$ we have using (12) that C is bounded. In fact

$$\|C\|^2 = |\gamma|^2 \exp(\|\beta\|^2 + \inf_{\beta^* + A^*\beta = (\mathbb{1} - A^* A)^{1/2} \eta_0} \|\eta_0\|^2) \quad \square$$

We now consider specific cases of type S mappings. If $\mathscr{H} = \mathscr{K}$ the set $\{(\gamma, \beta, \beta^*, A) | \gamma \in C, A \in \mathscr{B}(\mathscr{H}), \|A\| \leqq 1, \beta, \beta^* \in \mathscr{H}, \beta^* + A^* \beta \in \operatorname{Ran}(\mathbb{1} - A^* A)^{1/2}\}$, of type S operators generates a C^*-algebra determined by the following product and involution rules

$$\begin{aligned}
(\gamma, \beta, \beta^*, A)(\gamma', \beta', B^{*\prime}, A') &= (\gamma \gamma' \exp\langle \beta^* | \beta' \rangle, \beta + A\beta', \beta^{*\prime} + A'^* \beta^*, AA') \\
(\gamma, \beta, \beta^*, A)^* &= (\bar{\gamma}, \beta^*, \beta, A^*).
\end{aligned} \tag{13}$$

From this we immediately recover the type S representations of groups, which were considered by Guichardet [Gui], as mappings

$$\pi\colon \mathscr{G} \to \mathscr{U}(\mathscr{S}(\mathscr{H}))\colon\ g \mapsto (c_g, \xi_g, -U_{g^{-1}} \xi_g, U_g)$$

where

(i) $g \mapsto U_g$ is a unitary representation of $\mathscr{G}$ into $\mathscr{U}(\mathscr{H})$

(ii) $g \mapsto \xi_g \in \mathscr{H}$ satisfies the following cocycle relation

$$\xi_g \xi_{g'} = \xi_g + U_g \xi_{g'}. \tag{14}$$

(iii) $g \mapsto c_g \in \mathbb{C}$ satisfies

$$c_{gg'} = c_g c_{g'} \exp - \langle \xi_g | U_g \xi_{g'} \rangle.$$

Another class of type S mappings $\mathscr{S}(\mathscr{H}) \to \mathscr{S}(\mathscr{K})$ which will be of interest in the sequel are the type S isometries which are easily seen to be given by $(\gamma, \beta, -W^* \beta, W)$ where $W: \mathscr{H} \to \mathscr{K}$ is an isometry and $|\gamma|^2 \exp \|\beta\|^2 = 1$.

II.2. Some Preliminary Lemmas

It is well known [Pa] that continuous infinitely divisible positive definite functions on a connected group never vanish. We now generalise this property to C.P. triplets. Using the notation introduced just before Theorem I.9 we have the following lemma.

Lemma II.4. *Let $(\mathscr{H}, \Phi, \Omega)$ be a continuous infinitely divisible* C.P. *triplet on a connected group $\mathscr{G}$. For all $\Delta \in \tilde{\mathscr{G}}$ we have that*

$$|\langle \Omega | \Phi(\Delta) \Omega \rangle| > 0.$$

Proof. The proof uses an induction argument on the length $\#(\Delta)$ of the n-tuple $\Delta \in \tilde{\mathscr{G}}$.

(i) For $\#(\Delta) = 1$ the situation reduces to that of infinitely divisible positive definite functions. The proof can be found in [Pa].

(ii) Suppose that the result holds for n-tuples of length k then we show that it is also valid for n-tuples of length $k+1$. Let $\Delta \in \tilde{\mathscr{G}}$ be such an n-tuple and write $\Delta = \{g\} \times \Delta'$ where $g \in \mathscr{G}$ and Δ' is an n-tuple of length k.

Let $(\mathscr{K}^{1/n}, \pi^{1/n}, V^{1/n})$ be the Stinespring triplet associated with the n^{th}-root $(\mathscr{H}^{1/n}, \Phi^{1/n}, \Omega^{1/n})$ of $(\mathscr{H}, \Phi, \Omega)$. Choose $\lambda, \mu \in \mathbb{C}$ such that $|\lambda| = |\mu| = 1$ then by the triangle inequality:

$$\begin{aligned} &\|\lambda V^{1/n} \Omega^{1/n} - \mu \pi^{1/n}(g) V^{1/n} \Phi^{1/n}(\Delta') \Omega^{1/n}\|^2 \leq \\ &\quad 2 \|\lambda V^{1/n} \Omega^{1/n} - \pi^{1/n}(g) V^{1/n} \Omega^{1/n}\|^2 \\ &\quad + 2 \|\pi^{1/n}(g) V^{1/n} \Omega^{1/n} - \mu \pi^{1/n}(g) V^{1/n} \Phi^{1/n}(\Delta') \Omega^{1/n}\|^2. \end{aligned} \tag{15}$$

We now compute the different terms which appear in (15):

$$\begin{aligned} &\|\lambda V^{1/n} \Omega^{1/n} - \mu \pi^{1/n}(g) V^{1/n} \Phi^{1/n}(\Delta') \Omega^{1/n}\|^2 \\ &\quad = 1 + \|\Phi^{1/n}(\Delta') \Omega^{1/n}\|^2 - 2 \operatorname{Re} \bar{\lambda} \mu \langle \Omega^{1/n} | \Phi^{1/n}(\Delta) \Omega^{1/n} \rangle \end{aligned} \tag{16}$$

$$\|\lambda V^{1/n} \Omega^{1/n} - \pi^{1/n}(g) V^{1/n} \Omega^{1/n}\|^2 = 2 - 2 \operatorname{Re} \bar{\lambda} \langle \Omega^{1/n} | \Phi^{1/n}(g) \Omega^{1/n} \rangle \tag{17}$$

$$\begin{aligned} &\|\pi^{1/n}(g) V^{1/n} \Omega^{1/n} - \mu \pi^{1/n}(g) V^{1/n} \Phi^{1/n}(\Delta') \Omega^{1/n}\|^2 \\ &\quad = 1 + \|\Phi^{1/n}(\Delta') \Omega^{1/n}\|^2 - 2 \operatorname{Re} \mu \langle \Omega^{1/n} | \Phi^{1/n}(\Delta') \Omega^{1/n} \rangle. \end{aligned} \tag{18}$$

By choosing appropriate phases for λ and μ and inserting (16), (17), (18) in (15) we then obtain the following inequality:

$$1-|\langle\Omega^{1/n}|\Phi^{1/n}(\Delta)\,\Omega^{1/n}\rangle|\leqq 2(1-|\langle\Omega^{1/n}|\Phi^{1/n}(g)\,\Omega^{1/n}\rangle|) \\ +2(1-|\langle\Omega^{1/n}|\Phi^{1/n}(\Delta')\,\Omega^{1/n}\rangle|). \tag{19}$$

As $(\mathscr{H}^{1/n}, \Phi^{1/n}, \Omega^{1/n})$ is a n^{th}-root of $(\mathscr{H}, \Phi, \Omega)$ we have for any $X\in\mathscr{G}$

$$|\langle\Omega^{1/n}|\Phi^{1/n}(X)\,\Omega^{1/n}\rangle|=|\langle\Omega|\Phi(X)\,\Omega\rangle|^{1/n}. \tag{20}$$

Multiplying (19) by n, using (20) and taking the limit $n\to\infty$ we then obtain:

$$-\ln|\langle\Omega|\Phi(\Delta)\,\Omega\rangle|\leqq -2\ln|\langle\Omega|\Phi(g)\,\Omega\rangle|-2\ln|\langle\Omega|\Phi(\Delta')\,\Omega\rangle| \\ <\infty$$

by the induction hypothesis.

Hence

$$|\langle\Omega|\Phi(\Delta)\,\Omega\rangle|>0. \quad \square$$

Lemma II.5. *Let $f:X\to\mathbb{C}$ be a continuous function on an arcwise connected topological space X such that*

$$f(x)\neq 0 \quad \textit{for all } x\in X \quad \textit{and} \quad f(x_0)=1 \quad \textit{for some } x_0\in X. \tag{21}$$

Then for each $n\in\mathbb{N}_0$ there exists at most one continuous function f_n: $X\to\mathbb{C}$ such that $(f_n)^n=f$ and $f_n(x_0)=1$.

Furthermore, if for all $n\in\mathbb{N}_0$ such a function f_n with the properties mentioned above exists, then there exists a unique continuous function v: $X\to\mathbb{C}$ such that

$$f=\exp v, \qquad v(x_0)=0.$$

Moreover for all $x\in X$

$$v(x)=\lim_{n\to\infty} n(f_n(x)-1)$$

and

$$f_n(x)=\exp\frac{v(x)}{n}. \tag{22}$$

Proof. First remark that a continuous function f: $X\to\mathbb{C}$ satisfying (21) admits at most one continuous n^{th}-root f_n with $f_n(x_0)=1$, $n\in\mathbb{N}_0$.

Indeed suppose that both f_n and f_n' are continuous n^{th}-roots of f with $f_n(x_0)=f_n'(x_0)=1$. Define $\theta_n^{(\prime)}(x)$ by

$$f_n^{(\prime)}(x)=|f_n^{(\prime)}(x)|\exp i\theta_n^{(\prime)}(x).$$

As $(f_n)^n=(f_n')^n=f$ and $|f(x)|>0$ we have that $\exp in\,(\theta_n(x)-\theta_n'(x))=1$ and therefore there exists a $k(x)\in\mathbb{Z}$ such that $\theta_n(x)-\theta_n'(x)=\dfrac{2\pi k(x)}{n}$. But the function

$x \mapsto f_n(x)/f_n'(x) = \exp i(\theta_n(x) - \theta_n'(x)) = \exp \dfrac{2\pi i k(x)}{n}$ is continuous on a connected space and takes the value 1 in x_0. Therefore $k(x) \in n\mathbb{Z}$, so $f_n = f_n'$

To show the lemma fix now $x \in X$ and let τ: $[0,1] \to X$ be a path in X connecting x_0 and x. Then $t \mapsto \tilde{\tau}(t) = f(\tau(t))/[f(\tau(t))|$ is a path in the 1-dimensional torus with $\tilde{\tau}(0) = 1$ and hence by the Covering Path Property there exists a unique path ϕ: $[0,1] \to \mathbb{R}$ with $\phi(0) = 0$ such that $\tau = \exp i\phi$. Put now $\theta(x) = \phi(1)$. Then $f(x) = |f(x)| \exp i\theta(x)$. We have to show that $\theta(x)$ is independent on the choice of the path τ. Consider therefore another path τ' connecting x_0 and x and let ϕ' be the corresponding path in $\mathbb{R}$ and $\theta'(x)$ the corresponding number. As $f(x) = |f(x)| \exp i\theta(x) = |f(x)| \exp i\theta'(x)$ there exists a $k \in \mathbb{Z}$ such that

$$\theta'(x) = \theta(x) + 2\pi k. \tag{23}$$

The functions $t \mapsto |f_{|k|+1}(\tau(t))| \exp i\phi(t)/(|k|+1))$ and $t \mapsto f_{(k)+1}(\tau(t))$ are both continuous $(|k|+1)^{\text{th}}$ roots of $t \mapsto f(\tau(t))$ taking the value 1 in $t = 0$. Hence by the remark in the beginning of this proof, we have that $f_{|k|+1}(\tau(t)) = |f_{|k|+1}(\tau(t))|$ $\exp i\phi(t)/(|k|+1)$ and of course the same relation holds for τ' and ϕ'. Taking $t = 1$ and using (23) we then find

$$\begin{aligned} \exp i\theta(x)/(|k|+1) &= \exp i\theta'(x)/(|k|+1) \\ &= \exp[i\theta(x)/(|k|+1) + 2\pi i k/(|k|+1)] \end{aligned}$$

which implies $k = 0$ and therefore $\theta(x) = \theta'(x)$.

