Probab. Th. Rel. Fields 82, 545–563 (1989)

Probability Theory and Related Fields

© Springer-Verlag 1989

On the Infimum of the Local Time of a Wiener Process

Antónia Földes

Mathematical Institute of the Hungarian Academy of Sciences,
Reáltanoda u. 13–15, H-1053 Budapest, Hungary

Summary. Let $\{W(t),\ t \geqq 0\}$ be a standard Wiener process, and let $L(x,\ t)$ be its jointly continuous local time. Define

$$T_r = \inf\{t \geqq 0;\ L(0,t) \geqq r\}.$$

The upper and lower class behaviour of $\inf L(y,\ T_r)$ is investigated, where the infimum is taken on an interval, which is an appropriately chosen function of r.

Introduction

Let $\{W(t),\ t \geqq 0\}$ be a standard Wiener process and let $L(x,\ t)$ $(-\infty < x < \infty,\ 0 \leqq t)$ be its local time, which is jointly continuous a.s.,

Denote

$$L^*(t) = \sup_x L(x,t).$$

Kesten (1965) proved the following results:

$$\limsup_{t\to\infty} \frac{L^*(t)}{\sqrt{2t \log\log t}} = 1 \quad \text{a.s.}$$

$$\liminf_{t\to\infty} \frac{L^*(t)\sqrt{\log\log t}}{\sqrt{t}} = \gamma \quad \text{a.s.}$$

The exact value of γ was evaluated by Csáki and Földes (1986), namely $\gamma = j_1 \cdot \sqrt{2}$ where j_1 is the first positive root of the Bessel function $J_0(x) = I_0(ix)$.

Research supported by Hungarian National Foundation for Scientific Research Grant No. 1808

The problem of considering the infimum of $L(x, t)$ over an increasing interval was raised by Perkins (1981), who proved, with the notation

$$L_*(t, h(t)) = \inf_{|x| \leqq h(t)} L(x, t),$$

the following theorem.

Theorem (Perkins). *There is a nonincreasing function* $\theta(\alpha)$ $(\alpha \geqq 0)$ *such that*

(a) $\limsup_{t \to \infty} \frac{L_*\left(t, \alpha \sqrt{\frac{t}{2 \log \log t}}\right)}{\sqrt{2t \log \log t}} = \theta(\alpha)$ a.s. *for all* $\alpha \geqq 0$,

(b) $\theta(\alpha) \leqq \frac{1}{2\alpha} \wedge 1$ *for all* $\alpha \geqq 0$,

(c) $\theta(a) \geqq (1 - \sqrt{\alpha})^2$ *for all* $\alpha \leqq 1$.

The above results stimulated the investigation of the following problem.

Denote by $T_r = \inf\{t \geqq 0, L(0, t) \geqq r\}$, i.e., the first passage process associated with $L(0, t)$. What can one say about the upper and lower class behaviour of $\inf L(y, T_r)$ where the infimum is taken on an appropriately chosen interval. It is not at all surprising that the above question is strongly connected with the behaviour of

$$\phi_r = \max_{0 \leqq s \leqq T_r} W(s).$$

Theorem 1. *Let* $f(x)$ *be nondecreasing,* $\lim_{x \to \infty} f(x) = +\infty$ *and let*

$$I_1(f) = \int_1^\infty \frac{dx}{x f(x)}.$$

Then

$$P\{\phi_r > r f(r) \text{ i.o.}\} = \begin{cases} 0 & \text{if } I_1(f) < \infty \\ 1 & \text{if } I_1(f) = \infty. \end{cases}$$

Denote

$$\hat{\phi}_r = -\min_{0 < s < T_r} W(s)$$

and

$$\eta_r = \min(\phi_r, \hat{\phi}_r).$$

Theorem 2. *Let* $f(x)$ *be nondecreasing,* $\lim_{x \to \infty} f(x) = +\infty$, *and let*

$$I_2(f) = \int_1^\infty \frac{dx}{x f^2(x)}.$$

Then

$$P\{\eta_r \geqq r f(r) \text{ i.o.}\} = \begin{cases} 0 & \text{if } I_2(f) < \infty \\ 1 & \text{if } I_2(f) = \infty. \end{cases}$$

Theorem 3. *Let* $f(x)$ *be nondecreasing* $\lim_{x\to\infty} f(x) = +\infty$, $\frac{x}{f(x)} \nearrow +\infty$, *and*

$$I_3(f) = \int_1^\infty \frac{f(x)}{x} e^{-f(x)} dx.$$

Then

$$P\left\{\phi_r < \frac{r}{2f(r)} \text{ i.o.}\right\} = \begin{cases} 0 & \text{if } I_3(f) < \infty \\ 1 & \text{if } I_3(f) = \infty. \end{cases}$$

Remark 1. Theorem 3 holds for η_r too.

Remark. Theorem 1–3 can be formulated for simple symmetric random walk too. Denote by T_k^* the k-th return to the origin of the simple symmetric random walk S_n. It's easy to see, that

$$P(\max_{0 \leqq j \leqq T_k^*} S_j < l) = \left(1 - \frac{1}{2l}\right)^k.$$

Based on this observation it's not hard to prove the random walk analogons of Theorem 1–3. This observation gives the possibility of proving the above theorems first for random walk, and get the above theorems via invariance principle. However we do not know the random walk analogues of the next two theorems.

Theorem 4. *Let* $f(x)$ *be nondecreasing,* $\lim_{x\to\infty} f(x) = +\infty$. *If* $I_2(f) < +\infty$ *then*

$$P(\lim_{r\to\infty} \inf_{|x| < rf(r)} L(x, T_r) = 0) = 1.$$

If $I_2(f) = +\infty$ *then*

$$P\left(\overline{\lim_{r\to\infty}} \inf_{|x| < rf(r)} \frac{L(x, T_r)}{r} = 1\right) = 1.$$

Theorem 5. *For any* $0 < \delta \leqq 1$

$$P\left(\liminf_{r\to\infty} \inf_{|y| < \frac{r\delta}{2\log\log r}} \frac{L(y, T_r)}{r} = K(\delta)\right] = 1$$

where $K(\delta)=(1-\sqrt{\delta})^2$. *Moreover, in case* $\delta=1$ *we also have*

$$P\left(\liminf_{r\to\infty} \inf_{|y|<\frac{r}{2\log\log r}} L(y, T_r)=0\right)=1.$$

Remark. As a consequence of Theorem 5 one can get a new proof of Perkins' result (his statement c.). However I learnt from Perkins the following precise result:

Theorem A (Perkins). *For all* $\alpha>0$,

$$\overline{\lim_{t\to\infty}} \inf_{|x|<\alpha(t/2\log\log t)^{1/2}} L(x,t)(2t\log\log t)^{-1/2}=(\alpha^2+1)^{1/2}-\alpha \quad \text{a.s.}$$

Remark. On the other hand our theorems imply only the following less precise results.

