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Probability Theory and Related Fields

Convergence in Probability for Perturbed Stochastic Integral Equations

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Summary. In this work, one considers two stochastic integral equations indexed by some parameter ε and one studies the contiguity of their solutions when the parameter converges to some ε_0. Two types of behaviour are described; they lead to the notion of regular and singular perturbations. The method which is used also enables a study of the rate of convergence. Applications to time discretization of equations are given.

1. Introduction

On a probability space $(\Omega, \mathscr{A}, \mathbb{P})$, consider the differential equation

$$dX_t^\varepsilon = b(X_{t-}^\varepsilon)\,dt + g(X_{t-}^\varepsilon)\,dY_t^\varepsilon; \qquad X_0^\varepsilon = \xi. \tag{1.1}$$

In this equation, b and g are 'regular' functions and Y_t^ε is a process depending on a parameter ε belonging to some Haussdorf topological space; its paths are assumed to be càdlàg (right continuous with left-hand limits) and to have finite variation. Suppose that, as $\varepsilon \to \varepsilon_0$, the noise process Y_t^ε converges to a Wiener process y_t; what is the behaviour of X_t^ε? This question has been studied by several authors in the past years, and can actually be decomposed into at least three different problems, according to the type of approximation one looks for: weak convergence, pathwise convergence or convergence in probability. A powerful theory is now available for the first problem: one knows several sufficient conditions for the tightness of the laws of X^ε and it is generally not very difficult to identify the limit of a converging subsequence. In the second problem, one assumes that with probability 1, the trajectories of Y^ε converge to those of y, and one wonders whether the trajectories of X^ε also converge almost surely; if the approximations Y_t^ε are continuous and one-dimensional, then it is well-known since [33] that X^ε does converge to the solution of the Stratonovich equation

$$dx_t = b(x_t)\,dt + g(x_t) \circ dy_t; \qquad x_0 = \xi. \tag{1.2}$$

In the multidimensional case, it follows from [7] that the same result holds if the vector fields associated to the columns of g commute.

However, the condition of [7] is rather restrictive, so we are led to the third viewpoint: assuming that Y^ε converges in probability to y, does X^ε converge in probability and what is its limit? It is well known that the convergence holds at least for some special Y^ε. The two most classical, and probably simplest cases are the Euler stepwise approximation studied in [25] and the polygonal interpolation of y. These two approximations are defined respectively by $Y_t^\varepsilon = y_{k\varepsilon}$ for $k\varepsilon \leqq t < (k+1)\varepsilon$ and by

$$Y_t^\varepsilon = y_{k\varepsilon} + \frac{t-k\varepsilon}{\varepsilon}(y_{(k+1)\varepsilon} - y_{k\varepsilon}) \quad \text{for } k\varepsilon \leqq t < (k+1)\varepsilon. \tag{1.3}$$

The convergence can be proved for these two cases, and the limit of X^ε is respectively the solution of

$$dx_t = b(x_t)\,dt + g(x_t)\,dy_t; \qquad x_0 = \xi \tag{1.4}$$

and that of (1.2); some further results in this direction are given in [26] in the framework of the McShane stochastic calculus. Another classical example of continuous approximation Y^ε is the so-called class of mollifiers described in [23] and which appears to behave like the polygonal interpolation. On the other hand, as it was already noticed in Sect. VI.3 of [26], one may also meet a different behaviour; for instance, when Y^ε is in the class described in Sect. 6 of [11], then X^ε converges in probability and the limit x is solution of an equation of the type

$$dx_t = (b+c)(x_t)\,dt + g(x_t)\,dy_t. \tag{1.5}$$

The function c is often equal to the Itô-Stratanovich corrective term, but may be different. The same type of problems is dealt with by weak convergence techniques in [18]. In this work, we will say that an approximation scheme is regular if Itô's equations are stable with respect to it; thus the Euler scheme is regular; when a corrective term has to be added in the limit equation, we will say that the approximation is singular; singular approximations will also be divided into symmetric ones – their limit is (1.2) – and non symmetric ones – their limit is (1.5) for a function c which is generally not the Itô-Stratonovich corrective term.

Several authors have also considered more general semimartingales Y_t^ε and y_t. When y_t is continuous, one can check theorems which are similar to the Brownian case; for instance, a Euler discretization scheme is used in [16] and approximations which are close to the polygonal interpolation are considered in [28, 15]. Semimartingales with jumps have also been studied; approximations which converge to y for some strong topology on the space of semimartingales are shown to be regular in [8, 30]; other approximations, which are close to those used in this work are studied in [2, 3]. The problem of singular approximations is considered in [24, 31] and more recently by Mackevičius: see [19] for continuous processes and [21, 22] for the general case; the idea used in

[22] will be basic in our study of singular approximations; some of his results are generalized in [9].