Moreover, it can easily be seen that $x \mapsto \theta(x)$ is continuous.

Summarizing, we have now shown that there exists a unique continuous function $x \mapsto \theta(x)$ such that

$$\begin{aligned} &\theta(x_0) = 0 \\ &f(x) = |f(x)| \exp i\theta(x) \\ &f_n(x) = |f_n(x)| \exp i\frac{\theta(x)}{n}. \end{aligned}$$

Now it follows immediately that $x \mapsto v(x) = \ln|f(x)| + i\theta(x)$ is the unique continuous logarithm for f with $v(x_0) = 0$. Clearly also (22) holds. □

Applying the first part of Lemma II.5. to C.P. triplets we find:

Corollary II.6. *Let* $(\mathscr{H}, \Phi, \Omega)$ *be a continuous* C.P. *triplet on an arcwise connected group* $\mathscr{G}$. *Then* $(\mathscr{H}, \Phi, \Omega)$ *has, up to unitary equivalence, at most one continuous* n^{th} *root.*

Proof. Suppose $(\mathscr{H}_1, \Phi_1, \Omega_1)$ and $(\mathscr{H}_2, \Phi_2, \Omega_2)$ are two continuous n^{th} roots for $(\mathscr{H}, \Phi, \Omega)$. Then we have to show that

$$\langle \Omega_1 | \Phi_1(\Delta) \Omega_1 \rangle = \langle \Omega_2 | \Phi_2(\Delta) \Omega_2 \rangle$$

for all $\Delta \in \tilde{\mathscr{G}}$.

Take $\Delta \in \tilde{\mathscr{G}}$ with $\Delta \neq \phi$ and put $k = \# \Delta$. Let $X = \mathscr{G} \times \mathscr{G} \times \ldots \times \mathscr{G}$ (k times) and $x_0 = (e, e, \ldots, e)$. When X is equipped with the product topology of $\mathscr{G}$, it becomes an arcwise connected space. Define $f\colon X \to \mathbb{C}\colon \Delta \to \langle \Omega | \Phi(\Delta)\, \Omega \rangle$ and $f_j\colon X \to \mathbb{C}\colon \Delta \to \langle \Omega_j | \Phi_j(\Delta)\, \Omega_j \rangle (j=1,2)$.

Then f, f_1 and f_2 are continuous, $f(x_0)=f_1(x_0)=f_2(x_0)=1$, and $f=(f_1)^n =(f_2)^n$. From Lemma II.5. it now follows that $f_1=f_2$ which proves the result. □

Definition II.7. A continuous infinitely divisible C.P. triplet $(\mathscr{H}, \Phi, \Omega)$ on an arcwise connected group $\mathscr{G}$ is said to satisfy *condition C* if it has continuous n^{th} roots.

We have seen that the continuous n^{th} roots of a continuous infinitely divisible C.P. triplet on an arcwise connected group are necessarily unique (up to unitary equivalence). Moreover if the group, on which the triplet is defined, is continuously divisible, then condition C is always satisfied (cf. Theorem I.9.).

As an immediate consequence of the second part of Lemma II.5., we now have:

Corollary II.8. *Let $(\mathscr{H}, \Phi, \Omega)$ be a continuous infinitely divisible* C.P. *triplet on an arcwise connected group $\mathscr{G}$. If $(\mathscr{H}, \Phi, \Omega)$ satisfies condition C, then there exists a unique continuous function d*

$$d\colon \tilde{\mathscr{G}} \to \mathbb{C}\colon \quad \Delta \mapsto d_\Delta$$

such that

(i) $\langle \Omega | \Phi(\Delta)\, \Omega \rangle = \exp d_\Delta$

(ii) $d_{\Delta_e}=0$ *where Δ_e is any n-tuple of the form $(e, e, \ldots, e)$* (24)

Furthermore,

$$\langle \Omega^{1/n} | \Phi^{1/n}(\Delta)\, \Omega^{1/n} \rangle = \exp d_\Delta / n. \tag{25}$$

Using this function d we can now construct two positive kernels.

Lemma II.9. *Let $(\mathscr{H}, \Phi, \Omega)$ be an infinitely divisible* C.P. *triplet satisfying condition C on an arcwise connected group $\mathscr{G}$ ad let $d\colon \tilde{\mathscr{G}} \to \mathbb{C}$ be as above, then*

(i) $k_1(\Delta, \Delta') = d_{\Delta^* \times \Delta'} - d_{\Delta^*} - d_{\Delta'}$ *is a positive kernel on $\tilde{\mathscr{G}}$.* (26)

(ii) $k_2((g, \Delta), (g', \Delta')) = d_{\Delta^* \times (g^{-1} g') \times \Delta'} - d_{\Delta^* \times (g^{-1})} - d_{(g') \times \Delta'}$ (27)

is a positive kernel on $\mathscr{G} \times \tilde{\mathscr{G}}$.

Proof. (i) will immediately follow from (ii) by putting $g=g'=e$ in (27) and observing that $\Phi(e)=\mathbb{1}$.

(ii) Let $(\mathscr{H}^{1/n}, \Phi^{1/n}, \Omega^{1/n})$ be the n^{th} root of $(\mathscr{H}, \Phi, \Omega)$ and take

$$\chi_n = \sum_{g, \Delta} \lambda_{g, \Delta} (\Phi^{1/n}((g) \times \Delta)\, \Omega^{1/n} - \Omega^{1/n}) \in \mathscr{H}^{1/n}.$$

Then,

$$\begin{aligned} 0 \leqq n \|\chi_n\|^2 = n \sum_{\substack{g, \Delta \\ g', \Delta'}} \bar{\lambda}_{g, \Delta} \lambda_{g', \Delta'} [&\langle \Omega^{1/n} | \Phi^{1/n}(\Delta^* \times (g^{-1}) \times (g) \times \Delta')\, \Omega^{1/n} \rangle \\ &+ 1 - \langle \Omega^{1/n} | \Phi^{1/n}(\Delta^* \times (g^{-1}))\, \Omega^{1/n} \rangle - \langle \Omega^{1/n} | \Phi^{1/n}((g') \times \Delta')\, \Omega^{1/n} \rangle]. \end{aligned}$$

By the 2-positivity inequality (2) the first term can be majorized to get

$$0 \leqq n \sum_{\substack{g,\Delta \\ g',\Delta'}} \bar{\lambda}_{g,\Delta} \lambda_{g',\Delta'} [(\langle \Omega^{1/n} | \Phi^{1/n}(\Delta^* \times (g^{-1} g') \times \Delta') \Omega^{1/n} \rangle - 1)$$
$$- (\langle \Omega^{1/n} | \Phi^{1/n}(\Delta^* \times (g^{-1})) \Omega^{1/n} \rangle - 1)$$
$$- (\langle \Omega^{1/n} | \Phi^{1/n}((g') \times \Delta') \Omega^{1/n} \rangle - 1)].$$

Taking now the limit $n \to \infty$ and using (25) we get

$$\sum_{\substack{g,\Delta \\ g',\Delta'}} \bar{\lambda}_{g,\Delta} \lambda_{g',\Delta'} (d_{\Delta^* \times (g^{-1} g') \times \Delta} - d_{\Delta^* \times (g^{-1})} - d_{(g') \times \Delta}) \geqq 0. \quad \square$$

Given a positive kernel $(x, y) \in X \times X \mapsto k(x, y) \in \mathbb{C}$ on a set X one constructs in a standard way a Hilbert space. The kernel k extends to a positive sesquilinear form on the complex free vector space $V(X)$ generated by X: Let $V_0 = \{u \in V(X) | k(u, u) = 0\}$. By $\mathrm{hil}(X, k)$ we denote the completion of $V(X)/V_0$ for the scalar product induced by k. By abuse of notation we will denote the elements of $V(X)/V_0$ by u instead of $u + V_0$. Also the scalar product in $\mathrm{hil}(X, k)$ will be denoted in the conventional way $(\langle \cdot | \cdot \rangle)$.

II.3. Structure Theorem for Infinitely Divisible C.P. Triplets

We are now in a position to describe the general structure of infinitely divisible C.P. triplets and their Stinespring decomposition. More precisely we will show that any such mapping, as well as its Stinespring decomposition, extends to a mapping of type S.

Lemma II.10. *Let $(\mathscr{H}, \Phi, \Omega)$ be a continuous infinitely divisible* C.P. *triplet satisfying condition C on an arcwise connected group $\mathscr{G}$. Denote by (K, π, V) its Stinespring triplet. Let $H_1 = \mathrm{hil}(\tilde{\mathscr{G}}, k_1)$ and $H_2 = \mathrm{hil}(\mathscr{G} \times \tilde{\mathscr{G}}, k_2)$ where k_1 and k_2 are defined from $(\mathscr{H}, \Phi, \Omega)$ by* (26) *and* (27).

Then

(i) $V_1: \mathscr{H} \to \mathscr{S}(H_1)$: $\Phi(\Delta)\, \Omega \mapsto \langle \Omega | \Phi \Delta \Omega \rangle\, \mathrm{Exp}\, \Delta$ *extends to an isometry.*

(ii) V_2: $\mathscr{K} \to \mathscr{S}(H_2)$: $V\Phi(\Delta)\ \Omega \mapsto \langle \Omega | \Phi(\Delta)\ \Omega \rangle\ \mathrm{Exp}(e, \Delta)$ *extends to an isometry.*

(iii) W: $H_1 \to H_2$: $\Delta \mapsto (e, \Delta)$ *extends to an isometry. Its adjoint is given by*

$$W^*:\ H_2 \to H_1:\ (g, \Delta) \mapsto (g) \times \Delta$$

(iv) *The mapping* $U: \mathscr{G} \to \mathscr{U}(H_2)$: $g \mapsto U_g$ *with*

$$U_g(h, \Delta) = (gh, \Delta) - (g, \phi) \tag{28}$$

is a continuous unitary representation of $\mathscr{G}$ on H_2.