Corollary 1. *For any* $\eta>0$

$$P(\underline{\lim_{u\to\infty}} \inf_{|y|\leq\sqrt{u}(\log u)^{-1}(\log\log u)^{-2-\eta}} L(y,u)\,u^{-1/2}\log u(\log\log u)^{1+\eta}=+\infty)=1.$$

Corollary 2. *For any* $\eta>0$

$$P(\underline{\lim_{u\to\infty}} \inf_{|y|\leq\sqrt{u}(\log u)^{-1/2}(\log\log u)^{-\frac{1}{2}+\eta}} L(y,u)=0)=1.$$

Remark. As Theorem A is far more significant than Theorem 4, we omit the proof of the latter, and incorporate Perkin's proof of Theorem A.

§ 2. Preliminary Result

In what follows we list some well-known properties of $L(x, T_r)$ which will be used later on (see e.g. Bass-Griffin, Ito-McKean, Knight)

(A) $\{L(x, T_r);\ x\geqq 0\}$ is a diffusion in x on natural scale, started from $L(0, T_r)=r$ with generator $2x\dfrac{\partial^2 y}{dx^2}$.

(B) $$E(\exp\{-\beta L(x, T_r)\})=\exp\left\{-\frac{\beta r}{1+2\beta x}\right\}. \tag{2.1}$$

(C) Considering the equation

$$2x\frac{\partial^2 y}{\partial x^2}=\alpha y \qquad \alpha>0 \tag{2.2}$$

the increasing and decreasing solutions are $\sqrt{x}I_1(\sqrt{2\alpha x})$ and $\sqrt{x}K_1(\sqrt{2\alpha x})$, where $I_1(\cdot)$ and $K_1(\cdot)$ are the modified Bessel functions of the first and third kind respectively.

(D) Denoting by

$$\tau_b = \inf\{x \geqq 0, L(x, T_r) = b\}, \tag{2.3}$$

$$E_r(\exp\{-\alpha\tau_b\}) = \begin{cases} \sqrt{\dfrac{r}{b}}\dfrac{I_1(\sqrt{2\alpha r})}{I_1(\sqrt{2\alpha b})} & \text{if } r \leqq b \\ \sqrt{\dfrac{r}{b}}\dfrac{K_1(\sqrt{2\alpha r})}{K_1(\sqrt{2\alpha b})} & \text{if } r \geqq b. \end{cases} \tag{2.4}$$

(E) $\{L(y, T_r),\ y \leqq 0\}$ is also a diffusion with the same generator as $\{L(y, T_r), y \geqq 0\}$, and the processes are independent from each other.

(F) Scale-change property

$$L(x, T_r) \stackrel{d}{=} \frac{1}{c} L(cx, T_{cr}).$$

The following statement is also well known. It can be obtained for instance from Theorem 4.3.6 of Knight.

Lemma 2.1.

$$P(\max_{0 \leqq s \leqq T_r} W(s) < x) = P(L(x, T_r) = 0) = \exp\left\{-\frac{r}{2x}\right\}.$$

Remark 2.2. For any nondecreasing function $f(x)$ for which $\lim_{x \to \infty} f(x) = +\infty$

$$\sum_{k=1}^{\infty} \frac{1}{f(\rho^k)} \quad \text{and} \quad \int_1^{\infty} \frac{1}{xf(x)}\,dx \qquad (\rho > 1)$$

are equiconvergent.

The following lemma is frequently used and its proof is routine. (See e.g. in Csáki-Erdös-Révész, Lemma 4 and 5.)

Lemma 2.3. *Let $f(x)$ $x > 0$ non-decreasing and positive, and $n_k = \exp\left\{\dfrac{k}{\log k}\right\}$. Then*

$$\int_1^{\infty} \frac{f(x)}{x} e^{-f(x)}\,dx \quad \text{and} \quad \sum_{k=2}^{\infty} e^{-f(n_k)}$$

are equiconvergent.

The following Borel-Cantelli type lemma is due to Erdös and Rényi (cf. Rényi [6], p. 391).

Lemma 2.4. *If* $\sum_k P(A_k) = +\infty$ *and*

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^{n}\sum_{l=1}^{n} P(A_k A_l)}{\left(\sum_{k=1}^{n} P(A_k)\right)^2} \leqq 1, \tag{2.5}$$

then

$$P(A_k \text{ i.o.}) = 1.$$

Lemma 2.5. *The Laplace transform of*

$$P(\inf_{0\leqq y\leqq \gamma r} L(y, T_r) > \alpha r) \quad (\textit{in } \gamma) \textit{ is for } \quad \alpha \leqq 1$$

$$L(\eta) = \int_0^\infty e^{-\eta\gamma} P(\inf_{0\leqq y\leqq \gamma r} L(y, T_r) > \alpha r)\, d\gamma = \frac{1}{\eta}\left(1 - \frac{1}{\sqrt{\alpha}} \frac{K_1(\sqrt{2\eta})}{K_1(\sqrt{2\eta\alpha})}\right). \tag{2.6}$$

Remark. According to (F)

$$P(\inf_{0\leqq y\leqq \gamma r} L(y, T_r) > \alpha r) = P(\inf_{0\leqq z\leqq \gamma} L(z, T_1) > \alpha),$$

hence independent of r.

Proof. Based on the result quoted in (A) and (D) we get

$$L(\eta) = \int_{\gamma=0}^{\infty} e^{-\eta\gamma} P_1(\tau_\alpha > \gamma)\, d\gamma = E_1\left(\int_0^{\tau_\alpha} e^{-\eta\gamma} d\gamma\right) = \frac{1}{\eta}(1 - E_1(e^{-\eta\tau_\alpha}))$$

which gives (2.6) by (2.4).

For the Laplace transform of the probability

$$P(\inf_{0\leqq y\leqq \gamma r} L(y, T_r) \leqq \alpha r) \quad (\text{as a function of } \gamma)$$

the following estimate holds.