The first aim of this work is to give a systematic study of the regularly perturbed system

$$
\begin{aligned}
X_t^\varepsilon(\omega) &= R_t^\varepsilon(\omega) + \int_0^t F^\varepsilon(s, \omega, X_{s-}^\varepsilon(\omega))\, dW_s^\varepsilon, \\
x_t^\varepsilon(\omega) &= r_t^\varepsilon(\omega) + \int_0^t f^\varepsilon(s, \omega, x_{s-}^\varepsilon(\omega))\, dw_s^\varepsilon.
\end{aligned}
\tag{1.6}
$$

In this system, the càdlàg processes R^ε, r^ε, the semimartingales W^ε, w^ε and the random functions F^ε, f^ε depend on ε, and the underlying filtrations also depend on ε and are not necessarily the same for the two equations; the coefficients will not be assumed to be Lipschitz but only asymptotically monotone; we will suppose that as $\varepsilon \to \varepsilon_0$, $R^\varepsilon - r^\varepsilon$, $F^\varepsilon - f^\varepsilon$ and $W^\varepsilon - w^\varepsilon$ converge to 0 for some topology and we will look for conditions which ensure the convergence of $X^\varepsilon - x^\varepsilon$ to 0 for the same topology; this topology may be the convergence in probability for each fixed time t, the convergence in probability of $\sup_t |\cdot|$, or may be intermediate between these two possibilities; moreover, we will give conditions ensuring convergence in L^q. Note that both processes X^ε and x^ε depend on ε; in particular they are not necessarily defined on the same probability space, so that the situation is more general than the classical case (studied in [3]) where the equation for x^ε does not depend on ε. Applications to the Euler discretization scheme including results of [2] will be given. Allowing both equations of (1.6) to depend on ε leads to new results: we prove that the convergence of the Euler scheme is uniform over some families of equations and this technique may be applied to other situations. Our second aim is to take advantage of our framework in order to show that singular perturbations may often be reduced to regular ones; consider

$$
\begin{aligned}
X_t^\varepsilon(\omega) &= X_0^\varepsilon(\omega) + \int_0^t F^\varepsilon(s, \omega, X_{s-}^\varepsilon(\omega))\, dW_s^\varepsilon + \int_0^t G^\varepsilon(s, \omega, X_{s-}^\varepsilon(\omega))\, dY_s^\varepsilon, \\
x_t^\varepsilon(\omega) &= x_0^\varepsilon(\omega) + \int_0^t f^\varepsilon(s, \omega, x_{s-}^\varepsilon(\omega))\, dw_s^\varepsilon + \int_0^t g^\varepsilon(s, \omega, x_{s-}(\omega))\, dy_s^\varepsilon \\
&\quad + \int_0^t h^\varepsilon(s, \omega, x_{s-}^\varepsilon(\omega))\, dz_s^\varepsilon.
\end{aligned}
\tag{1.7}
$$

In this system, the semimartingales W^ε and w^ε will behave as in (1.6) but Y^ε will be a singular approximation of y^ε; roughly speaking, the singularity generally comes from the fact that the variation of the finite variation part of Y^ε explodes as $\varepsilon \to \varepsilon_0$. We will prove that for a good choice of the coefficient h^ε and the semimartingale z^ε, the process $X^\varepsilon - x^\varepsilon$ again converges in probability to 0. Note that in the singular case the driving processes R_t^ε and r_t^ε are replaced by initial conditions. Note also that we will assume more regularity on G^ε and g^ε than on F^ε and f^ε; in particular the dependence of $G^\varepsilon(s, \omega, x)$ on (s, ω) may not be quite general; in [22], the coefficient was a function of x only; a more general case involving some semimartingale dependence is dealt with in [9] and here, we will generalize [22] in another direction. Examples will include polygonal interpolation of continuous semimartingales and approximations of Brownian motion consisting of absolutely continuous Gaussian processes. Another important problem is the estimation of the rate of convergence of $X^\varepsilon - x^\varepsilon$ to 0 in L^q; with additional smoothness conditions on the coefficients, we will prove that the rate of convergence of the various data of (1.6) or (1.7) is transmitted to the solutions. The particular case of absolutely continuous approximations of a Brownian motion is studied in [29] and we will give some additional results here.

In Sect. 2, we will first introduce the topology for which our convergence results will be proved and will give the definitions used in this paper; then we will state the results concerning regular perturbations (1.6) and will apply them to the Euler discretization scheme. We will also mention some applications to weak convergence problems. The proofs of these results will be detailed in Sect. 3. In Sect. 4, we will study the singular perturbations (1.7) and some examples. The rate of convergence will be dealt with in Sect. 5.

Throughout this paper, we will adopt the following notational convention: since nearly all the functions and processes depend on ε, we will drop the superscript ε in the notation; the word 'family' of processes, filtrations, ... will mean that the object depends on ε. If Z is a càdlàg process, the process of its left-hand limits will be denoted by Z_- with the convention $Z_{0-} = Z_0$, and ΔZ_t will denote the jump $Z_t - Z_{t-}$. The filtrations will always be assumed to satisfy the usual conditions (they are right continuous and contain the negligible sets of $(\Omega, \mathscr{A}, \mathbb{P})$) and $\mathscr{P}(\mathscr{F})$ will denote the σ-algebra of $\mathbb{R}_+ \times \Omega$ consisting of $\mathscr{F}_t$ predictable events; when dealing with semimartingales, we will always suppose that they are càdlàg and special: they can be canonically decomposed into the sum of a predictable process with finite variation and of a local martingale; moreover, we will assume that the local martingale parts of these semimartingales are locally square integrable. In (1.6) and (1.7), the processes X_t, W_t and Y_t, as well as x_t, w_t and y_t will be vector-valued and their respective dimensions d_1, d_2 and d_3 will be fixed integers. The constant numbers involved in the calculations will be denoted by C, though they may change from line to line. The convergence in probability will also be called convergence in L^0. If U is a nonnegative variable and $K > 0$, the mean of $\exp KU$ will be called the exponential moment of U of order K. For all technical details concerning the general theory of processes and semimartingales, we refer to [5, 6] or to [13].