(v) $\tilde{\pi}: \mathscr{G} \to \mathscr{U}(\mathscr{S}(H_2))$: $g \mapsto \tilde{\pi}$ *with*

$$\tilde{\pi}_g\, \mathrm{Exp}\, \eta = \langle \Omega | \Phi(g)\, \Omega \rangle \exp \langle (g^{-1}, \phi) | \eta \rangle\, \mathrm{Exp}(U_g \eta + (g, \phi)), \qquad \eta \in H_2$$

defines a continuous unitary representation of type S of $\mathscr{G}$ *on* $\mathscr{S}(H_2)$. *In terms of the quadruplet notation for type S mapping we introduced in* (13) *we have*

$$\tilde{\pi}_g = (\langle \Omega | \Phi(g) \Omega \rangle, (g, \phi), (g^{-1}, \phi), U_g) \tag{29}$$

(vi) *The mapping* $A: \mathscr{G} \to \mathscr{B}(H_1): g \mapsto A_g$ *given by*

$$A_g \Delta = (g) \times \Delta - (g)$$

is continuous, normalised and C.P.

(vii) $\tilde{\Phi}: \mathscr{G} \to \mathscr{B}(\mathscr{S}(H_1)): g \mapsto \tilde{\Phi}(g)$ *where*

$$\tilde{\Phi}(g) \operatorname{Exp} \zeta = \langle \Omega | \Phi(g) \Omega \rangle \exp \langle (g^{-1}) | \zeta \rangle \operatorname{Exp}(A_g \zeta + (g)), \qquad \zeta \in H_1$$

is a continuous, normalised C.P. *mapping of types S. In terms of the quadruplet notation we have*

$$\tilde{\Phi}(g) = (\langle \Omega | \Phi(g) \Omega \rangle, (g), (g^{-1}), A_g).$$

(viii) *The objects above are related by the following equations*

$$A_g = W^* U_g W \tag{30}$$

$$\tilde{\Phi}(g) = \operatorname{Exp} W^* \tilde{\pi}_g \operatorname{Exp} W \quad \textit{where} \ \operatorname{Exp} W = (1, 0, 0, W) \tag{31}$$

$$V = V_2^* \operatorname{Exp} W V_1 \tag{32}$$

$$\pi(g) = V_2^* \tilde{\pi}_g V_2 \tag{33}$$

$$\Phi(g) = V_1^* \tilde{\Phi}(g) V_1 . \tag{34}$$

Proof. (i), (ii) and (iii) follow from (24), (26) and (27) and the observation that the normalisation of Φ implies that $d_{\Delta \times (e) \times \Delta'} = d_{\Delta \times \Delta}$, for all $\Delta, \Delta' \in \tilde{\mathscr{G}}$. For instance (ii) is shown as follows: for all $\Delta, \Delta' \in \tilde{\mathscr{G}}$ we have that

$$\begin{aligned}
&\langle \langle \Omega | \Phi(\Delta) \Omega \rangle \operatorname{Exp}(e, \Delta) | \langle \Omega | \Phi(\Delta') \Omega \rangle \operatorname{Exp}(e, \Delta') \rangle \\
&= \langle \Omega | \Phi(\Delta^*) \Omega \rangle \langle \Omega | \Phi(\Delta') \Omega \rangle \exp \langle (e, \Delta) | (e, \Delta') \rangle \\
&= \langle \Omega | \Phi(\Delta^*) \Omega \rangle \langle \Omega | \Phi(\Delta') \Omega \rangle \exp(d_{\Delta^* \times (e) \times \Delta'} - d_{\Delta^* \times (e)} - d_{(e) \times \Delta'}) \\
&= \langle \Phi(\Delta) \Omega | \Phi(\Delta') \Omega \rangle = \langle V \Phi(\Delta) \Omega | V \Phi(\Delta') \Omega \rangle.
\end{aligned}$$

(iv) Using (27) one checks that for all $g, h, h' \in \mathscr{G}$ and $\Delta, \Delta' \in \tilde{\mathscr{G}}$

$$\langle (gh, \Delta) - (g, \phi) | (gh', \Delta') - (g, \phi) \rangle = \langle (h, \Delta) | (h', \Delta') \rangle.$$

Hence U_g is well defined by (28) and isometric. Also $U_g U_{g'} = U_{gg'}$ and as $U_e = \mathbb{1}$, U is a unitary representation. The continuity of U follows immediately from the continuity of d

(v) One verifies straightforwardly that the quadruplet (29) satisfies condition (14).

(viii) For $\Delta \in \tilde{\mathscr{G}}$ we have that

$$\begin{aligned}
W^* U_g W \Delta = W^* U_g(e, \Delta) &= W^*((g, \Delta) - (g, \phi)) \\
&= (g) \times \Delta - (g),
\end{aligned}$$

which proves (30) and by Theorem 1.3 as well statement (vi). One derives (31) and (vii) easily from

$$\begin{aligned}
&\operatorname{Exp} W^* \pi_g \operatorname{Exp} W \operatorname{Exp} \zeta \\
&\quad = \operatorname{Exp} W^* \pi_g \operatorname{Exp} W\zeta \\
&\quad = \operatorname{Exp} W^* \langle \Omega | \Phi(g)\, \Omega\rangle \exp\langle (g^{-1}, \phi) | W\zeta\rangle \operatorname{Exp}(U_g W\zeta + (g, \phi)) \\
&\quad = \langle \Omega | \Phi(g)\, \Omega\rangle \exp\langle W^*(g^{-1}, \phi) | \zeta\rangle \operatorname{Exp}(W^* U_g W\zeta + W^*(g, \phi)) \\
&\quad = \langle \Omega | \Phi(g)\, \Omega\rangle \exp\langle (g^{-1}) | \zeta\rangle \operatorname{Exp}(A_g \zeta + (g)).
\end{aligned}$$

The statements (32), (33) and (34) are proven by analogous calculations. □

The preceding results can be summarised in the following theorem.

Theorem II.11. *Let $(\mathscr{H}, \Phi, \Omega)$ be a continuous infinitely divisible* C.P. *triplet satisfying condition C on an arcwise connected group $\mathscr{G}$. Denote by $(\mathscr{K}, \pi, V)$ its Stinespring triplet.*

Then there exist

- *symmetric Hilbert spaces $\mathscr{S}(H_1)$ and $\mathscr{S}(H_2)$,*
- *an isometry $W: H_1 \to H_2$*
- *a continuous* C.P. *mapping of type S: $\tilde{\Phi}: \mathscr{G} \to \mathscr{B}(\mathscr{S}(H_1))$*
- *a continuous unitary representation of type S:*

$$\tilde{\pi}: \mathscr{G} \to \mathscr{U}(\mathscr{S}(H_2))$$

such that

- *$(\mathscr{H}, \Phi, \Omega)$ is unitarily equivalent to $(\mathscr{S}(H_1)_c, \tilde{\Phi}_c, \operatorname{Exp} 0)$ where*

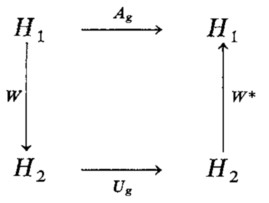
$$\begin{aligned}
&\mathscr{S}(H_1)_c = [\{\tilde{\Phi}(\mathscr{G})\}'' \operatorname{Exp} 0] \\
&\tilde{\Phi}_c(g) = \tilde{\Phi}(g)|_{\mathscr{S}(H_1)_c}.
\end{aligned} \tag{35}$$

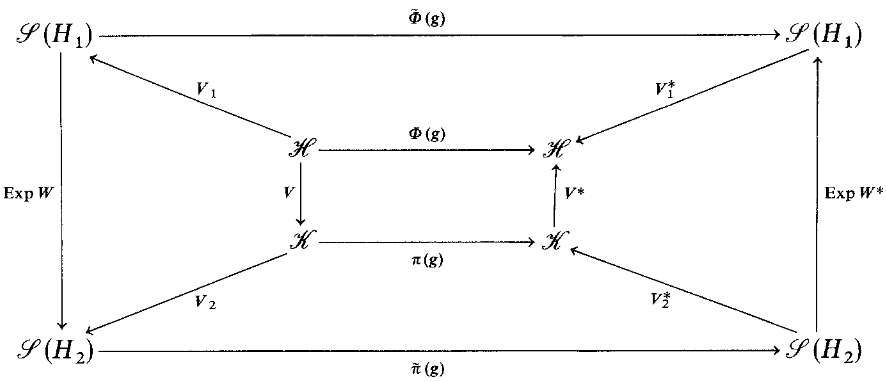
- *$(\mathscr{K}, \pi, V)$ is unitarily equivalent to $(\mathscr{S}(H_2)_c, \tilde{\pi}_c, (\operatorname{Exp} W)_c)$ where*

$$\begin{aligned}
&\mathscr{S}(H_2)_c = [\{\tilde{\pi}(\mathscr{G})\}'' \operatorname{Exp} W \mathscr{S}(H_1)_c] \\
&\tilde{\pi}_c(g) = \tilde{\pi}(g)|_{\mathscr{S}(H_2)_c} \\
&(\operatorname{Exp} W)_c = \operatorname{Exp} W|_{\mathscr{S}(H_1)_c}.
\end{aligned} \tag{36}$$

Moreover the minimal objects $(\mathscr{S}(H_1), \tilde{\Phi}, \operatorname{Exp} 0)$ and $(\mathscr{S}(H_2), \tilde{\pi}, \operatorname{Exp} W)$ which satisfy (35) *and* (36), *are unique up to unitary equivalence. We call them the canonical forms of the triplet and the Stinespring decomposition. Their explicit construction is given in the previous lemma.*

The various relations between the objects introduced in Lemma II.10 and Theorem II.11 can be summarised in the following diagrams.





II.4. Example

Here we will illustrate Theorem II.11 for the infinitely divisible triplet ($\mathscr{S}(H_R)$, $\Phi_{R,Q,c}$, Exp 0) introduced in Sect. I.5.

One easily computes that

$$d_\Delta = ic \sum_{j=1}^{k} \theta_j - \frac{1}{2} \sum_{j=1}^{k} \langle \xi_j | (R^*R+Q)\, \xi_j \rangle - \sum_{1 \leq i < j \leq k} \langle R\xi_i | R\xi_j \rangle \tag{37}$$

for $\Delta = ((\xi_1, \theta_1), \ldots, (\xi_k, \theta_k)) \in \tilde{\mathscr{G}}$.

Now it follows immediately that the triplet is already in its canonical form. Indeed, one has

$$k_1(\Delta, \Delta') = \langle R \sum_{i=1}^{k} \xi_i | R \sum_{j=1}^{k'} \xi'_j \rangle.$$

Hence, using the notation of Theorem II.11 and Sect. I.5, we have $H_1 = H_R$ and $\tilde{\Phi} = \Phi_{R,Q,c}$.