Lemma 2.6. *Given any* $0 < \alpha < 1$, $c > 0$ *there exist* c_1 *and* c_2 *depending only on* c *such that*

$$\begin{aligned} c_1 \frac{e^{-\sqrt{2}(1-\sqrt{\alpha})\sqrt{\eta}}}{\eta \sqrt[4]{\alpha}} &\leqq \int_0^\infty e^{-\eta\gamma} P(\inf_{0\leqq y\leqq \gamma r} L(y, T_r) \leqq \alpha r)\, d\gamma \\ &\leqq \frac{c_2\, e^{-\sqrt{2}(1-\sqrt{\alpha})\sqrt{\eta}}}{\eta\sqrt[4]{\alpha}} \end{aligned} \tag{2.7}$$

holds if $\eta > \dfrac{c}{\alpha}$.

Proof. From (2.6) it is obvious that for $0<\alpha<1$

$$\int_0^\infty e^{-\eta\gamma} P(\inf_{0\leqq y\leqq \gamma r} L(y, T_r)\leqq \alpha r)\, d\gamma = \frac{1}{\eta\sqrt{\alpha}} \frac{K_1(\sqrt{2\eta})}{K_1(\sqrt{2\eta\alpha})}. \tag{2.8}$$

From the asymptotic expansion of Erdélyi et al. 1953 (Vol. 2, p. 24, formula (4))

$$K_1(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}\left(1+0\left(\frac{1}{|z|}\right)\right) \qquad z\to\infty, \tag{2.9}$$

(2.7) follows easy computation.

Let us define the function $F(h,\alpha)$ by

$$F(h,\alpha) = P(\inf_{0\leqq y\leqq hr} L(y, T_r)\leqq \alpha r) = P(\inf_{0\leqq y\leqq h} L(y, T_1)\leqq \alpha).$$

then clearly $F(h,\alpha)$ is monotone increasing in h.

Lemma 2.7. *For any* $0<\alpha<1$, $h>0$, *such that* $\dfrac{\alpha(1-\sqrt{\alpha})^2}{2h^2}>1$ *we have*

$$F(h,\alpha) \leqq \frac{c_2}{\sqrt[4]{\alpha}} \exp\left\{-\frac{1-\sqrt{\alpha})^2}{2h}\right\} \tag{2.10}$$

where c_2 *is the constant of Lemma* 2.6.

Proof. Starting from the obvious equality

$$F(h,\alpha) = F(h,\alpha)\cdot \eta e^{h\eta} \int_h^\infty e^{-\eta u}\, du \leqq \eta e^{\eta h} \int_h^\infty F(u,\alpha)\, e^{-\eta u}\, du$$

$$\leqq \eta e^{\eta h} \int_0^\infty F(u,\alpha)\, e^{-\eta u}\, du \leqq e^{\eta h} c_2 \exp\{-\sqrt{2}(1-\sqrt{\alpha})\sqrt{\eta}\}\cdot \frac{1}{\sqrt[4]{\alpha}}$$

by the monotonicity of $F(u,\alpha)$ and (2.7). Computing the minimum of this function in η we get (2.10), where the minimum is taken for $\eta = \dfrac{(1-\sqrt{\alpha})^2}{2h^2}$.

Lemma 2.8. *For any given* $\varepsilon^*>0$ *there is a* $h_0(\varepsilon^*)$ *such that if* $\alpha(1-\sqrt{\alpha})^2/2h^2>1$, $0<\alpha<1-\varepsilon^*$ *and* $h<h_0(\varepsilon^*)$, *then*

$$F(h,\alpha) \geqq D \exp\left\{-\frac{(1-\sqrt{\alpha})^2}{2h}(1+\varepsilon^*)\right\} \tag{2.11}$$

where D *is an absolute constant.*

Proof. For a given ε^* choose an $\varepsilon>0$ such that

$$4\varepsilon<\varepsilon^*, \quad \text{and} \quad (1-3\varepsilon)(2\varepsilon(1+\varepsilon))^{-1}\geqq 1 \tag{2.12}$$

should hold. Then $\varepsilon=\varepsilon(\varepsilon^*)$ is fixed, when ε^* is fixed. Introduce the following notations

$$\int_0^\infty e^{-\eta\gamma}F(\gamma,\alpha)\,d\gamma=\int_0^{h(1-\varepsilon)}+\int_{h(1-\varepsilon)}^{h(1+\varepsilon)}+\int_{h(1+\varepsilon)}^{h(2+\varepsilon)}+\int_{h(2+\varepsilon)}^{\infty}$$
$$=I_1+I_2+I_3+I_4 \tag{2.13}$$

Put $\eta=\dfrac{(1-\sqrt{\alpha})^2}{2h^2}$. According to (2.7)

$$\int_0^\infty \exp\left\{-\frac{\gamma(1-\sqrt{\alpha})^2}{2h^2}\right\}F(\gamma,\alpha)\,d\gamma\geqq\frac{2c_1h^2}{(1-\sqrt{\alpha})^2\sqrt[4]{\alpha}}\exp\left\{-\frac{(1-\sqrt{\alpha})^2}{h}\right\}. \tag{2.14}$$

Denote

$$D=\min_{0<\alpha<1}\frac{2c_1}{(1-\sqrt{\alpha})^2\sqrt[4]{\alpha}}$$

to get

$$\int_0^\infty \exp\left\{-\frac{\gamma(1-\sqrt{\alpha})^2}{2h^2}\right\}F(\gamma,\alpha)\,d\gamma\geqq Dh^2\exp\left\{-\frac{(1-\sqrt{\alpha})^2}{h}\right\}. \tag{2.15}$$

In order to get a lower estimate for $F(h,\alpha)$, estimate I_1, I_2, I_3 and I_4 from above. Based on (2.10)

$$I_1\leqq\int_0^{h(1-\varepsilon)}\frac{c_2}{\sqrt[4]{\alpha}}\exp\left\{-\frac{(1-\sqrt{\alpha})^2}{2\gamma}-\frac{(1-\sqrt{\alpha})^2}{2h^2}\gamma\right\}d\gamma. \tag{2.16}$$

Observe that the integrand in (2.16) is monotone increasing for $\gamma<h$, implying

$$I_1\leqq\frac{c_2}{\sqrt[4]{\alpha}}h\exp\left\{-\frac{(1-\sqrt{\alpha})^2}{2h}\left(1-\varepsilon+\frac{1}{1-\varepsilon}\right)\right\}. \tag{2.17}$$

Now being $1-\varepsilon+\dfrac{1}{1-\varepsilon}=2+\dfrac{\varepsilon^2}{1-\varepsilon}$,

$$I_1\leqq\frac{c_2}{\sqrt[4]{\alpha}}h\exp\left\{-\frac{\varepsilon^2}{1-\varepsilon}\frac{(1-\sqrt{\alpha})^2}{2h}\right\}\exp\left\{-\frac{(1-\sqrt{\alpha})^2}{h}\right\}. \tag{2.18}$$