2. Regular Perturbations

In this section, we first give the definitions which will be involved in all our subsequent results: more precisely, in Sect. 2.1, we define the topology of convergence in probability or in L^q on the stopping times of some filtration; in Sect. 2.2, after listing some domination properties for semimartingales W_t, we study the convergence of $\int_0^t \psi_s\,dW_s$ as $\psi \to 0$; in Sect. 2.3, the problem of comparing $\int_0^t \psi_s\,dW_s$ and $\int_0^t \psi_s\,dw_s$ when W and w are close is taken up, and in Sect. 2.4, we state the main convergence theorems for regularly perturbed systems; the technical proofs will be given in Sect. 3; in Sect. 2.5, as an application of these results, we get the convergence of the Euler scheme for stochastic integral equations, the coefficients of which are monotone; in Sect. 2.6, the results of Sect. 2.4 are also applied to the simultaneous discretization of a family of equations.

2.1 Description of the Topologies

Let us fix the probability space $(\Omega, \mathscr{A}, \mathbb{P})$ and let $\mathscr{H}_t \subset \mathscr{A}$ be some filtration; for $T>0$, let $\mathscr{T}(\mathscr{H}, T)$ be the set of $\mathscr{H}_t$ stopping times which are uniformly bounded by T. If $Z_t(\omega)$ is a measurable process and $q \geqq 1$, we put

$$|Z|_{\mathscr{H},q,T} = \sup\{\|Z_\tau\|_q;\ \tau \in \mathscr{T}(\mathscr{H}, T)\} \tag{2.1.1}$$

where $\|\cdot\|_q$ denotes the $L^q(\Omega, \mathscr{A}, \mathbb{P})$ norm. This defines a seminorm on the subspace of processes where it is finite and we consider the topology defined by the family of these seminorms as $T>0$: the resulting topological space will be denoted $L^q(\mathscr{H})$; it becomes a Haussdorf space when restricted to left continuous or right continuous processes. A family of processes Z depending on ε converges as $\varepsilon \to \varepsilon_0$ to a process z in $L^q(\mathscr{H})$ if and only if for any uniformly bounded family of $\mathscr{H}_t$ stopping times τ, $Z_\tau - z_\tau$ converges to 0 in L^q; if we have only the convergence in probability of $Z_\tau - z_\tau$, we will say that Z converges to z in $L^0(\mathscr{H})$; this is equivalent to

$$\lim_{\varepsilon \to \varepsilon_0} \sup\{\mathbb{P}[|Z_\tau - z_\tau| > \mu];\ \tau \in \mathscr{T}(\mathscr{H}, T)\} = 0 \tag{2.1.2}$$

for any T and $\mu > 0$. Then let $\mathscr{T}_p(\mathscr{H}, T)$ be the subset of $\mathscr{T}(\mathscr{H}, T)$ consisting of $\mathscr{H}_t$ predictable times; we define

$$\|Z\|_{\mathscr{H},q,T} = \sup\{\|Z_\tau\|_q;\ \tau \in \mathscr{T}_p(\mathscr{H}, T)\}. \tag{2.1.3}$$

If T describes $(0, \infty)$, we obtain a space $\mathbb{L}^q(\mathscr{H})$; we can also define as previously a space $\mathbb{L}^0(\mathscr{H})$.

If the filtration $\mathscr{H}_t$ also depends on ε, we can still study the convergence of $|Z-z|_{\mathscr{H},q,T}$ to 0; if it holds, we will still say that Z converges to z in $L^q(\mathscr{H})$, though this is not a convergence in a topological space; we apply the same convention for $L^0(\mathscr{H})$, $\mathbb{L}^q(\mathscr{H})$, $\mathbb{L}^0(\mathscr{H})$. If both Z and z depend on ε, we obtain the notion of contiguity.

Definition 2.1.1. Let $\mathscr{H}_t$ be a family of filtrations, Z and z be families of measurable processes; for $q \geqq 1$ or $q=0$, we will say that Z and z are contiguous in $L^q(\mathscr{H})$ (resp. in $\mathbb{L}^q(\mathscr{H})$) if for any fixed T and any family of times τ in $\mathscr{T}(\mathscr{H}, T)$ (resp. in $\mathscr{T}_p(\mathscr{H}, T)$), the variables $Z_\tau - z_\tau$ converge to 0 in L^q.

Subsequently, except otherwise stated, the filtration $\mathscr{H}_t$ will always be allowed to depend on ε. We now give some properties of these notions. First note that the choice of $\mathscr{H}_t$ enables the study of different types of convergence: the bigger $\mathscr{H}_t$ is, the stronger the topologies are; for instance, if $\mathscr{H}_t$ is almost surely trivial, we get a notion of convergence on deterministic times; on the other hand, if Z and z are $\mathscr{H}_0$ measurable, we have

$$|Z-z|_{\mathscr{H},q,T}=\|Z-z\|_{\mathscr{H},q,T}=\|\sup_{t\leqq T}|Z_t-z_t|\|_q. \tag{2.1.4}$$

We now prove that a result of the same kind holds when Z and z are only assumed to be $\mathscr{H}_t$ optional or predictable.