We now compute the canonical form of the Stinespring decomposition. First observe that

$$k_2((g, \Delta), (g', \Delta')) = \langle S_1 \xi | S_1 \xi' \rangle + \langle S_2 \xi' | S_2 \xi \rangle + \left\langle R\left(\xi + \sum_{i=1}^{k} \xi_i\right) \middle| R\left(\xi' + \sum_{j=1}^{k'} \xi_j\right) \right\rangle,$$

where $g = (\xi, \theta) \in \mathscr{G}$, Δ and Δ' are as above, and S_1 (resp. S_2) is the positive square root of $Q + c - R^*R$ (resp. $Q - c + R^*R$). Hence H_2 can be identified as follows. Let H_{S_1} (resp. H_{S_2}) be the closure of Ran S_1 (resp. Ran S_2). We denote by $\bar{H}_{S_2}$ the conjugate Hilbertspace of H_{S_2} i.e., $\bar{H}_{S_2}$ is as a set the same set as H_{S_2}, the identification mapping bein denoted by $\eta \in H_{S_2} \mapsto \bar{\eta} \in \bar{H}_{S_2}$. The addition in $\bar{H}_{S_2}$ is given by $\bar{\eta}_1 + \bar{\eta}_2 = \overline{\eta_1 + \eta_2}$, the scalar multiplication by $\lambda \bar{\eta} = \overline{\bar{\lambda}\eta}$ and the inner product by $\langle \bar{\eta}_1 | \bar{\eta}_2 \rangle = \langle \eta_2 | \eta_1 \rangle$. Then H_2 is the subspace of $H_{S_1} \otimes \bar{H}_{S_2} \otimes H_R$ generated by $\{(S_1 \xi, \overline{S_2 \xi}, R\xi') | \xi, \xi' \in H\}$.

Furthermore, the representation $\tilde{\pi}$ of the group H_σ on $\mathscr{S}(H_2)$ reads as

$$\begin{aligned} &\tilde{\pi}(\xi, \theta)\, \mathrm{Exp}(\eta_1, \bar{\eta}_2, \eta_3) \\ &\quad = \exp[ic\theta - \tfrac{1}{2}\langle \xi | (R^* R + Q)\, \xi \rangle - \langle (S_1 \xi, \overline{S_2 \xi}, R\xi) | (\eta_1, \bar{\eta}_2, \eta_3) \rangle] \\ &\quad \cdot \mathrm{Exp}(\eta_1 + S_1 \xi, \bar{\eta}_2 + \overline{S_2 \xi}, \eta_3 + R\xi). \end{aligned}$$

Moreover the isometry V: $\mathscr{S}(H_1) \to \mathscr{S}(H_2)$ maps $\mathrm{Exp}\, \eta$ onto $\mathrm{Exp}(0, 0, \eta)$. One may check that indeed $\tilde{\Phi}(\cdot) = V^* \tilde{\pi}(\cdot) V$.

III. Logarithms of Infinitely Divisible Completely Positive Triplets

In this section we will show how we can construct a logarithm for an infinitely divisible completely positive triplet. We hereby generalise the well known result that an infinitely divisible positive definite function is the exponential of a conditionally positive definite function. Due to the noncommutativity arising from consideration of general mappings instead of functions, the logarithm will not only be conditionally completely positive (in a suitable sense) but also enjoy properties which remained hidden in the case of functions where they trivialise.

In a first subsection we will list the definitions of the objects and their properties which will naturally arise in the construction of the logarithm. In the second we will characterise the infinitely divisible triplets as those admitting a suitable logarithm. Finally in the last section we will make the explicit construction of the logarithm for an example.

III.1. Conditionally Completely Positive Triplets

Definition III.1. (i) A mapping Ψ: $\mathscr{G} \to \mathscr{L}_D(\mathscr{K})$ (=set of possibly unbounded linear operators on some Hilbertspace $\mathscr{K}$ having some common dense domain $D \subset \mathscr{K}$) is said to be *conditionally completely positive* if

$$\sum_{g, g' \in \mathscr{G}} \langle \xi_g | \Psi(g^{-1} g')\, \xi_{g'} \rangle \geqq 0$$

for all choices of almost zero functions $g \mapsto \xi_g \in D$ such that $\sum_{g \in \mathscr{G}} \xi_g = 0$.

(ii) We call $(\mathscr{K}, \Psi, \Omega^0)$ *a conditionally completely positive triplet* on a group $\mathscr{G}$ *if* Ψ: $\mathscr{G} \to \mathscr{L}(\mathscr{K})$ is a mapping from $\mathscr{G}$ into the (possibly unbounded) linear operators on a hilbertspace $\mathscr{K}$ and $\Omega^0 \in \mathscr{K}$ is a normalized vector such that:
- $\Omega^0 \in \mathrm{Dom}\, \Psi(g)$ and $\Psi(\Delta)\, \Omega^0 \in \mathrm{Dom}\, \Psi(g)$ for all $g \in \mathscr{G}$, $\Delta \in \tilde{\mathscr{G}}$
- $D = \mathrm{span}\{\Psi(\Delta)\, \Omega^0 | \Delta \in \tilde{\mathscr{G}}\}$ is dense in $\mathscr{K}$
- Ψ: $\mathscr{G} \to \mathscr{L}_D(\mathscr{K})$ is conditionally completely positive in the sense of (i)

– Ψ is normalised in the sense that $\Psi(e)=0$
– $D\subset \mathrm{Dom}\,\Psi(g)^*$ for all $g\in\mathscr{G}$.

(iii) A conditionally C.P. triplet $(\mathscr{K},\Psi,\Omega^0)$ is *continuous* if $g\mapsto\langle\xi|\Psi(g)\eta\rangle$ is continuous for all $\xi,\eta\in D$.

(iv) A conditionally C.P. triplet $(\mathscr{K},\Psi,\Omega^0)$ is called *hermitian* if $\Psi(g^{-1})\subset\Psi(g)^*$.

Remark that the case of conditionally positive definite functions is covered by the notion of conditionally C.P. triplet by taking $\mathscr{K}=\mathbb{C}$ and $\Omega^0=1\in\mathbb{C}$.

Definition III.2. A conditionally C.P. triplet $(\mathscr{K},\Psi,\Omega^0)$ on $\mathscr{G}$ is called *infinitely additive* if for all $n\in\mathbb{N}_0$, there exists a conditionally C.P. triplet $(\mathscr{K}_n,\Psi_n,\Omega_n^0)$ on $\mathscr{G}$ and an isometry $U_n\colon\mathscr{K}\to\otimes^n\mathscr{K}_n$ such that

$$-U_nD\subset\otimes^nD_n\qquad\text{where } D_n=\mathrm{span}\{\Psi_n(\Delta)\,\Omega_n^0\,|\,\Delta\in\tilde{\mathscr{G}}\}$$
$$-U_n\Omega^0=\otimes^n\Omega_n^0$$
$$-\Psi(\Delta)=U_n^*\prod_{g\in\Delta}^{\rightarrow}(\Psi_n(g)\otimes\mathbb{1}\otimes\ldots\otimes\mathbb{1}+\ldots+\mathbb{1}\otimes\ldots\otimes\mathbb{1}\otimes\Psi_n(g))\,U_n\qquad\text{for all }\Delta\in\tilde{\mathscr{G}}. \tag{38}$$

We call $(\mathscr{K}_n,\Psi_n,\Omega_n^0)$ an n^{th}-*part* of $(\mathscr{K},\Psi,\Omega^0)$.

Notice that in the case of conditionally positive definite functions $\Psi\colon\mathscr{G}\to\mathbb{C}$ the notion of infinite additivity trivialises. Indeed take $\mathscr{K}_n=\mathbb{C}$, $\Omega_n^0=1\in\mathbb{C}$, U_n the isomorphism between $\mathbb{C}$ and $\otimes^n\mathbb{C}$ and $\Psi_n(g)=\Psi(g)/n$.

Definition III.3. Given a mapping $\Psi\colon\mathscr{G}\to\mathscr{L}(\mathscr{K})$ and a normalized vector $\Omega^0\in\mathscr{K}$ such that Ω^0 and $\psi(\Delta)\,\Omega^0$ belong to domain of $\Psi(g)$ for all $g\in\mathscr{G}$, $\Delta\in\tilde{\mathscr{G}}$, the *cumulants* P_Λ^Ψ, $\Lambda\in\tilde{\mathscr{G}}$ of Ψ with respect to Ω^0 are inductively defined by

$$P_\phi^\Psi=0$$
$$\langle\Omega^0|\Psi(\Delta)\,\Omega^0\rangle=\sum_{p\in\mathscr{P}_\Delta}\prod_{\Lambda\in p}P_\Lambda^\Psi. \tag{39}$$

where $\mathscr{P}_\Delta$ is the set of ordered partitions p of Δ into non empty sets Λ.

Remarks. (1) Clearly P_Λ^Ψ is a homogeneous polynomial in $\langle\Omega^0|\Psi(X)\,\Omega^0\rangle$ of degree $\#(\Lambda)$ if we put $\deg\langle\Omega^0|\Psi(X)\,\Omega^0\rangle=\#(X)$, e.g.,

$$P_{(g)}^\Psi=\langle\Omega^0|\Psi(g)\,\Omega^0\rangle$$
$$P_{(g,h)}^\Psi=\langle\Omega^0|\Psi(g)\,\Psi(h)\,\Omega^0\rangle-\langle\Omega^0|\Psi(g)\,\Omega^0\rangle\langle\Omega^0|\Psi(h)\,\Omega^0\rangle.$$

(2) P_Λ^Ψ generalizes the usual cumulants P_n which are defined by

$$\langle\Omega|e^{itA}\,\Omega\rangle=\exp\sum_{n\geq1}\frac{(it)^n}{n!}P_n,\qquad A\in\mathscr{L}(\mathscr{K}).$$

Indeed, take $\Psi(g)=A$ for all $g\in\mathscr{G}$, then $P_n=P_\Lambda^\Psi$ whenever $\#(\Lambda)=n$.

(3) If Ψ is a complex valued function, then $P_\Lambda^\Psi=0$ for all Λ with $\#(\Lambda)>1$.

Lemma III.4. *If* $(\mathscr{K}, \Psi, \Omega^0)$ *is an infinitely additive conditionally* C.P. *triplet with* n^{th} *parts* $(\mathscr{K}_n, \Psi_n, \Omega_n^0)$ *on a group* $\mathscr{G}$, *then the cumulants of* Ψ w.r.t. Ω^0 *satisfy*

$$\text{(i)} \quad P_\Delta^\Psi = \lim_{n\to\infty} n \langle \Omega_n^0 | \Psi_n(\Delta)\, \Omega_n^0 \rangle, \qquad \Delta \in \tilde{\mathscr{G}} \backslash \{\phi\} \tag{40}$$

$$\text{(ii)} \quad P_\Delta^{\Psi_n} = \frac{1}{n} P_\Delta^\Psi, \qquad \Delta \in \tilde{\mathscr{G}} \tag{41}$$

Proof. First we prove the existence of the limit in (40) by induction on $\#(\Delta)$. For $\#(\Delta)=1$, say $\Delta=(g)$, we have by infinite additivity of $(\mathscr{K}, \Psi, \Omega^0)$ that $n\langle \Omega_n^0 | \Psi_n(g)\, \Omega_n^0\rangle = \langle \Omega^0 | \Psi(g)\, \Omega^0\rangle$. Suppose now that the limit exists for all $\Delta \in \tilde{\mathscr{G}}$ with $\#(\Delta) \leq m$. Take then a $\Delta \in \tilde{\mathscr{G}}$ with $\#(\Delta)=m+1$. Now note that by infinite additivity we have for all $\Delta \in \tilde{\mathscr{G}}$

$$\begin{aligned}\langle \Omega^0 | \Psi(\Delta)\, \Omega^0\rangle &= \left\langle \otimes^n \Omega_n^0 \middle| \overrightarrow{\prod_{g\in\Delta}} \left(\sum_{j=1}^n \mathbb{1} \otimes \ldots \otimes \Psi_n(g)_j \otimes \ldots \otimes \mathbb{1}\right) \otimes^n \Omega_n^0 \right\rangle \\ &= \sum_{p\in\mathscr{P}_\Delta} \frac{n!}{(n-\#(p))!} \prod_{\Lambda\in p} \langle \Omega^0 | \Psi_n(\Lambda)\, \Omega_n^0\rangle. \end{aligned} \tag{42}$$

Hence

$$\begin{aligned} n\langle \Omega_n^0 | \Psi_n(\Delta)\, \Omega_n^0\rangle &= \langle \Omega^0 | \Psi(\Delta)\, \Omega^0\rangle \\ &\quad - \sum_{\substack{p\in\mathscr{P}_\Delta \\ p\neq\{\Delta\}}} \frac{n!}{(n-\#(p))!} \prod_{\Lambda\in p} \langle \Omega_n^0 | \Psi_n(\Lambda)\, \Omega_n^0\rangle \end{aligned} \tag{43}$$

The limit for $n\to\infty$ of the right hand side of (43) exists by the induction hypothesis since all Λ's that appear have $\#(\Lambda)\leq m$; so the limit of the left hand side of (43) must exist as well. Denoting this limit by P_Λ^Ψ and taking the limit $n\to\infty$ of (42) it is also clear that (39) holds.