From the condition $\frac{\alpha(1-\sqrt{\alpha})^2}{2h^2}>1$ we get that $\alpha^{-1/4}<h^{-1/2}$. Now being $0<\alpha<1-\varepsilon^*$

$$(1-\sqrt{a})^2 \geqq (1-\sqrt{1-\varepsilon^*})^2=\bar{\varepsilon}>0. \tag{2.19}$$

Moreover

$$c_2\sqrt{h}\exp\left\{-\frac{\varepsilon^2}{1-\varepsilon}\cdot\frac{\bar{\varepsilon}}{2h}\right\}<D\varepsilon h^2$$

if h is small enough, say $h<h_1(\varepsilon^*)$ (as both ε and $\bar{\varepsilon}$ is a function of ε^*). Hence

$$I_1 \leqq D\varepsilon h^2 \exp\left\{-\frac{(1-\sqrt{\alpha})^2}{h}\right\} \tag{2.20}$$

if $h<h_1(\varepsilon^*)$. Being $F(\gamma,\alpha)$ monotone increasing in γ one gets for I_2

$$I_2 \leqq 2h\varepsilon F(h(1+\varepsilon),\alpha)\exp\left\{-\frac{(1-\sqrt{\alpha})^2}{2h}(1-\varepsilon)\right\}. \tag{2.21}$$

The estimation of I_3 is similar to I_1, by (2.10)

$$I_3 \leqq \frac{c_2}{\sqrt[4]{\alpha}}\int_{h(1+\varepsilon)}^{h(2+\varepsilon)} \exp\left\{-\frac{(1-\sqrt{\alpha})^2}{2\gamma}-\frac{(1-\sqrt{\alpha})^2}{2h^2}\gamma\right\}d\gamma.$$

The above integrand is monotone decreasing in γ leading to the estimation

$$I_3 \leqq \frac{c_2}{\sqrt[4]{\alpha}}h\exp\left\{-\frac{(1-\sqrt{\alpha})^2}{2h}\left(\frac{1}{1+\varepsilon}+1+\varepsilon\right)\right\}.$$

Using the same argument as for I_1 we get that

$$I_3 \leqq \varepsilon h^2 D\exp\left\{-\frac{(1-\sqrt{\alpha})^2}{h}\right\} \tag{2.22}$$

if $h \leqq h_2(\varepsilon^*)$.

To estimate I_4 observe that $F(\gamma,\alpha)\leqq 1$ (being a probability). Thus

$$I_4 \leqq \int_{h(2+\varepsilon)}^{\infty} e^{-\eta\gamma}d\gamma=\int_{h(2+\varepsilon)}^{\infty}\exp\left\{-\frac{(1-\sqrt{\alpha})^2}{2h^2}\right\}d\gamma$$
$$=\frac{2h^2}{(1+\sqrt{\alpha})^2}\exp\left\{-\frac{(1-\sqrt{\alpha})^2}{2h}\varepsilon\right\}\exp\left\{-\frac{(1-\sqrt{\alpha})^2}{h}\right\}.$$

According to (2.19) we have

$$I_4 \leqq \frac{2h^2}{\bar{\varepsilon}} \exp\left\{-\frac{\bar{\varepsilon}\varepsilon}{2h}\right\} \exp\left\{-\frac{(1-\sqrt{\alpha})^2}{h}\right\}$$

$$\leqq \varepsilon h^2 D \exp\left\{-\frac{(1-\sqrt{\alpha})^2}{h}\right\} \tag{2.23}$$

if $h \leqq h_3(\varepsilon^*)$. Now from (2.15)–(2.23) if $h \leqq \min_{1 \leqq i \leqq 3} h_i(\varepsilon^*)$ we have

$$Dh^2 \exp\left\{-\frac{(1-\sqrt{\alpha})^2}{h}\right\} \leqq 3\varepsilon D h^2 \exp\left\{-\frac{(1-\sqrt{\alpha})^2}{h}\right\}$$

$$+2\varepsilon h F(h(1+\varepsilon),\alpha) \exp\left\{-\frac{(1-\sqrt{\alpha})^2}{2h}(1-\varepsilon)\right\} \tag{2.24}$$

implying

$$F(h(1+\varepsilon),\alpha) \geqq \frac{1-3\varepsilon}{2\varepsilon} Dh \exp\left\{-\frac{(1-\sqrt{\alpha})^2}{2h}(1+\varepsilon)\right\}.$$

Denote $h(1+\varepsilon)=h^*$, then by (2.12)

$$F(h^*,\alpha) \geqq \frac{1-3\varepsilon}{2\varepsilon(1+\varepsilon)} h^* D \exp\left\{-\frac{(1-\sqrt{\alpha})^2}{2h^*}(1+\varepsilon)^2\right\}$$

$$\geqq h^* D \exp\left\{-\frac{(1-\sqrt{\alpha})^2}{2h^*}(1+\varepsilon)^2\right\}$$

$$= D \exp\left\{-\frac{(1-\sqrt{\alpha})^2}{2h^*}(1+\varepsilon)\right\} \cdot h^* \exp\left\{\frac{(1-\sqrt{\alpha})^2}{2h^*}((1+\varepsilon^*)-(1+\varepsilon)^2)\right\}. \tag{2.25}$$

Now again by (2.12) and (2.19)

$$h^* \exp\left\{\frac{(1-\sqrt{\alpha})^2}{2h^*}[(1+\varepsilon^*)-(1+\varepsilon)^2]\right\} \geqq \exp\left\{\frac{(1-\sqrt{\alpha})^2}{2h^*}(\varepsilon^*-2\varepsilon-\varepsilon^2)\right\}$$

$$\geqq h^* \exp\left\{\frac{\bar{\varepsilon}(\varepsilon^*-3\varepsilon)}{2h^*}\right\} \geqq h^* \exp\left\{\frac{\bar{\varepsilon}\varepsilon}{2h^*}\right\} = h(1+\varepsilon)\exp\left\{\frac{\bar{\varepsilon}\varepsilon}{2h(1+\varepsilon)}\right\} > 1$$

when $h < h_4(\varepsilon^*)$ (as ε and $\bar{\varepsilon}$ is a function of ε^*). Define $h_0(\varepsilon^*) = \min_{1 \leqq i \leqq 4} h_i(\varepsilon^*)$. Then we get from (2.25), that

$$F(h^*,\alpha) \geqq D \exp\left\{-\frac{(1-\sqrt{\alpha})^2}{2h^*}(1+\varepsilon^*)\right\}$$

if $h^* < h_0(\varepsilon^*)$ (as $h^* < h_0(\varepsilon^*)$ implies that $h < h_0(\varepsilon^*)$) and the lemma is proved.