Lemma 2.1.2. *Let $\mathscr{H}_t$ be a family of filtrations indexed by ε.*

(a) *For any family Z of $\mathscr{H}_t$ optional processes and any $T>0$, one has*

$$\mathbb{P}[\sup_{t\leqq T}|Z_t|\geqq\mu]=\sup\{\mathbb{P}[|Z_\tau|\geqq\mu];\ \tau\in\mathscr{T}(\mathscr{H},T)\} \tag{2.1.5}$$

for $\mu\geqq 0$ and

$$\|\sup_{t\leqq T}|Z_t|\|_{q'}\leqq C\,|Z|_{\mathscr{H},q,T} \tag{2.1.6}$$

for $1\leqq q'<q$.

(b) *If Z and z are $\mathscr{H}_t$ optional processes, then the contiguity in $L^0(\mathscr{H})$ is equivalent to the convergence in probability of $\sup_{t\leqq T}|Z_t-z_t|$ to 0 for any T; moreover, for $q>1$, the contiguity in $L^q(\mathscr{H})$ implies the convergence of $\sup_{t\leqq T}|Z_t-z_t|$ to 0 in $L^{q'}$ for every $q'<q$.*

(c) *If Z and z are $\mathscr{H}_t$ predictable, analogous results hold for $\mathbb{L}^0(\mathscr{H})$ and $\mathbb{L}^q(\mathscr{H})$.*

Proof. We are going to prove (a); (b) is an immediate corollary and (c) is proved with a similar method. Fix T and $\mu>0$; since the set $\{|Z_t(\omega)|\geqq\mu\}$ is optional, we deduce from the optional section theorem (Theorem IV.84 of [5]) that for any $\alpha>0$, there exists a $\mathscr{H}_t$ stopping time $\tau\leqq T$ such that

$$\mathbb{P}[\sup_{t\leqq T}|Z_t|\geqq\mu]\leqq\mathbb{P}[|Z_\tau|\geqq\mu]+\alpha \tag{2.1.7}$$

so the non trivial part of (2.1.5) is proved. Let us now check (2.1.6); assuming that the right-hand side is finite, one gets from (2.1.5) and the Bienaymé-Chebychev inequality

$$\mathbb{P}[\sup_{t\leqq T}|Z_t|\geqq\mu]\leqq\frac{|Z|^q_{\mathscr{H},q,T}}{\mu^q}\wedge 1. \tag{2.1.8}$$

Thus

$$\|\sup_{t\leqq T}|Z_t|\|^{q'}_{q'}=q'\int_0^\infty\mathbb{P}[\sup_{t\leqq T}|Z_t|\geqq\mu]\,\mu^{q'-1}\,d\mu\leqq\frac{q}{q-q'}|Z|^{q'}_{\mathscr{H},q,T} \tag{2.1.9}$$

so the proof is complete. □

If our processes are not $\mathscr{H}_t$ optional, we have to put additional assumptions to get results similar to Lemma 2.1.2. We will consider càdlàg processes, so let D^n be the space of càdlàg functions defined on $[0, \infty)$ and with values in $\mathbb{R}^n$; it is endowed with the Skorohod topology and we choose some bounded compatible metric d. We also let C^n be the subset consisting of continuous functions. The next lemma implies that even if $\mathscr{H}_t$ is trivial, the contiguity in $\mathbb{L}^0(\mathscr{H})$ and the tightness in D^n imply the contiguity in probability in D^n.

Lemma 2.1.3. *Let Z and z be families of measurable càdlàg processes.*

(a) *We suppose that the laws of Z and z considered as probability laws on D^n are tight and that for all but a countable number of times t, $Z_t - z_t$ converges in probability to 0. Then $d(Z, z)$ converges in probability to 0; if moreover z is C^n tight, then $\sup_{t \leqq T} |Z_t - z_t|$ converges in probability to 0.*

(b) *Suppose that for some $q > 1$, $p > 1$, and some monotone deterministic continuous function $\Phi(t)$, one has*

$$\mathbb{E}[|Z_t - Z_s|^q] \leqq |\Phi(t) - \Phi(s)|^p. \tag{2.1.10}$$

Suppose also that Z_0 is bounded in L^q. Then for any $q' < q$, $\sup_{t \leqq T} |Z_t|^{q'}$ is uniformly integrable.

(c) *If Z and z satisfy* (2.1.10), *and if for all t, $Z_t - z_t$ converges in probability to* 0, *then $\sup_{t \leqq T} |Z_t - z_t|$ converges to 0 in $L^{q'}$ for any $q' < q$.*