Moreover, again by an induction argument on $\#(\Delta)$ it follows immediately from (43) that P_Δ^Ψ is homogeneous polynomial of degree $\#(\Delta)$ in $\langle \Omega^0 | \Psi(X)\, \Omega^0\rangle$, $X\subset\Delta$. Hence, the P_Δ^Ψ's given by (40) are actually the cumulants of Ψ w.r.t. Ω^0.

It is straightforwardly checked that $(\mathscr{K}_n, \Psi_n, \Omega_n^0)$ is infinitely additive and that its k^{th} parts are given by $(\mathscr{K}_{nk}, \Psi_{nk}, \Omega_{nk}^0)$. So by (i):

$$P_\Delta^{\Psi_n} = \lim_{k\to\infty} k\langle \Omega_{nk}^0 | \Psi_{nk}(\Delta)\, \Omega_{nk}^0\rangle = \frac{1}{n} \lim_{k\to\infty} nk \langle \Omega_{nk}^0 | \Psi_{nk}(\Delta)\, \Omega_{nk}^0\rangle = \frac{1}{n} P_\Delta^\Psi. \quad \square$$

III.2. Construction of the Logarithm of Infinitely Divisible Triplets

Lemma III.6. *Let* $(\mathscr{H}, \Phi, \Omega)$ *be a continuous infinitely divisible* C.P. *triplet satisfying condition C on an arcwise connected group* $\mathscr{G}$. *Denote its* n^{th} *root by* $(\mathscr{H}^{1/n}, \Phi^{1/n}, \Omega^{1/n})$. *Then,*

(i) $f(\Delta) \equiv \lim_{n\to\infty} \langle \otimes^n \Omega^{1/n} | \Gamma_n(\Delta) \otimes^n \Omega^{1/n}\rangle$ exists *where*

$$\Gamma_n(g) = (\Phi^{1/n}(g) - \mathbb{1}) \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} + \mathbb{1} \otimes (\Phi^{1/n}(g) - \mathbb{1}) \otimes \ldots \otimes \mathbb{1} + \ldots + \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes (\Phi^{1/n}(g) - \mathbb{1}). \tag{44}$$

Moreover

$$f(\Delta) = \sum_{p \in \mathscr{P}_\Delta} \prod_{\Lambda \in p} \sum_{X \subset \Lambda} (-1)^{\#(\Lambda \backslash X)} d_X \tag{45}$$

where $\mathscr{P}_\Delta$ is the set of ordered partition p of Δ into non empty sets Λ and where d_X is given by (24).

(ii) $\tilde{f}(\Delta, \Delta') = f(\Delta^* \times \Delta')$ *is a positive kernel on* $\tilde{\mathscr{G}}$.

Remark. If Φ is a function (i.e., $\mathscr{H} = \mathbb{C}$), then $f(\Delta) = \prod_{g \in \Delta} v(g)$, where $\Phi = e^v$.

Proof. (i) Using the notation

$$\eta_n(g) = \Phi^{1/n}(g) - \mathbb{1} \tag{46}$$

we can write

$$f(\Delta) = \lim_{n \to \infty} \langle \otimes^n \Omega^{1/n} \Big| \overrightarrow{\prod_{g \in \Delta}} \sum_{j=1}^{n} (\mathbb{1} \otimes \ldots \otimes \eta_n(g)_j \otimes \ldots \mathbb{1}) \otimes^n \Omega^{1/n} \rangle$$

$$= \lim_{n \to \infty} \sum_{p \in \mathscr{P}_\Delta} \frac{n!}{(n - \#(p))!} \prod_{\Lambda \in p} \langle \Omega^{1/n} | \eta_n(\Lambda) \, \Omega^{1/n} \rangle$$

$$= \lim_{n \to \infty} \sum_{p \in \mathscr{P}_\Delta} \frac{n!}{(n - \#(p))!} \prod_{\Lambda \in p} \sum_{X \subset \Lambda} (-1)^{\#(\Lambda \backslash X)} \langle \Omega^{1/n} | \Phi^{1/n}(X) \, \Omega^{1/n} \rangle.$$

Now, as $\Lambda \in p$ is a non-empty set, we have $\sum_{X \subset \Lambda} (-1)^{\#(\Lambda \backslash X)} = 0$ and we can, using (25), rewrite $f(\Delta)$ as follows

$$f(\Delta) = \lim_{n \to \infty} \sum_{p \in \mathscr{P}_\Delta} \frac{n!}{(n - \#(p))!} \prod_{\Lambda \in p} \sum_{X \subset \Lambda} (-1)^{\#(\Lambda \backslash X)} [\exp(d_X/n) - 1]$$

$$= \sum_{p \in \mathscr{P}_\Delta} \prod_{\Lambda \in p} \sum_{X \subset \Lambda} (-1)^{\#(\Lambda \backslash X)} d_X$$

which proves (i).

(ii) The proof of (ii) is straightforward. □

Theorem III.7. *Let $(\mathscr{H}, \Phi, \Omega)$ be a continuous infinitely divisible* C.P. *triplet satisfying condition C on an arcwise connected group $\mathscr{G}$. Using the notation of lemma III.6, let* $\mathscr{K} = \mathrm{hil}(\tilde{\mathscr{G}}, \tilde{f})$.

(i) *For all $g \in \mathscr{G}$*

$$\Psi(g)\colon \ \Delta \in \mathscr{K} \mapsto (g) \times \Delta \in \mathscr{K} \tag{47}$$

defines a linear operator on the dense subspace $D = \mathrm{span}\{\Delta \mid \Delta \in \tilde{\mathscr{G}}\}$ *of* $\mathscr{K}$

(ii) *Put $\Omega^0 = \phi \in \mathscr{K}$, then $(\mathscr{K}, \Psi, \Omega^0)$ is a continuous hermitian conditionally* C.P. *triplet*

(iii) *$(\mathscr{K}, \Psi, \Omega^0)$ is infinitely additive and has continuous hermitian n^{th} parts.*

(iv) $(\mathscr{K}, \Psi, \Omega^0)$ *satisfies an additional positivity condition:*

$$((g, \Delta), (g' \Delta')) \mapsto \sum_{A \subset \Delta^* \times (g^{-1} g') \times \Delta'} P_A^{\Psi} \tag{48}$$

is a conditionally positive kernel on $\mathscr{G} \times \widetilde{\mathscr{G}}$.

Proof. (i) If $\sum_\Delta \lambda_\Delta \Delta = 0$ in $\mathscr{K}$, then $\sum_\Delta \lambda_\Delta (g) \times \Delta = 0$ as well since

$$\begin{aligned}\|\sum_\Delta \lambda_\Delta (g) \times \Delta\|^2 &= \sum_{\Delta, \Delta'} \bar{\lambda}_\Delta \lambda_{\Delta'} f(\Delta^* \times (g^{-1}, g) \times \Delta)^* \times \Delta') \\ &= \sum_{\Delta, \Delta'} \lambda_\Delta \lambda_{\Delta'} f(((g^{-1}, g) \times \Delta') \\ &= \langle \sum_\Delta \lambda_\Delta (g^{-1}, g) \times \Delta | \sum_{\Delta'} \lambda_{\Delta'} \Delta' \rangle = 0.\end{aligned}$$

Hence $\Psi(g)$ is well defined by (47).

(ii) By construction $\Omega^0 = \phi \in \mathrm{Dom}\ \Psi(g)$ and $\Delta = \Psi(\Delta)\ \Omega^0 \in \mathrm{Dom}\ \Psi(g)$. Also $D = \mathrm{span}\{\Psi(\Delta)\, \Omega^0 | \Delta \in \widetilde{\mathscr{G}}\}$ is dense in $\mathscr{K}$.

Furthermore, for all $g \to \xi_g = \sum_\Delta \lambda_{g, \Delta}\ \Delta \in D$ with $\sum_g \xi_g = 0$, we have, using notation (44),

$$\begin{aligned}&\sum_{g, g'} \langle \xi_g | \Psi(g^{-1} g') \xi_{g'} \rangle \\ &= \sum_{\substack{g, g' \\ \Delta, \Delta'}} \bar{\lambda}_{g, \Delta} \lambda_{g', \Delta'} f(\Delta^* \times (g^{-1} g') \times \Delta') \\ &= \lim_{k \to \infty} \sum_{\substack{g, g' \\ \Delta, \Delta'}} \bar{\lambda}_{g, \Delta} \lambda_{g', \Delta'} \langle \Gamma_k(\Delta) \otimes^k \Omega^{1/k} | \Gamma_k(g^{-1} g')\, \Gamma_k(\Delta') \otimes^k \Omega^{1/k} \rangle \\ &= \lim_{k \to \infty} \sum_{\substack{g, g' \\ \Delta, \Delta'}} \bar{\lambda}_{g, \Delta} \lambda_{g', \Delta'} \langle \Gamma_k(\Delta) \otimes^k \Omega^{1/k} \\ &\quad \Big| \sum_{j=1}^{k} (\mathbb{1} \otimes \ldots \otimes \Phi^{1/k}(g^{-1} g')_j \otimes \ldots \otimes \mathbb{1})\, \Gamma_k(\Delta') \otimes^k \Omega^{1/k} \rangle \\ &- \lim_{k \to \infty} \sum_{\substack{g, g' \\ \Delta, \Delta'}} \bar{\lambda}_{g, \Delta} \lambda_{g', \Delta'} \langle \Gamma_k(\Delta) \otimes^k \Omega^{1/k} | \Gamma_k(\Delta') \otimes^k \Omega^{1/k} \rangle.\end{aligned}$$

The first term is positive by complete positivity of $\Phi^{1/k}$, whereas the second term tends to

$$\sum_{\substack{g, g' \\ \Delta, \Delta'}} \bar{\lambda}_{g, \Delta} \lambda_{g', \Delta'} f(\Delta^* \times \Delta') = \|\sum_g \xi_g\|^2 = 0.$$

Hence $(\mathscr{K}, \Psi, \Omega^0)$ is a conditionally C.P. triplet.