In our next lemma we give the well-known upper and lower class results for T_r (see Chung and Hunt, Fristedt).

Lemma 2.9.

(a) *If* $\int_1^\infty \frac{1}{\sqrt{h(s)}} ds = +\infty$ *then* $\overline{\lim}_{s\to\infty} \frac{T_s}{h(s)} = +\infty$ a.s.

(b) *If* $\int_1^\infty \frac{1}{\sqrt{h(s)}} ds < +\infty$ *then* $\overline{\lim}_{s\to\infty} \frac{T_s}{h(s)} = 0$ a.s.

(c) $P\left(T_s \leqq \frac{\beta s^2}{\log\log s} \text{ i.o.}\right) = \begin{cases} 1 & \text{if } \beta > \frac{1}{2} \\ 0 & \text{if } \beta < \frac{1}{2} \end{cases}$

§ 3. Proofs of the Results

The proof of the exact $\overline{\lim}$ result is based on the theory of Donsker and Varadhan (1977); the following proof is due to E. Perkins (letter to the Author).

Proof of Theorem A. Let $\underset{=}{A} = \{f: (-\infty, \infty) \to [0, \infty): f$ absolutely continuous,

$$\int_{-\infty}^{\infty} f(y)\,dy \leqq 1, \quad I(f) = \tfrac{1}{8} \int_{-\infty}^{\infty} f'(y)^2/f(y)\,dy \leqq 1\}.$$

Let $\rho_t = (t/\log\log t)^{1/2}$. Theorem 3.9 of Donsker-Varadhan (1977) implies that w.p.1 the limit points (in the compact-open topology on $C(\mathbf{R}, \mathbf{R})$) of $\{x \to (t\log\log t)^{-1/2} L(\rho_t x, t)\}$ is $\underset{=}{A}$, and hence (apply Cor. 3.11 of Donsk.-Var. with $\Phi(g) = \inf_{|s| \leqq \alpha/\sqrt{2}} g(x)$)

$$\begin{aligned} &\overline{\lim}_{t\to\infty} \inf_{|x| \leqq \alpha(t/2\log\log t)^{1/2}} L(x,t)(2t\log\log t)^{-1/2} \\ &= \frac{1}{\sqrt{2}} \sup\{\inf_{|z| \leqq \alpha\sqrt{2}} f(z): f \in \underset{=}{A}) \\ &\equiv c(\alpha\sqrt{2})/\sqrt{2}. \end{aligned} \tag{3.1}$$

Fix $f \in \underset{=}{A}$ and let $c = \inf_{|z| \leqq \alpha'} f(z)$, $\alpha' = \alpha/\sqrt{2}$. Then

$$\begin{aligned} 1 &\geqq \tfrac{1}{8}\Big[\int_{|z|>\alpha'} (f'/f)^2 f(z)\,dz \Big/ \int_{|z|>\alpha'} f(z)\,dz\Big] \int_{|z|>\alpha'} f(z)\,dz \qquad (0/0 = 0) \\ &\geqq \tfrac{1}{8}\Big[\int_{|z|>\alpha'} |f'/f| f(z)\,dz \Big/ \int_{|z|>\alpha'} f(z)\,dz\Big]^2 \int_{|z|>\alpha'} f(z)\,dz \qquad \text{(Jensen)} \\ &\geqq \tfrac{1}{8}\Big[\Big(\int_{|z|>\alpha'} |f'(z)|\,dz\Big)^2 \Big/ \int_{|z|>\alpha'} f(z)\,dz\Big] \\ &\geqq \tfrac{1}{8}(2c)^2/(1 - \int_{|z|<\alpha'} f(z)\,dz) \end{aligned}$$

(Note that $\int_{\alpha'}^{\infty} |f'(z)|\,dz \geqq f(\alpha') \geqq c$). Hence

$$1 \geqq (c^2/2)/(1-2\alpha' c) \Rightarrow c^2 \leqq 2-4c\alpha' \Rightarrow c \leqq -2\alpha' + (4\alpha'^2+2)^{1/2} \equiv c_0(\alpha'). \tag{3.2}$$

Therefore $\sup\{\inf_{|z|\leqq\alpha'} f(z): f\in \underline{A}\} \leqq -2\alpha' + (4\alpha'^2+2)^{1/2}$.

To prove equality in (3.2) we simply have to find the right f in $\underline{A}$. An examination of the above argument shows we want $f'/f = \text{constant}$ on $|y|>\alpha'$, f constant on $|y|\leqq\alpha'$ and $I(f)=1$. This leads us to define

$$f(z) = \begin{cases} c_0(\alpha') & \text{if } |x|\leqq\alpha' \\ c_0(\alpha')\,e^{-\beta(x-\alpha')} & \text{if } x>\alpha',\ \beta = 2c_0(\alpha')(1-2\alpha' c_0(\alpha'))^{-1} \\ c_0(\alpha')\,e^{+\beta(x+\alpha')} & \text{if } x<-\alpha'. \end{cases}$$

A routine calculation shows $f\in\underline{A}$ and hence

$$\sup\{\inf_{|z|\leqq\alpha'} f(z):\ f\in\underline{A}\} = c_0(\alpha').$$

Therefore $c(\alpha/\sqrt{2})/\sqrt{2} = (-\alpha\sqrt{2} + (2\alpha^2+2)^{1/2})/\sqrt{2} = -\alpha + (\alpha^2+1)^{1/2}$ and (3.1) gives the result.

Proof of Theorem 1. Convergent part: According to Lemma 2.1

$$\frac{1}{4x} < P(\max_{0\leqq s\leqq T_r} W(s) > rx) = 1 - \exp\left\{-\frac{1}{2x}\right\} < \frac{1}{2x} \tag{3.3}$$

for $x>\frac{1}{2}$. Let $r_k = e^k$, $k=1, 2, \ldots$. Then

$$P(\max_{0\leqq s\leqq T_{r_{k+1}}} W(s) > r_k f(r_k)) = P(\max_{0<s\leqq T_{r_{k+1}}} W(s) > r_{k+1}\frac{r_k}{r_{k+1}} f(r_k)) \leqq \frac{r_{k+1}}{r_k}\,\frac{1}{2f(r_k)} = \frac{e}{2}\,\frac{1}{f(r_k)}.$$

Hence the convergence of $I_1(f)$, Remark 2.2 and Borel Cantelli lemma implies that for $k>k_0(\omega)$

$$\max_{0\leqq s\leqq T_{r_{k+1}}} W(s) \leqq r_k f(r_k)$$

implying for $r_k < r \leqq r_{k+1}$

$$\max_{0\leqq s\leqq T_r} W(s) < r f(r) \tag{3.4}$$

by monotonicity.