Proof. Consider the space $D^n \times D^n$ with its product topology and let (ξ^1, ξ^2) be its canonical process. The laws of (Z, z), considered on $D^n \times D^n$ are tight; so let (Z', z') be some subsequence, the law of which converges to some probability Q. Then except on a countable subset of $[0, \infty)$, (Z'_t, z'_t) converges in law to (ξ^1_t, ξ^2_t) (considered under Q); thus $Z'_t - z'_t$ converges in law to $\xi^1_t - \xi^2_t$, but it also converges in probability to 0 so $\xi^1_t = \xi^2_t$ Q-a.s., and therefore $\xi^1 = \xi^2$ Q-a.s.; in particular, the mean of $d(Z', z')$ converges to 0. This proves the convergence in probability of $d(Z, z)$ to 0. Now if z is C^n tight, ξ^2 is continuous so $Q(C^n \times C^n) = 1$; since the map

$$(\xi^1, \xi^2) \mapsto \sup_{t \leqq T} |\xi^1_t - \xi^2_t|$$

is continuous at each point of $C^n \times C^n$, we can deduce the convergence in probability of $\sup_{t \leqq T} |Z'_t - z'_t|$ to 0. Thus (a) is proved. If the family Z_t satisfies (2.1.10), we can deduce from the proof of Theorem 12.3 of [4] that

$$\mathbb{P}[\sup_{t \leqq T} |Z_t| \geqq \mu] \leqq \frac{C}{\mu^q}. \tag{2.1.11}$$

By using this inequality in the first line of (2.1.9), we get (b). Since (2.1.10) implies the C^n tightness of Z_t, the statement (c) is a consequence of (a) and (b). □

Assuming again that our processes are càdlàg, we now give a result which permits to compare the left-hand limits of Z and z when they are contiguous in $L^0(\mathscr{H})$ or $L^q(\mathscr{H})$.

Lemma 2.1.4. *Let Z be a family of measurable càdlàg processes; for $q \geqq 1$,*

$$\|Z_-\|_{\mathscr{H},q,T} \leqq |Z|_{\mathscr{H},q,T} \tag{2.1.12}$$

and for every $\mu > 0$,

$$\sup\{\mathbb{P}[|Z_{\tau-}| > \mu];\ \tau \in \mathscr{T}_p(\mathscr{H}, T)\} \leqq \sup\{\mathbb{P}[|Z_\tau| > \mu];\ \tau \in \mathscr{T}(\mathscr{H}, T)\}. \tag{2.1.13}$$

Proof. We prove (2.1.12), the other inequality is similar. Fix ε and $\tau \in \mathscr{T}_p(\mathscr{H}, T)$; there exists a nondecreasing sequence of $\mathscr{H}_t$ stopping times τ_k which converges to τ and such that $\tau_k < \tau$ on $\{\tau > 0\}$. Then Z_{τ_k} converges almost surely to $Z_{\tau-}$, so from Fatou's lemma,

$$\|Z_{\tau-}\|_q \leqq \liminf_{k \uparrow \infty} \|Z_{\tau_k}\|_q. \tag{2.1.14}$$

The conclusion is then easy. $\square$

Since stopping times are limits of nonincreasing sequences of predictable times, we can also check that the topologies of $L^q(\mathscr{H})$ and $\mathbb{L}^q(\mathscr{H})$ coincide on right continuous processes.

2.2 Estimation of Stochastic Integrals

Suppose that we are given a semimartingale W_t and a family of adapted processes ψ_t indexed by ε which converges in some sense to 0 as $\varepsilon \to \varepsilon_0$. To prove the convergence of the stochastic integrals $\int_0^t \psi_s \, dW_s$ to 0, we can use a stochastic dominated convergence theorem (Theorem VIII.14 of [6]); the aim of this subsection is to study the same convergence when W_t also depends on ε, provided that it satisfies some domination properties; we will state a general theorem in a form which will be convenient for subsequent results; the proof of this theorem will be given in Sect. 3.2.

We first need some definitions which make precise some domination properties. The first one is the basic

Definition 2.2.1. (a) A family Z_t of vector-valued measurable processes will be said to be bounded in probability if for any T

$$\lim_{\mu \to \infty} \sup_{\varepsilon} \mathbb{P}[\sup_{t \leqq T} |Z_t| \geqq \mu] = 0. \tag{2.2.1}$$

(b) If $\mathscr{F}_t$ is a family of filtrations, sequences of $\mathscr{F}_t$ stopping times τ_k^ε will be said to be admissible localization times if for any T

$$\lim_{k \uparrow \infty} \sup_{\varepsilon} \mathbb{P}[\tau_k \leqq T] = 0. \tag{2.2.2}$$

Let Z_t be measurable processes; if there exist admissible localization times τ_k such that $|Z_{t\wedge\tau_k}|$ is less than k on $\{\tau_k>0\}$, then Z_t is easily shown to be bounded in probability. If Z_t are $\mathscr{F}_t$ predictable, the converse also holds; if Z_t is only $\mathscr{F}_t$ optional, it holds for instance when the jumps of $\sup_{s\leqq t}|Z_s|$ are uniformly bounded, but not in the general case.