From (45) and the continuity of $X \in \widetilde{\mathscr{G}} \mapsto d_X \in \mathbb{C}$, it follows that

$$g \in \mathscr{G} \mapsto f(\Delta^* \times (g) \times \Delta') = \langle \Delta | \Psi(g)\, \Delta' \rangle$$

is continuous for all $\Delta, \Delta' \in \tilde{\mathscr{G}}$. Hence $(\mathscr{K}, \Psi, \Omega^0)$ is continuous.

Moreover, as

$$\langle \Delta | \psi(g) \Delta' \rangle = f(\Delta^* \times ((g) \times \Delta') = f((g^{-1}) \times \Delta)^* \times \Delta') = \langle \Psi(g^{-1}) \Delta | \Delta' \rangle,$$

$(\mathscr{K}, \Psi, \Omega^0)$ is hermitian.

Finally, since $f(\Delta) = 0$ as soon as $e \in \Delta$, it is clear that $\Psi(e) = 0$.

(iii) Let $(\mathscr{H}^{1/n}, \Phi^{1/n}, \Omega^{1/n})$ be the continuous n^{th} root of $(\mathscr{H}, \Phi, \Omega)$. Because $(\mathscr{H}^{1/n}, \Phi^{1/n}, \Omega^{1/n})$ is also infinitely divisible and has continuous roots, we can by (i) and (ii) construct with it a continuous hermitian conditionally C.P. triplet which we denote by $(\mathscr{K}_n, \Psi_n, \Omega_n^0)$. We prove that $(\mathscr{K}_n, \Psi_n, \Omega_n^0)$ is the n^{th} part of $(\mathscr{K}, \Psi, \Omega^0)$. Note therefore that (use notation (44))

$$\begin{aligned}
&\langle \Psi(\Delta) \Omega^0 | \Psi(\Delta') \Omega^0 \rangle \\
&\quad = \lim_{k \to \infty} \langle \otimes^k \Omega^{1/k} | \Gamma_k(\Delta^* \times \Delta') \otimes^k \Omega^{1/k} \rangle \\
&\quad = \lim_{k \to \infty} \langle \otimes^{kn} \Omega^{1/kn} | \Gamma_{kn}(\Delta^* \otimes \Delta') \otimes^{kn} \Omega^{1/kn} \rangle \\
&\quad = \lim_{k \to \infty} \left\langle \otimes^{kn} \Omega^{1/kn} \right| \overrightarrow{\prod_{g \in \Delta^* \times \Delta'}} \sum_{l=0}^{n-1} \left(\sum_{j=lk+1}^{(l+1)k} \mathbb{1} \otimes \ldots \otimes \eta_{kn}(g)_j \otimes \ldots \otimes \mathbb{1} \right) \otimes^{kn} \Omega^{1/kn} \Big\rangle \\
&\quad = \lim_{k \to \infty} \sum_{p \in \mathscr{P}_{\Delta^* \times \Delta'}} \frac{n!}{(n - \#(p))!} \prod_{\Lambda \in p} \langle \otimes^k \Omega^{1/kn} \Big| \\
&\qquad \overrightarrow{\prod_{g \in \Lambda}} \sum_{j=1}^{k} (\mathbb{1} \otimes \ldots \otimes \eta_{kn}(g)_j \otimes \ldots \otimes \mathbb{1}) \otimes^k \Omega^{1/kn} \rangle \\
&\quad = \lim_{k \to \infty} \sum_{p \in \mathscr{P}_{\Delta^* \times \Delta'}} \frac{n!}{(n - \#(p))!} \prod_{\Lambda \in p} \langle \otimes^k \Omega^{1/k} | \Gamma_{kn}(\Lambda) \otimes^k \Omega^{1/kn} \rangle \\
&\quad = \sum_{p \in \mathscr{P}_{\Delta^* \times \Delta'}} \frac{n!}{(n - \#(p))!} \prod_{\Lambda \in p} \langle \Omega_n^0 | \Psi_n(\Lambda) \Omega_n^0 \rangle \\
&\quad = \left\langle \otimes^n \Omega_n^0 \right| \overrightarrow{\prod_{g \in \Delta^* \times \Delta'}} \left(\sum_{j=1}^{n} \mathbb{1} \otimes \ldots \otimes \Psi_n(g)_j \otimes \ldots \otimes \mathbb{1} \right) \otimes^n \Omega_n^0 \Big\rangle \\
&\quad = \Big\langle \overrightarrow{\prod_{g \in \Delta}} \left(\sum_{j=1}^{n} \mathbb{1} \otimes \ldots \otimes \Psi_n(g)_j \otimes \ldots \mathbb{1} \otimes^n \Omega_n^0 \right| \\
&\qquad \overrightarrow{\prod_{g' \in \Delta'}} \left(\sum_{j=1}^{n} \mathbb{1} \otimes \ldots \otimes \Psi_n(g')_j \otimes \ldots \otimes \mathbb{1} \right) \otimes^n \Omega_n^0 \Big\rangle.
\end{aligned}$$

This implies that the mapping

$$\Psi(\Delta) \Omega^0 \in \mathscr{K} \mapsto \overrightarrow{\prod_{g \in \Delta}} \left(\sum_{j=1}^{n} \mathbb{1} \otimes \ldots \otimes \Psi_n(g)_j \otimes \ldots \mathbb{1} \right) \otimes^n \Omega_n^0 \in {}^n \mathscr{K}_n$$

is well defined and can be extended to an isometry U_n: $\mathscr{K} \to \otimes^n \mathscr{K}_n$. It is now clear that (38) is satisfied. Hence $(\mathscr{K}, \Psi, \Omega^0)$ is infinitely additive and has continuous n^{th} parts.

(iv) By comparing (39) and (45), recalling that $f(\Delta)=\langle \Omega^0 | \Psi(\Delta)\, \Omega^0\rangle$ and observing that $P_\phi^\psi = d_\phi = 0$, we have for all $\Lambda \in \tilde{\mathscr{G}}$

$$P_\Lambda^\Psi = \sum_{X \subset \Lambda} (-1)^{\#(\lambda \setminus X)} d_X. \tag{49}$$

So, summing (49) over $\Lambda \subset \Delta$ one gets

$$\sum_{\Lambda \subset \Delta} P_\Lambda^\Psi = \sum_{\Lambda \subset \Delta} \sum_{X \subset \Lambda} (-1)^{\#(\Lambda \setminus X)} d_X = \sum_{X \subset \Delta} \left(\sum_{X \subseteq \Lambda \subseteq \Delta} (-1)^{\#(\Lambda \setminus X)} \right) d_X$$

Since $\sum_{X_1 \subset \Lambda \subset X_2} (-1)^{\#(\Lambda)} = 0$ if $X_1 \neq X_2$, we end up with

$$d_\Delta = \sum_{\Lambda \subset \Delta} P_\Lambda^\Psi. \tag{50}$$

Hence in order to prove (iv), we have to show that $((g, \Delta), (g', \Delta')) \mapsto d_{\Delta^* \times (g^{-1} g') \times \Delta'}$ is a conditionally positive kernel. But this follows immediately from the fact it is hermitian and that its exponential (i.e., $((g, \Delta), (g' \Delta')) \mapsto \langle \Omega | \Phi(\Delta^* \times (g^{-1} g') \times \Delta)\, \Omega\rangle$) is a positive kernel. □

Remark. In the case of conditionally positive definite functions Ψ: $\mathscr{G} \to \mathbb{C}$ the propositions (iii) and (iv) of the preceding theorem are trivially satisfied, since then the notion of infinite additivity trivialises and the additional positivity condition turns out to be equivalent with conditional positive definiteness of Ψ itself. However, for mappings ($\dim \mathscr{K} \geqq 1$) (iii) and (iv) are non-trivial properties.

Definition III.8. Let $(\mathscr{H}, \Phi, \Omega)$ and $(\mathscr{K}, \Psi, \Omega^0)$ be as in theorem III.7. We call $(\mathscr{K}, \Psi, \Omega^0)$ *the logarithm* of $(\mathscr{H}, \Phi, \Omega)$. Notation: $(\mathscr{K}, \Psi, \Omega^0) = \ln(\mathscr{H}, \Phi, \Omega)$.

Clearly in the special case of a continuous infinitely divisible complex valued function $f = e^v$, we recover the usual definition of the logarithm: $\ln(\mathbb{C}, f, 1) = (\mathbb{C}, v, 1)$.

As we have now found an infinitely additive conditionally C.P. triplet as a logarithm for an infinitely divisible C.P. triplet, an obvious question arises: can we conversely "exponentiate" an infinitely additive conditionally C.P. triplet in some way, to end up with an infinitely divisible C.P. triplet? This will be the problem we will solve in the sequel.

To find a way to construct an exponential of a hermitian infinitely additive conditionally C.P. triplet, we consider the special case of complex valued functions. If v: $\mathscr{G} \to \mathbb{C}$ is a conditionally positive definite function with $v(e) = 0$ and $\overline{v(g)} = v(g^{-1})$, then

$$e^v = \lim_{n \to \infty} \left(1 + \frac{v}{n}\right)^n$$

is an infinitely divisible normalized positive definite function. If $(\mathscr{K}, \Psi, \Omega^0)$ is a hermitian infinitely additive conditionally C.P. triplet with n^{th} part $(\mathscr{K}_n, \Psi_n, \Omega_n^0)$, it is clear that $\frac{v}{n}$ should be replaced by Ψ_n. Therefore we consider

$$w-\lim_{n\to\infty} \otimes^n(\mathbb{1}+\Psi_n(g)).$$

Lemma III.9. *Let $(\mathscr{K}, \Psi, \Omega^0)$ be a hermitian infinitely additive conditionally* C.P. *triplet with n^{th} parts $(\mathscr{K}_n, \Psi_n, \Omega_n^0)$ on a group $\mathscr{G}$.*
Then

$$\text{(i)}\ \ F(\Delta)=\lim_{n\to\infty}\left\langle \otimes^n \Omega_n^0 \middle| \overrightarrow{\prod_{g\in\Delta}} (\otimes^n(\mathbb{1}+\Psi_n(g))\otimes^n \Omega_n^0\right\rangle$$

exists and

$$F(\Delta)=\exp\sum_{\Lambda\in\Delta} P_\Lambda^\Psi. \tag{51}$$

(ii) $\tilde{F}(\Delta, \Delta')=F(\Delta^*\times\Delta')$ *is a positive kernel on* $\tilde{\mathscr{G}}$.