Divergent part:

Let $r_k = e^k$, $k=1, 2, \ldots$. Let us define the events

$$B_k = \max_{0 \leqq s \leqq T_{r_k}} W(s) > r_k f(r_k), \qquad B_k^* = \max_{T_{r_{k-1}} \leqq s \leqq T_{r_k}} W(s) > r_k f(r_k).$$

Then clearly $\{B_k^*\}_{k=1}^{\infty}$ are independent and $B_k^* \subseteq B_k$. On the other hand, observe that

$$\begin{aligned} P(B_k^*) &= P(\max_{0 \leqq s \leqq T_{r_k - r_{k-1}}} W(s) > r_k f(r_k)) \\ &= P(\max_{0 \leqq s \leqq T_{r_k - r_{k-1}}} W(s) > (r_k - r_{k-1}) \cdot \frac{r_k}{r_k - r_{k-1}} f(r_k)) \\ &\geqq \frac{1}{4} \frac{r_k - r_{k-1}}{r_k} \frac{1}{f(r_k)} = \frac{1}{4} \frac{e-1}{e} \frac{1}{f(e^k)} \end{aligned} \tag{3.5}$$

by (3.3). Using again Remark 2.2 and the Borel-Cantelli lemma we get that if $I_1(f) = +\infty$

$$P(B_k^* \text{ i.o.}) = 1 \qquad \text{hence } P(B_k \text{ i.o.}) = 1.$$

Proof of Theorem 2. Observe that

$$\begin{aligned} P(\eta_r \geqq rx) &= P(\min(\phi_r, \hat{\phi}_r) \geqq rx) \\ &= P(\max_{0 \leqq s \leqq T_r} W(s) \geqq rx, -\min_{0 < s \leqq T_r} W(s) \geqq rx) \\ &= P(\max_{0 \leqq s \leqq T_r} W(s) \geqq rx, \min_{0 \leqq s \leqq T_r} W(s) \leqq -rx) \\ &= P(L(xr, T_r) \neq 0, L(-xr, T_r) \neq 0) = P^2(L(xr, T_r) \neq 0), \end{aligned}$$

where the last equality holds by property (E). Consequently,

$$\frac{1}{16x^2} \leqq P(\eta_r \geqq rx) = \left(1 - e^{-\frac{1}{2x}}\right)^2 \leqq \frac{1}{4x^2} \tag{3.6}$$

where the left-hand side inequality holds for $x > \frac{1}{2}$. Based on (3.6) a repetition of the argument of the proof of Theorem 1 gives the proof of Theorem 2.

The method of the next proof goes back to the well-known Kolmogorov-Erdös-Feller-Petrovski integral test. A detailed version of this technique can be found e.g. in Csáki, Erdös and Révész (1985). Therefore we give only a brief

Outline of the Proof of Theorem 3. Let $r_1 = 1$, $r_k = \exp\left\{\frac{k}{\log k}\right\}$ $(k=2,3,\ldots)$. Then according to Lemma 2.3 $I_3(f)$ and $\sum_k \exp\{-f(r_k)\}$ converge or diverge together.

Convergent part:

Split the indices into the following two sets:

$$A_f=\{k; f(r_k)\leqq C\log\log r_k\},\qquad B_f=\{k; f(r_k)>C\log\log r_k\} \tag{3.7}$$

where $C>2$ is fixed. Let

$$D_k=\left\{\max_{0\leqq s\leqq T_{r_k}} W(s)<\frac{1}{f(r_{k+1})}\frac{r_{k+1}}{2}\right\}$$

by Lemma 2.1

$$P(D_k)=\exp\left\{-\frac{r_k}{r_{k+1}}f(r_{k+1})\right\}. \tag{3.8}$$

Being

$$\frac{r_k}{r_{k+1}}>\exp\left\{-\frac{1}{\log(k+1)}\right\} \tag{3.9}$$

it is easy to see that

$$P(D_k)\leqq\begin{cases}\left(\dfrac{\log(k+1)}{k+1}\right)^{\frac{C}{2}} & \text{if } k+1\in B_f\\ \exp\{-f(r_{k+1})+C\} & \text{if } k+1\in A_f.\end{cases} \tag{3.10}$$

(3.10) implies our statement by the Borel-Cantelli lemma.

Divergent part:

It is easy to see that without the loss of generality one can assume that

$$\tfrac{1}{5}\log\log n\leqq f(n)\leqq 2\log\log n. \tag{3.11}$$

(see in Csáki, Erdös and Révész Lemma 9). Let

$$A_k=\left(\max_{0\leqq s\leqq T_{r_k}} W(s)<\frac{r_k}{2f(r_k)}\right). \tag{3.12}$$

By Lemma 2.1

$$P(A_k)=\exp\{-f(r_k)\}. \tag{3.13}$$

It is easy to observe that for $k<l$

$$P(A_k A_l)\leqq P(A_k)\,P\left(\max_{0\leqq s\leqq T_{r_l-r_k}} W(s)<\frac{r_l}{2f(r_l)}\right)=P(A_k)\exp\left\{-\frac{r_l-r_k}{r_l}f(r_l)\right\}. \tag{3.14}$$

Now for fixed k, split the indices l $(k<l\leqq n)$ into three parts

$$L_1=\{l:\ 0<l-k\leqq \log l\}$$
$$L_2=\{l:\ \log\ l<l-k<\log^2 l\}$$
$$L_3=\{l:\ \log^2 l<l-k\}.$$

Based on (3.11)–(3.14) one has to show that for arbitrary $k>k_0$

$$\sum_{l\in L_1} P(A_k A_l)<CP(A_k) \tag{3.15}$$

$$\sum_{l\in L_2} P(A_k A_l)<CP(A_k) \tag{3.16}$$

$$P(A_k A_l)<(1+\varepsilon)\,P(A_k)\,P(A_l) \quad \text{for } l\in L_3. \tag{3.17}$$

Clearly (3.15)–(3.17) and Lemma 2.4 imply our statement. To see that the above three conditions are fulfilled, one has to show that for $l\in L_1$

$$\frac{r_l-r_k}{r_l}f(r_l)\geqq C'(l-k) \tag{3.18}$$

implying (3.15), for $l\in L_2$

$$\frac{r_l-r_k}{r_l}\geqq C'' \tag{3.19}$$

which together with (3.11) implies (3.16).