Now consider a filtration $\mathscr{F}_t$ and a $\mathscr{F}_t$ semimartingale W_t. Our semimartingales will always be assumed to be special so that they have a canonical decomposition $W_t=V_t+M_t$, where V_t is a $\mathscr{F}_t$ predictable process with finite variation and M_t is a local $\mathscr{F}_t$ martingale; if W is such a semimartingale, $|W|_t$ will denote $\int_0^t |dV_s|$, $[\![W, W]\!]_t$ will be the trace of the symmetric matrix $[W, W]_t$ and $\langle\!\langle W, W\rangle\!\rangle_t^c$ will be $[\![M^c, M^c]\!]_t$, where M^c is the 'continuous martingale' part of M. If M is locally square integrable (and this will be always assumed), $\langle W, W\rangle_t$ and $\langle\!\langle W, W\rangle\!\rangle_t$ will be the predictable compensators of $[W, W]_t$ and $[\![W, W]\!]_t$. For $1\leqq q<\infty$, if $\sum_{s\leqq t}|\Delta W_s|^q$ is locally integrable, the process $\langle\!\langle W\rangle\!\rangle_t^{(q)}$ will denote its predictable compensator and actually, the local integrability of $\sum|\Delta W_s|^q$ will be implicitly assumed whenever we refer to $\langle\!\langle W\rangle\!\rangle_t^{(q)}$. The processes $|W|_t$, $\langle\!\langle W, W\rangle\!\rangle_t$ and $\langle\!\langle W\rangle\!\rangle_t^{(q)}$ will be sometimes called predictable characteristics of W_t. If $2\leqq q'\leqq q$, one has

$$x^{q'}\leqq C(x^2+x^q) \tag{2.2.3}$$

for $x\geqq 0$, so it is easy to deduce that the increments of $\langle\!\langle W\rangle\!\rangle_t^{(q')}$ are dominated by the sum of increments of $\langle\!\langle W, W\rangle\!\rangle_t$ and $\langle\!\langle W\rangle\!\rangle_t^{(q)}$ (with a multiplicative positive constant number). Now we can set the

Definition 2.2.2. Fix two families $\mathscr{F}_t$ and $\mathscr{H}_t$ of filtrations.

(a) A family of admissible $\mathscr{H}_t$ dominating processes will be a family of $\mathscr{H}_t$ predictable right continuous nondecreasing processes L_t which satisfy $L_0=0$ and $L_\infty\leqq 1$.

(b) A family of $\mathscr{F}_t$ semimartingales W_t will be said to be $\mathscr{H}_t$ dominated if there exist $\mathscr{H}_t$ dominating processes L_t and admissible $\mathscr{F}_t$ localization times τ_k such that the process

$$A_t=|W|_t+\langle\!\langle W, W\rangle\!\rangle_t \tag{2.2.4}$$

is absolutely continuous with respect to L_t and for any k, there exists $p>1$ such that

$$\sup_\varepsilon \mathbb{E}\int_0^{\tau_k}\left|\frac{dA_t}{dL_t}\right|^p dL_t<\infty. \tag{2.2.5}$$

The above definition may be completed by the following one; a family of $\mathscr{F}_t$ semimartingales W_t is said to be prelocally $\mathscr{H}_t$ dominated if for any T and $\alpha>0$, there exist $\mathscr{H}_t$ dominated $\mathscr{F}_t$ semimartingales $\bar{W}_t$ such that

$$\mathbb{P}[\exists t\leqq T, W_t\neq\bar{W}_t]\leqq\alpha. \tag{2.2.6}$$

Generally, one looks for semimartingales $\bar{W}_t$ which have uniformly bounded jumps, so that one constructs $\bar{W}_t$ from W_t by eliminating the big jumps (see the classical truncation method in [13]); for instance, if W_t is nondecreasing and $|W|_t$ is $\mathscr{H}_t$ dominated, then W_t is prelocally $\mathscr{H}_t$ dominated. Most subsequent convergence results may be proved for prelocally dominated semimartingales; however, in order to avoid too awkward statements, we will consider only dominated semimartingales (the generalization is not hard). The class of $\mathscr{H}_t$ dominated $\mathscr{F}_t$ semimartingales is of course stable by stopping at $\mathscr{F}_t$ times. Note also that if W_t is $\mathscr{H}_t$ dominated, then the process A_t defined in (2.2.4) is bounded in probability; the converse statement is not always true but it holds when $\mathscr{H}_t=\mathscr{F}_t$; in this case, the above definition may be simplified by means of the

Proposition 2.2.3. *Let W_t be $\mathscr{F}_t$ semimartingales such that the process A_t defined in (2.2.4) is $\mathscr{H}_t$ predictable.*

(a) *Suppose that A_t is bounded in probability; then W_t is $\mathscr{H}_t$ dominated.*

(b) *If $|W|_t$ is bounded in probability and if there exist admissible localization times τ_k such that*

$$\mathbb{E}[\sup_{t\leqq\tau_k}|W_t-W_0|^2]\leqq k, \tag{2.2.7}$$

the semimartingale W_t is $\mathscr{H}_t$ dominated.

Proof. In order to prove (a), choose $L_t=\arctan A_t$, define

$$\bar{\tau}_k=\inf\{t; A_t\geqq k\} \tag{2.2.8}$$

and let τ_k be admissible localization $\mathscr{F}_t$ stopping times such that $\tau_k<\bar{\tau}_k$ (such a family exists since $\bar{\tau}_k$ are admissible localization $\mathscr{F}_t$ predictable times); then $A_{t\wedge\tau_k}$ is less than k, so (2.2.5) is easily checked (this procedure using the predictability of A_t will be often applied subsequently). Let us prove (b); since $|W|_t$ is $\mathscr{F}_t$ predictable and bounded in probability, there exist (from the above argument) localization times τ'_k such that $|W|_{\tau'_k}$ is less than $\sqrt{k}$; put $\tau''_k=\tau_k\wedge\tau'_k$. If $W=V+M$ is the canonical decomposition of W, the process M stopped at τ''_k is therefore a square integrable martingale and

$$\begin{aligned}\mathbb{E}\langle\!\langle W,W\rangle\!\rangle_{\tau''_k}&=\mathbb{E}|W_{\tau''_k}-W_0|^2-2\,\mathbb{E}\int_0^{\tau''_k}|W_{s-}-W_0|\,dV_s\\&\leqq\mathbb{E}\sup_{t\leqq\tau_k}|W_t-W_0|^2+2\sqrt{k}\,\mathbb{E}\sup_{t\leqq\tau_k}|W_t-W_0|\\&\leqq 3k.\end{aligned} \tag{2.2.9}$$