Proof. (i) We have

$$\begin{aligned}
&\left\langle \otimes^n \Omega_n^0 \middle| \overrightarrow{\prod_{g\in\Delta}} \otimes^n(\mathbb{1}+\Psi_n(g))\otimes^n \Omega_n^0\right\rangle \\
&=\left\langle \Omega_n^0 \middle| \overrightarrow{\prod_{g\in\Delta}} (\mathbb{1}+\Psi_n(g))\, \Omega_n^0\right\rangle^n \\
&=\Big[\sum_{\Lambda\subset\Delta} \langle \Omega_n^0 | \Psi_n(\Lambda)\, \Omega_n^0\rangle\Big]^n \\
&=\left[1+\frac{1}{n}\sum_{\substack{\Lambda\subset\Delta\\ \Lambda\neq\phi}} n\langle \Omega_n^0 | \Psi_n(\Lambda)\, \Omega_n^0\rangle\right]^n.
\end{aligned}$$

Now use (40) and the fact that $P_\phi^\Psi=0$ to get (51). □
(ii) follows straightforwardly.

Theorem III.10. *Let $(\mathscr{K}, \Psi, \Omega_0)$ be a hermitian infinitely additive conditionally* C.P. *triplet on a group $\mathscr{G}$ such that $\Psi(e)=0$ and the additional positivity* (48) *is satisfied. Using the notation of Lemma III.9, let $\mathscr{H}=$*hil$(\tilde{\mathscr{G}}, \tilde{F})$.

(i) *For all $g\in\mathscr{G}$, $\Phi(g)$: $\Delta\in\mathscr{H}\to(g)\times\Delta\in\mathscr{H}$ defines a bounded linear operator on $\mathscr{H}$.*

(ii) *Put $\Omega=\phi\in\mathscr{H}$, then $(\mathscr{H}, \Phi, \Omega)$ is an infinitely divisible C.P. triplet on $\mathscr{G}$.*

(iii) *If $(\mathscr{K}, \Psi, \Omega^0)$ is continuous and has continuous n^{th} parts, then also $(\mathscr{H}, \Phi, \Omega)$ is continuous and it has continuous roots.*

Proof. (i) If $\sum_{\Delta} \lambda_\Delta \Delta = 0$ in $\mathscr{H}$, then $\sum_{\Delta} \lambda_\Delta((g)\times\Delta)=0$ as well, because

$$\|\sum_{\Delta} \lambda_\Delta((g)\times\Delta)\|^2 = \sum_{\Delta,\Delta'} \bar{\lambda}_\Delta \lambda_{\Delta'} F(\Delta^* \times (g^{-1}, g) \times \Delta')$$
$$= \langle \sum_{\Delta} \lambda_\Delta \Delta \,|\, \sum_{\Delta'} \lambda_{\Delta'} (g^{-1}, g) \times \Delta' \rangle = 0.$$

Hence $\Phi(g)$ is well defined as a linear operator on the dense subspace $D = \text{span}\{\Delta \,|\, \Delta \in \tilde{\mathscr{G}}\}$. Since $F(\Delta^* \times (g) \times \Delta') = F(((g^{-1}) \times \Delta)^* \times \Delta')$, it is clear that $\Phi(g^{-1}) \subset \Phi(g)^*$.

Moreover, as $\Psi(e)=0$ we have $\Psi_n(e)=0$ and so

$$\langle \Delta \,|\, \Phi(e)\,\Delta' \rangle = F(\Delta^* \times (e) \times \Delta')$$
$$= \lim_{n\to\infty} \left\langle \Omega_n^0 \,\middle|\, \prod_{g\in\Delta^*}^{\rightarrow} (\mathbb{1} + \Psi_n(g))(\mathbb{1} - \Psi_n(e)) \prod_{g'\in\Delta'}^{\rightarrow} (\mathbb{1} + \Psi_n(g'))\,\Omega_n^0 \right\rangle^n$$
$$= \lim_{n\to\infty} \left\langle \Omega_n^0 \,\middle|\, \prod_{g\in\Delta^*\times\Delta'}^{\rightarrow} (\mathbb{1} + \Psi_n(g))\,\Omega_n^0 \right\rangle^n$$
$$= F(\Delta^* \times \Delta') = \langle \Delta \,|\, \Delta' \rangle$$

which means that $\Phi(e) = \mathbb{1}$.

To prove boundedness of $\Phi(g)$, we first show complete positivity of $g \mapsto \Phi(g)$ on D. Let $\xi_g = \sum_{\Delta} \lambda_{g,\Delta} \Delta \in D$, then

$$\sum_{g,g'} \langle \xi_g \,|\, \Phi(g^{-1}g')\,\xi_{g'} \rangle = \sum_{g,g'} \bar{\lambda}_{g,\Delta} \lambda_{g',\Delta'} F(\Delta^* \times (g^{-1}g') \times \Delta)$$
$$= \sum_{\substack{g,g'\\ \Delta,\Delta'}} \bar{\lambda}_{g,\Delta} \lambda_{g',\Delta} \exp \sum_{\Lambda \subset \Delta^* \times (g^{-1}g') \times \Delta'} P_\Lambda^\Psi \geqq 0 \qquad (52)$$

where the inequality follows from the additional positivity property of and the fact that the exponential of a hermitian conditionally positive kernel is positive.

Now (52) implies for all $\xi, \eta \in D$

$$\langle \xi \,|\, \Phi(g)\,\eta \rangle + \langle \Phi(g)\,\eta \,|\, \xi \rangle + \langle \xi \,|\, \xi \rangle + \langle \eta \,|\, \eta \rangle \geqq 0.$$

Take $\|\xi\| = \|\eta\| = 1$ and multiply ξ and η with an appropriate phase factor to get that $|\langle \xi \,|\, \Phi(g)\,\eta \rangle| \leqq 1$ for all normalized $\xi, \eta \in D$ and since D is dense in $\mathscr{H}$ this implies $\|\Phi(g)\| \leqq 1$.

(ii) In the proof of (i) we have already shown that $g \mapsto \Phi(g)$ is C.P. on D and by continuity also on the whole of $\mathscr{H}$. By construction, $\|\Omega\| = 1$ and Ω is cyclic for $\{\Phi(\mathscr{G})\}''$. Hence $(\mathscr{H}, \Phi, \Omega)$ is a C.P. triplet.

Now we show that it is infinitely divisible. The n^{th} part $(\mathscr{K}_n, \Psi_n, \Omega_n^0)$ of $(\mathscr{K}, \Psi, \Omega^0)$ is clearly infinitely additive as well and by (41) it also satisfies the additional positivity condition (48). Therefore we can construct with it a C.P. triplet $(\mathscr{H}^{1/n}, \Phi^{1/n}, \Omega^{1/n})$ in the same way as $(\mathscr{H}, \Phi, \Omega)$ was made out of $(\mathscr{K}, \Psi, \Omega^0)$.

It can easily be seen that $(\mathscr{H}^{1/n}, \Phi^{1/n}, \Omega^{1/n})$ is an n^{th} root for $(\mathscr{H}, \Phi, \Omega)$. Indeed, notice that

$$\begin{aligned}\langle \Omega | \Phi(\Delta)\, \Omega \rangle &= F(\Delta) \\ &= \exp \sum_{\Lambda \subset \Delta} P_\Lambda^\Psi \\ &= \exp n \sum_{\Lambda \subset \Delta} P_\Lambda^{\Psi_n} \\ &= F_n(\Delta)^n \\ &= \langle \Omega^{1/n} | \Phi^{1/n}(\Delta)\, \Omega^{1/n} \rangle^n = \langle \otimes^n \Omega^{1/n} | \otimes^n \Phi^{1/n}(\Delta) \otimes^n \Omega^{1/n} \rangle.\end{aligned}$$

Hence $(\mathscr{H}, \Phi, \Omega) = \otimes^n (\mathscr{H}^{1/n}, \Phi^{1/n}, \Omega^{1/n})$.

(iii) The continuity of $(\mathscr{K}, \Psi, \Omega^0)$ and $(\mathscr{K}_n, \Psi_n, \Omega_n)$ yields the continuity of $g \mapsto F(\Delta^* \times (g) \times \Delta')$ and $g \mapsto F_n(\Delta^* \times (g) \times \Delta')$ and this clearly implies the continuity of Φ and $\Phi^{1/n}$.

Definition III.11. Let $(\mathscr{K}, \Psi, \Omega^0)$ and $(\mathscr{H}, \Phi, \Omega)$ be as in theorem III.10. We call $(\mathscr{H}, \Phi, \Omega)$ *the exponential* of $(\mathscr{K}, \Psi, \Omega^0)$. Notation: $(\mathscr{H}, \Phi, \Omega) = \exp(\mathscr{K}, \Psi, \Omega^0)$.

Remark. In the special case of a conditionally positive definite function $v: G \to \mathbb{C}$ we have $\exp(\mathbb{C}, v, 1) = (\mathbb{C}, e^v, 1)$.

The logarithmic construction of theorem III.7 and the exponential of Theorem III.10 are mutually inverse. In fact we have:

Theorem III.12. (i) *If $(\mathscr{H}, \Phi, \Omega)$ is a continuous infinitely divisible* C.P. *triplet satisfying condition C on an arcwise connected group, we have*

$$\exp(\ln(\mathscr{H}, \Phi, \Omega)) = (\mathscr{H}, \Phi, \Omega)$$

(up to unitary equivalence).

(ii) *If $(\mathscr{K}, \Psi, \Omega^0)$ is a continuous hermitian infinitely additive conditionally* C.P. *triplet on an arcwise connected group, satisfying $\Psi(e) = 0$ and the additional positivity condition* (48) *and having continuous parts, then we have*

$$\ln(\exp(\mathscr{K}, \Psi, \Omega_0) = (\mathscr{K}, \Psi, \Omega^0)$$

(up to unitary equivalence).

Proof. (i) Let $(\mathscr{K}, \Psi, \Omega^0) = \ln(\mathscr{H}, \Phi, \Omega)$ and $(\tilde{\mathscr{H}}, \tilde{\Phi}, \tilde{\Omega}) = \exp(\mathscr{K}, \Psi, \Omega^0)$. Then using (51) and (50) we have

$$\langle \tilde{\Omega} | \tilde{\Phi}(\Delta)\, \tilde{\Omega} \rangle = \exp \sum_{\Lambda \subset \Delta} P_\Lambda^\Psi = \exp d_\Delta = \langle \Omega | \Phi(\Delta)\, \Omega \rangle$$

Hence $(\tilde{\mathscr{H}}, \tilde{\Phi}, \tilde{\Omega}) = (\mathscr{H}, \Phi, \Omega)$.