Finally for $l\in L_3$, it is easy to show that

$$\frac{r_k}{r_l}f(r_l)\to 0 \quad \text{as } k\to\infty, \tag{3.20}$$

implying (3.17) for large enough k and hence the theorem.

Proof of Theorem 5. The $\delta=1$ case is a trivial consequence of the divergent part of Theorem 3. Hence we only deal with $0<\delta<1$.

Convergent part:

Let $r_k=\rho^k$, $\rho>1$. For any $0<\delta<1$ and for an arbitrary small $\varepsilon>0$ define

$$A_k=\left\{\inf_{|y|<\frac{r_{k+1}\delta}{2\log\log r_{k+1}}} L(y,T_{r_k})<(1-\varepsilon)\,K(\delta)\,r_{k+1}\right\} \tag{3.21}$$

where $K(\delta)=(1-\sqrt{\delta})^2$. If $\sum_{k=1}^{\infty} P(A_k)<\infty$ then for $k>k_0(\omega)$ and any $r_k<r\leqq r_{k+1}$

$$\inf_{|y|<\frac{r\delta}{2\log\log r}} L(y,T_r)>\inf_{|y|<\frac{r_{k+1}\delta}{2\log\log r_{k+1}}} L(y,T_{r_k})>(1-\varepsilon)\,K(\delta)\,r_k$$
$$>(1-\varepsilon)\,K(\delta)\,r \quad \text{a.s.} \tag{3.22}$$

which implies our statement. Thus it is enough to prove the convergence of $\sum_{k=1}^{\infty} P(A_k)$. To get an upper bound for $P(A_k)$ one has to apply Lemma 2.7. With the notation of Lemma 2.7 and using property E

$$P(A_k) \leqq 2F\left(\frac{\rho\delta}{2\log\log r_{k+1}}, (1-\varepsilon)\rho K(\delta)\right). \tag{3.23}$$

By (2.10) if k is big enough, then

$$\begin{aligned} P(A_k) &\leqq \frac{c_2}{\sqrt[4]{(1-\varepsilon)\rho K(\delta)}} \exp\left\{-\frac{(1-\sqrt{(1-\varepsilon)\rho K(\delta)})^2}{\rho\delta}\log\log r_{k+1}\right\} \\ &\leqq C^* \exp\left\{-\frac{(1-(1-\sqrt{\delta})\sqrt{\rho}(1-\varepsilon'))^2}{\rho\delta}\log\log r_{k+1}\right\} \\ &= C^*((k+1)\log\rho)^{-B(\rho,\delta,\varepsilon)} \end{aligned} \tag{3.24}$$

where C^* is a constant depending on ε, ρ and δ but not on k, $\varepsilon' = 1-\sqrt{1-\varepsilon}$. What remains to show is, that if $\rho > 1$ is small enough, then

$$B(\rho, \delta, \varepsilon) = \frac{(1-(1-\sqrt{\delta})\sqrt{\rho}(1-\varepsilon'))^2}{\rho\delta} \geqq 1+\phi \tag{3.25}$$

with some $\phi > 0$. But (3.25) is equivalent to

$$1-(1-\sqrt{\delta})\sqrt{\rho}(1-\varepsilon') \geqq (1+\psi)\sqrt{\rho\delta} \tag{3.26}$$

(with a convenient $\psi > 0$). (3.26) holds if

$$\sqrt{\rho} < \frac{1}{1-\varepsilon'(1-\sqrt{\delta})+\psi\sqrt{\delta}}. \tag{3.27}$$

Being $0<\delta<1$, for an arbitrary $\varepsilon'>0$ one can choose a small enough ψ such that the right-hand side of (3.27) should be greater than 1. Thus one can choose a $\rho > 1$ satisfying (3.27), hence (3.25). This implies the convergence of $\sum_{k=1}^{\infty} P(A_k)$.

Divergent part:

Let $0<\delta<1$ fixed. Choose an $\varepsilon(\delta)>0$ such that $\left(1+\frac{\varepsilon(\delta)}{2}\right)K(\delta)<1$.

Let $r_k = e^{\rho k \log k}$, where ρ will be chosen later on. One has to show that for any $\varepsilon > 0$, $0<\varepsilon<\varepsilon(\delta)$ for the events

$$A_k = \left\{\inf_{|y| < \frac{r_k\delta}{2\log\log r_k}} L(y, T_{r_k}) < (1+\varepsilon)K(\delta)r_k\right\}$$

$$P(A_k \text{ i.o.}) = 1. \tag{3.28}$$

To this end first observe that

$$\inf_{|y|<\frac{r_{k+1}\delta}{2\log\log r_{k+1}}} L(y, T_{r_{k+1}}) \leq \inf_{|y|<\frac{r_{k+1}\delta}{2\log\log r_{k+1}}} (L(y, T_{r_{k+1}})-L(y, T_{r_k}))+\sup_y L(y, T_{r_k}). \tag{3.29}$$

For the second term one can easily show from

$$P(\sup_{0<y} L(y, T_r)>rh)=\frac{1}{h} \quad (h>1) \tag{3.30}$$

(which is a simple consequence of property (A)) and the Borel-Cantelli lemma that for any $\varepsilon'>0$ and big enough r

$$\sup_y L(y, T_r)\leqq r(\log r)^{1+\varepsilon'} \quad \text{a.s.} \tag{3.31}$$

First we show that if ρ is chosen appropriately in the definition of r_k, then

$$\sup_y L(y, T_{r_k})\leqq r_{k+1}\frac{\varepsilon}{2}K(\delta) \tag{3.32}$$

if k is big enough. To see (3.32), observe that according to (3.31)

$$\sup_y L(y, T_{r_k})\leqq r_k(\log r_k)^{1+\varepsilon'}=e^{\rho k\log k}(\rho k\log k)^{1+\varepsilon'} \tag{3.33}$$

and

$$r_{k+1}\frac{\varepsilon}{2}K(\delta)\geqq(e^{\rho(k+1)\log k})\frac{\varepsilon}{2}K(\delta) \tag{3.34}$$

furthermore

$$(\rho k\log k)^{1+\varepsilon'}\leqq(e^{\rho\log k})\frac{\varepsilon}{2}K(\delta)=k^{\rho}\frac{\varepsilon}{2}K(\delta) \tag{3.35}$$

if ρ is big enough, implying (3.32).