Since τ''_k are admissible localization times, the process $A_t=|W|_t+\langle\!\langle W,W\rangle\!\rangle_t$ is bounded in probability, so the proof is complete. □

Remark. In place of (b), if one assumes only that W_t and $|W|_t$ are bounded in probability, one can check that W_t is prelocally $\mathscr{F}_t$ dominated, but it is not necessarily $\mathscr{F}_t$ dominated; as an example, let $\varepsilon>0$ and $\varepsilon_0=0$, consider $W_t^\varepsilon=(N_{\varepsilon t}-N'_{\varepsilon t})/\varepsilon$ where N_t and N'_t are standard independent Poisson processes, and let $\mathscr{F}_t^\varepsilon$ be the filtration generated by W_t^ε; then $|W|_t=0$, $\langle\!\langle W, W\rangle\!\rangle_t=2t/\varepsilon$ and W_t is bounded in probability.

Then our stochastic dominated convergence theorem (proved in Sect. 3.2) can be stated as

Proposition 2.2.4. *Let $\mathscr{F}_t$ and $\mathscr{H}_t$ be two families of filtrations, W a family of $\mathbb{R}^{d_2}$ valued $\mathscr{F}_t$ semimartingales and ψ_t a family of $\mathscr{F}_t$ predictable locally bounded processes with values in $\mathbb{R}^{d_1}\otimes\mathbb{R}^{d_2}$.*

(a) *For $2\leqq q<\infty$, there exists a constant number C depending only on q such that*

$$\mathbb{E}\sup_{t\leqq T}|W_t|^q\leqq C\,\mathbb{E}[|W|_T^q+\langle\!\langle W, W\rangle\!\rangle_T^{q/2}+\langle\!\langle W\rangle\!\rangle_T^{(q)}]. \tag{2.2.10}$$

(b) *Consider $2\leqq q<\infty$, suppose that there exist admissible $\mathscr{H}_t$ dominating processes L_t such that*

$$\sup_\varepsilon \mathbb{E}\int_0^T\left(\left|\frac{d|W|_t}{dL_t}\right|^p+\left|\frac{d\langle\!\langle W, W\rangle\!\rangle_t}{dL_t}\right|^{p/2}+\left|\frac{d\langle\!\langle W\rangle\!\rangle_t^{(q)}}{dL_t}\right|^{p/q}\right)dL_t<\infty \tag{2.2.11}$$

for some $q\leqq p\leqq\infty$ (when $p=\infty$, the above condition means by convention that the three densities are uniformly bounded by a constant number). Then

$$\left\|\sup_{t\leqq T}\left|\int_0^t\psi_s\,dW_s\right|\right\|_q\leqq C\,\|\psi\|_{\mathscr{H},q',T} \tag{2.2.12}$$

where C does not depend on ψ, and $1/q=1/p+1/q'$.

(c) *Suppose that ψ_t is bounded in probability and that W_t is $\mathscr{H}_t$ dominated; then $\int_0^t\psi_s\,dW_s$ is bounded in probability. If moreover ψ_t converges to 0 in $\mathbb{L}^0(\mathscr{H})$, then $\sup_{t\leqq T}\left|\int_0^t\psi_s\,dW_s\right|$ converges to 0 in probability.*

Remark 1. When $\mathscr{H}_t=\mathscr{F}_t$, the last statement says that for $\mathscr{F}_t$ dominated semimartingales, the convergence of $\sup_t|\psi_t|$ to 0 implies the convergence of $\sup_t\left|\int_0^t\psi_s\,dW_s\right|$ to 0; this property can actually be proved to be equivalent to the condition (*) of [32] on the family $W_{t\wedge T}$ (see also [14]).

Remark 2. Proposition 2.2.4(c) is immediately extended to prelocally $\mathscr{H}_t$ dominated semimartingales W_t (see the discussion following Definition 2.2.2).

The role of the filtration $\mathscr{H}_t$ may be emphasized by means of the above convergence result: the bigger $\mathscr{H}_t$ is, the weaker the assumptions on W_t are and the stronger the assumptions on ψ_t are; moreover, if one studies the rate of convergence with (2.2.12), it is desirable to choose $\mathscr{H}_t$ as small as possible in order to improve the rate. As an application, we are going to check that if the family of semimartingales is dominated, then its convergence to 0 implies that its quadratic variation also converges to 0 (see also Theorem VI.6.1 of [13]).