(ii) Let $(\mathscr{H}, \Phi, \Omega) = \exp(\mathscr{K}, \Psi, \Omega^0)$ and $(\tilde{\mathscr{K}}, \tilde{\Psi}, \tilde{\Omega}^0) = \ln(\mathscr{H}, \Phi, \Omega)$. Let $d: \mathscr{G} \to \mathbb{C}$ be the function satisfying $\langle \Omega | \Phi(X)\, \Omega \rangle = \exp d_X$. Then it follows from the construction of Φ and (51) that

$$d_X = \sum_{Y \subset X} P_Y^\Psi .$$

Hence, using (45) and (39) one gets

$$\begin{aligned}<\tilde{\Omega}^0 | \tilde{\Psi}(\Delta) \tilde{\Omega}^0\rangle &= \sum_{p \in \mathscr{P}_\Delta} \prod_{\Lambda \in p} \sum_{X \subset \Lambda} (-1)^{\#(\Lambda \setminus X)} d_X \\ &= \sum_{p \in \mathscr{P}_\Delta} \prod_{\Lambda \in p} \sum_{X \subset \Lambda} (-1)^{\#(\Lambda \setminus X)} \sum_{Y \subset X} P_Y^\Psi \\ &= \sum_{p \in \mathscr{P}_\Delta} \prod_{\Lambda \in p} \sum_{Y \subset \Lambda} \Big(\sum_{Y \subset X \subset \Lambda} (-1)^{\#(\Lambda \setminus X)} P_Y^\Psi \Big) \\ &= \sum_{p \in \mathscr{P}_\Delta} \prod_{\Lambda \in p} P_\Lambda^\Psi \\ &= \langle \Omega^0 | \Psi(\Delta) \Omega^0 \rangle\end{aligned}$$

which means $(\tilde{\mathscr{K}}, \tilde{\Psi}, \tilde{\Omega}^0) = (\mathscr{K}, \Psi, \Omega^0)$.

III.3. Example

We will illustrate the construction of the logarithm for the triplet $(\mathscr{S}(H_R), \Phi_{R,Q,c}, \operatorname{Exp} 0)$ introduced in Sect. I.5.

We show that the logarithm of this triplet is given by

$$(\mathscr{S}(H_R), \Psi_{R,Q,c}, \Omega^0)$$

where

$$\Omega^0 = \operatorname{Exp} 0$$

and

$$\Psi_{R,Q,c}(\xi, \theta) = iB(R\xi) + (ic\theta - \tfrac{1}{2}\langle \xi | (R^* \cdot R + Q)\xi \rangle) \mathbb{1}$$

where $B(\eta)$ is the infinitesimal generator of the strongly continuous unitary group $\{W(\lambda\eta) | \lambda \in \mathbb{R}\}$ on $\mathscr{S}(H)$ with

$$W(\lambda\eta) \operatorname{Exp} \xi = \exp(-\tfrac{1}{2}\lambda^2 \|\eta\|^2 - \lambda \langle \eta | \xi \rangle) \operatorname{Exp}(\lambda\eta + \xi)$$

or in terms of the quadruplet notation introduced in (13)

$$W(\lambda\eta) = (\exp - \tfrac{1}{2}\lambda^2 \|\eta\|^2, \lambda\eta, -\lambda\eta, \mathbb{1}).$$

First we compute the cumulants P_Λ^Ψ. Clearly, by (37) and (49) we have for $g_i = (\xi_i, \theta_i) \in H_\sigma (i = 1, 2)$.

$$\begin{aligned}P_{(g_1)}^\Psi &= ic\theta_1 - \tfrac{1}{2}\langle \xi_1 | (R^* R + Q)\xi_1 \rangle \\ P_{(g_1, g_2)}^\Psi &= -\langle R\xi_1 | R\xi_2 \rangle.\end{aligned}$$

For $\#(A)>2$ we have that

$$\begin{aligned} P_A^\Psi &= \sum_{X\subset A} (-1)^{\#(A\setminus X)} d_X \\ &= \sum_{g\in A} \Big(- \sum_{Y\subset A\setminus\{g\}} (-1)^{\#(A\setminus Y)}\Big) P_{(g)}^\Psi \\ &\quad + \sum_{(g,g')\subset A} \Big(\sum_{Y\subset A\setminus\{g,g'\}} (-1)^{\#(A\setminus Y)}\Big) P_{(g,g')}^\Psi \\ &= 0. \end{aligned}$$

It is well known [BR] that

$$\langle \operatorname{Exp} 0 | B(\eta_1)\ldots B(\eta_{2n+1}) \operatorname{Exp} 0\rangle = 0$$

and

$$\langle \operatorname{Exp} 0 | B(\eta_1)\ldots B(\eta_{2n}) \operatorname{Exp} 0\rangle = \sum \langle \eta_{i_1} | \eta_{i_2}\rangle \ldots \langle \eta_{i_{2n-1}} | \eta_{i_{2n}}\rangle$$

where the summation runs over all partitions of $(1, 2, \ldots, 2n)$ into sets $(i_1, i_2), \ldots (i_{2n-1}, i_{2n})$ with $i_1 < i_3 < \ldots < i_{2n-1}$ and $i_{2k-1} < i_{2k}$ for $k=1, \ldots, n$.

Hence it suffices to show that

$$\langle \Omega^0 | \tilde{\Psi}(g_1)\ldots \tilde{\Psi}(g_{2n+1})\, \Omega^0\rangle = 0 \tag{53}$$

and

$$\langle \Omega^0 | \tilde{\Psi}(g_1)\ldots \tilde{\Psi}(g_{2n})\, \Omega^0\rangle = (-1)^n \sum \langle R\xi_{i_1} | R_{i_2}\xi\rangle \ldots \langle R\xi_{i_{2n-1}} | R\xi_{i_{2n}}\rangle \tag{54}$$

where the summation is taken as above and

$$\tilde{\Psi}(g) = \Psi(g) - (ic\theta - \tfrac{1}{2}\langle \xi | (R^*R+Q)\,\xi\rangle)\,\mathbb{1}.$$

To prove (53) and (54), consider $A=(g_1, \ldots, g_m)$. Then we make the following summation rearrangements:

$$\begin{aligned} \langle \Omega^0 | \tilde{\Psi}(A)\, \Omega^0\rangle &= \Big\langle \Omega^0 \Big| \prod_{g\in A}^{\rightarrow} (\Psi(g) - P_{(g)}^\Psi \mathbb{1})\, \Omega^0\Big\rangle \\ &= (-1)^{\#(A)} \sum_{X\subset A} (-1)^{\#(X)} \langle \Omega^0 | \Psi(X)\, \Omega^0\rangle \prod_{g\in X^c} P_{(g)}^\Psi. \end{aligned}$$

Since $\langle \Omega^0 | \Psi(X)\, \Omega^0\rangle = \sum_{p\in\mathscr{P}_X} \prod_{Y\in p} P_Y^\Psi$ and $P_Y^\Psi = 0$ if $\#(Y)>2$ this can be rewritten as

$$(-1)^{\#(A)} \sum_q \Big(\prod_{Z\in q} P_Z^\Psi\Big) \prod_{g\in q^c} P_{(g)}^\Psi \sum_{U\subset q^c} (-1)^{\#(q^c\setminus U)} \tag{55}$$

where the summation $\sum_q$ means the sum over all sets q of subsets of Λ of the form $q=\{(g_{i_1}, g_{i_2}), \ldots, (g_{i_{2k-1}}, g_{i_{2k}})\}$; $k \in \mathbb{N}$; $g_i \in \Lambda$, $i_{2l-1} < i_{2l}$ and $i_1 < i_3 < \ldots < i_{2k-1}$, and where q^c is a shorthandnotation for the set $\Lambda \setminus (\bigcup_{Z \in q} Z)$.

If $\#(\Lambda)$ is odd, then q^c contains at least one element and therefore (55) vanishes. On the other hand if $\#(\Lambda)$ is even, the only terms in the $\sum_q$-summation that contribute are those for which $q^c = \phi$. Hence we recover (54).

Finally, the n^{th} part of $(\mathscr{S}(H_R), \Psi_{R,Q,c}, \Omega^0)$ is easily seen to be $(\mathscr{S}(H_R), \Psi_{R/\sqrt{n}, Q/n, C/n}, \Omega^0)$.

References

[AcBa] Accardi, L., Bach, A.: Quantum central limit for strongly mixing random variables. Z. Wahrscheinlichkeitstheor. Verw. Geb. **68**, 393–402 (1985)

[Ar] Araki, H.: Factorisable representation of current algebra. Publ. Res. Inst. Math. Sci. **5**, 361–422 (1970)

[ArWo] Araki, H., Woods, E.J.: Complete boolean algebras of type I factors. Publ. Res. Inst. Math. Sci. **2**, 157–242 (1966)

[BR] Bratteli, O., Robinson, D.W.: Operator algebras and quantum statistical mechanics II. New York Berlin Heidelberg: Springer 1979

[CoGuHu] Cockroft, A.A., Gudder, P., Hudson, R.L.: A quantum mechanical functional limit theorem. J. Multivariate Anal. **7** 125–148, (1977)

[DVV] Demoen, B., Vanheuverzwijn, P., Verbeure, A.: Completely positive maps on the CCR algebra. Lett. Math. Phys. **2**, 161–166 (1977)

[FQ] Fannes, M., Quagebeur, J.: Central limits of product mappings between CAR-algebra. Publ. Res. Inst. Math. Sci. **19**, 469–491 (1983)

[GvW] Giri, N., von Waldenfels, W.: An algebraic version of the central limit theorem. Z. Wahrscheinlichkeitstheor. Verw. Geb. **42**, 129–134 (1978)

[Gui] Guichardet, A.: Symmetric hilbert spaces and related topics. (Lect. Notes Math. Vol. 261) Berlin Heidelberg New York: Springer 1972

[He] Heyer, H.: Probability measures on locally compact groups. New York Berlin Heidelberg: Springer 1977

[HuWiPe] Hudson, R.L., Wilkinson, M.D., Peck, S.B.: Translation-invariant integrals and Fourier analysis on Clifford and Grassmann algebras. J. Funct. Anal. **37**, 68–67 (1980)

[MaStr] Mathon, D., Streater, R.F.: Infinitely divisible representation of Clifford algebras. Z. Wahrscheinlichkeitstheor. Verw. Geb. **20**, 308–316 (1971)

[MZ] Mongomery, D., Zippin, L.: Topological transformation groups. New York: Interscience (Wiley) 1955

[Pa] Parthasarathy, K.R.: Infinitely divisible representations and positive definite functions on a compact group. Comm. Math. Phys. **16**, 148–156 (1970)

[PaSch] Parthasarathy, K.R., Schmidt, K.: Positive definite kernels, continuous tensor products, and central limit theorems of probability theory. (Lect. Notes vol. 272) Berlin Heidelberg New York: Springer 1972

[Q] Quaegebeur, J.: A noncommutative central limit theorem for CCR-algebras. J. Funct. Anal. **57**, 1–20 (1984)

[Sch] Schürmann, M.: Positive and conditionally positive linear functionals on coalgebras. In: Accardi, L., Waldenfels, W. von (eds.) Quantum probability and applications II. Proceedings, Heidelberg 1984. Lect. Notes Math. vol. 1136, pp 475–492 New York Berlin Heidelberg: Springer 1985

[Sti] Stinespring, W.F.: Positive functions on C^*-algebras, Proc. Amer. Math. Soc. **6**, 211–216 (1955)

[Str] Streater, R.F.: Current commutation relations, continuous tensor products and infinitely divisible group representations; Rend. Sci. Ist. Fis. Enrico Fermi **11**, 247–263 (1969)

[Tak] Takesaki, M.: Theory of operator algebras I. Berlin Heidelberg New York: Springer 1979

[vW] Waldenfels, W. von: An algebraic version of the central limit theorem in the anticommuting case. Z. Wahrscheinlichkeitstheor. Verw. Geb. **42**, 135–140 (1978)

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