On the other hand the events

$$B_k=\left\{\inf_{|y|<\frac{r_{k+1}\delta}{2\log\log r_{k+1}}} (L(y, T_{r_{k+1}})-L(y, T_{r_k}))<\left(1+\frac{\varepsilon}{2}\right)K(\delta)\, r_{k+1}\right\}, \quad k=0,1,2,\ldots$$

and independent, and

$$P(B_k)=P\left(\inf_{|y|<\frac{r_{k+1}\delta}{2\log\log r_{k+1}}} L(y, T_{r_{k+1}-r_k})<\left(1+\frac{\varepsilon}{2}\right)K(\delta)\, r_{k+1}\right\}. \tag{3.36}$$

In order to get a lower estimate for $P(B_k)$ we use Lemma 2.8. Using again property (E) we get that if k is big enough, then

$$P(B_k) > \frac{3}{2} F\left(\frac{\delta}{2\log\log r_{k+1}} \frac{r_{k+1}}{r_{k+1}-r_k}, \left(1+\frac{\varepsilon}{2}\right) K(\delta) \frac{r_{k+1}}{r_{k+1}-r_k}\right) \tag{3.37}$$

where $F(h, \alpha)$ was defined in Lemma 2.7. Observe first that

$$\frac{r_{k+1}}{r_{k+1}-r_k} = 1+\eta_k \quad \text{where } \eta_k \to 0. \tag{3.38}$$

This implies by (2.11) with a small enough ε^* and $k > k_0(\varepsilon^*)$ that

$$P(B_k) > C \exp\left\{-\frac{\left(1-\sqrt{\left(1-\frac{\varepsilon}{2}\right)(1+\eta_k) K(\delta)}\right)^2}{\delta}(1+\varepsilon^*) \log\log r_{k+1}\right\} \tag{3.39}$$

To see that $\sum_k P(B_k) = +\infty$ we have to show that for any $0<\delta<1$ fixed, and for an arbitrary $0<\varepsilon<\varepsilon(\delta)$, one can find a small enough $\varepsilon^*>0$ such that

$$\frac{\left(1-\sqrt{\left(1+\frac{\varepsilon}{2}\right)(1+\eta_k) K(\delta)}\right)^2}{\delta}(1+\varepsilon^*) < 1. \tag{3.40}$$

As $0<\delta<1$ and $\eta_k \to 0$, it can be easily seen that (3.40) holds if k is big enough and $\varepsilon^*>0$ is small enough. Thus for k big enough

$$P(B_k) \geqq C \exp\{-\log\log r_{k+1}\} = \frac{1}{\rho(k+1)\log(k+1)} \tag{3.41}$$

implying the divergence of $\sum P(B_k)$. Thus $P(B_k \text{ i.o.}) = 1$. This together with (3.29) and (3.32) implies (3.28) and hence the theorem.

Proof of Corollary 1. According to Lemma 2.9b, for any $\eta>0$, $\rho>0$

$$T_r \leqq r^2(\log r)^2(\log\log r)^{2(1+\eta)}\rho \tag{3.42}$$

if $r > r_0(\omega)$. On the other hand from Theorem 5 we have, that for every $\varepsilon>0$, $0<\delta\leqq 1$

$$\inf_{|y|<\frac{r\delta}{2\log\log r}} L(y, T_r) > K(\delta)(1-\varepsilon) r \quad \text{if } r > r_0^*(\omega). \tag{3.43}$$

Consequently

$$\inf_{|y|<\frac{r\delta}{2\log\log r}} L(y, r^2(\log r)^2 (\log\log r)^{2(1+\eta)}\rho) \geqq K(\delta)(1-\varepsilon) r.$$

Denoting $r^2(\log r)^2(\log\log r)^{1+\eta}\rho=u$ and taking into account that ρ can be chosen arbitrary small we get that

$$P\Big(\varliminf_{u\to\infty}\inf_{|y|<\sqrt{u}(\log u)^{-1}(\log\log u)^{-2-\eta}} L(y,u)\,u^{-\frac{1}{2}}(\log u)(\log\log u)^{1+\eta}=+\infty\Big)=1$$

and this was to be proved.

Proof of Corollary 2 is similar to the proof of Corollary 1, the only difference is that one has to combine Lemma 2.9a, with the convergent part of Theorem 2, keeping in mind that

$$\{\eta_r<x\}=\{\inf_{|s|\le x} L(s,T_r)=0\}.$$

References

Bass, R.F., Griffin, P.S.: The most visited site of Brownian motion and simple random walk. Z. Wahrscheinlichkeitstheor. Verw. Geb. **70**, 417–436 (1985)

Chung, K.L., Hunt, G.A.: On the zeros of $\sum_1^n \pm 1$. Ann. Math. **50**, 385–400 (1949)

Csáki, E., Erdös, P., Révész, P.: On the length of the longest excursion. Z. Wahrscheinlichkeitstheor. Verw. Geb. **68**, 365–382 (1985)

Csáki, E., Földes, A., How small are the increments of the local time of a Wiener process? Ann. Probab. **14**, 533–546 (1986)

Donsker, M.D., Varadhan, S.R.S.: On laws of the iterated logarithm for local times. Commun. Pure Appl. Math. **30**, 707–793 (1977)

Erdélyi, A.: (ed.) Higher Transcendental Functions, Vol. 2. New York Toronto London: McGraw-Hill 1953

Erdélyi, A.: (ed.) Tables of Integral Transforms, Vol. 1. New York Toronto London: McGraw-Hill 1954

Feller, W.: An introduction to Probability Theorem and Its Applications, Vol. 2. New York London Sidney: John Wiley & Sons, Inc. 1966

Fristedt, B.: The behaviour of increasing stable processes for both small and large times. J. Math. Mech. **13**, 849–856 (1964)

Fristedt, B.: Sample functions of stochastic processes with stationary, independent increments, In: Advances in probability and related topics Vol. 3, 241–397. New York: Marcel Dekker 1974

Gradshteyn, I.S., Ryzhik, I.M.: Table of integrals, series and products. New York: Academic Press, Inc. 1980

Itô, K., McKean, H.: Diffusion processes and their sample paths. Berlin Heidelberg New York: Springer 1965

Kesten, H.: An iterated logarithm law for local time. Duke Math. J. **32**, 447–456 (1965)

Knight, F.B.: Essentials of Brownian Motion and Diffusion. Am. Math. Soc. Providence, Rhode Island 1981

Perkins, E.: On the iterated logarithm law for local time. Proc. Am. Math. Soc. **81**, 470–472 (1981)

Rényi, A.: Wahrscheinlichkeitsrechnung. Berlin: D.V. d. Wissenschaften 1962

Received December 15, 1987