Corollary 2.2.5. *Let W_t be a family of $\mathscr{F}_t$ semimartingales which are $\mathscr{H}_t$ dominated. If W_t converges to 0 in $L^0(\mathscr{H})$, then for every T, $[\![W, W]\!]_T$ converges to 0 in probability.*

Proof. We apply Proposition 2.2.4(c) twice: firstly with $\psi_t = 1$ in order to show that W_t is bounded in probability, secondly with $\psi_t = W_{t-}$ which converges to 0 in $\mathbb{L}^0(\mathscr{H})$ from Lemma 2.1.4; thus we have proved that $\int_0^T W_{s-}^* \, dW_s$ converges in probability to 0. Then, developing $|W_T|^2$ with Itô's formula, we easily deduce that $[\![W, W]\!]_T$ converges to 0. $\square$

Corollary 2.2.5 implies that if W and w are $\mathscr{F}_t$ semimartingales which are $\mathscr{H}_t$ dominated and contiguous in $L^0(\mathscr{H})$, then their continuous martingale parts are strongly related: the quadratic variation of their difference must converge to 0; for instance, under this framework, a non constant continuous martingale cannot be approximated by a process with finite variation. This is why we will consider subsequently processes W and w which are semimartingales with respect to different filtrations; as we will see, this allows much more general approximations.

2.3 Approximation by Step Processes

In previous subsection, we have stated results concerning the continuity of the stochastic integral $\int_0^t \psi_s \, dW_s$ with respect to ψ; we now want to obtain the continuity with respect to W. This will be achieved by approximating the stochastic integrals by means of Riemann sums; thus, we are going to use classes of processes ψ_t which will be uniformly approximated by step processes, as in the classical Riemann theory of integration.

Definition 2.3.1. Let $\mathscr{G}_t$ and $\mathscr{H}_t$ be two families of filtrations; a family of $\mathscr{G}_t$ predictable processes z_t will be said to be $\mathscr{H}_t$ Riemann if for any T and $\delta > 0$, there exist an integer N bounded uniformly in ε, families of $\mathscr{G}_t \cap \mathscr{H}_t$ stopping times τ_i and τ'_i, $i \leqq N$, such that τ'_i is $\mathscr{G}_t$ and $\mathscr{H}_t$ predictable, families of variables ϕ_i and ϕ'_i which are respectively $\mathscr{G}_{\tau_i}$ and $\mathscr{G}_{\tau'_i -}$ measurable such that if

$$\bar{z}_t = \sum_{i=0}^{N} 1_{(\tau_i, \infty)}(t) \, \phi_i + \sum_{i=0}^{N} 1_{[\tau'_i]}(t) \, \phi'_i, \tag{2.3.1}$$

then

$$\mathbb{E}[|z_\tau - \bar{z}_\tau| \wedge 1] \leqq \delta \tag{2.3.2}$$

for any $\tau \in \mathscr{T}_p(\mathscr{H}, T)$.

The class of $\mathscr{H}_t$ Riemann processes is linearly stable; in particular, one can study each component of z separately. One can also check that if z is left continuous and $\mathscr{H}_t$ Riemann, one can always choose $\phi'_i=0$ on $\{\tau'_i>0\}$ (because if $\bar{z}_t$ satisfies (2.3.2), then $\bar{z}_{t-}$ also satisfies it). If $\mathscr{H}_t\supset\mathscr{G}_t$, from Lemma 2.1.2, one can replace in the Definition (2.3.2) by

$$\mathbb{E}[\sup_{t\leq T}|z_t-\bar{z}_t|\wedge 1]\leq\delta. \tag{2.3.3}$$

We will prove in Sect. 3.1 that the processes described in the following propositions are $\mathscr{H}_t$ Riemann.

Proposition 2.3.2. *Assume that the $\mathscr{H}_t$ optional projection of a $\mathscr{G}_t$ optional process is $\mathscr{G}_t$ optional and let $z_t=x_{t-}$, where x_t are $\mathscr{G}_t$ adapted real càdlàg processes such that*

(i) *the process x_t is bounded in probability,*

(ii) *for every real numbers $u<v$, the number of upcrossings of x from u to v on a time interval $[0, T]$ is bounded in probability.*

Then z is $\mathscr{H}_t$ Riemann. In particular, if x_t are $\mathscr{G}_t$ semimartingales such that x_t and $|x|_t$ are bounded in probability, or if the laws of x_t are tight for the Skorohod topology, then z is $\mathscr{H}_t$ Riemann.

Remark. Conditions (i) and (ii) exactly mean that the law of x_t satisfies the tightness criterion for the pseudopath topology given in the corollary of Theorem 2 of [27].

Proposition 2.3.3. *Assume that the $\mathscr{H}_t$ optional projection of a $\mathscr{G}_t$ predictable process is both $\mathscr{G}_t$ and $\mathscr{H}_t$ predictable; if z_t are làdlàg (left-hand limited and right-hand limited at each time) $\mathscr{G}_t$ predictable processes which are zero except for a countable number of times, and if for any T and $\delta>0$, the number of times $t\leq T$ such that $|z_t|\geq\delta$ is bounded in probability, then z_t is $\mathscr{H}_t$ Riemann.*

If now z_t are general làdlàg processes, one can try to apply Proposition 2.3.2 to z_{t-}, and Proposition 2.3.3 to z_t-z_{t-}; in particular, if z is làdlàg and does not depend on ε, it is $\mathscr{H}_t$ Riemann. Note also that the assumption about $\mathscr{G}_t$ and $\mathscr{H}_t$ in Proposition 2.3.2 is satisfied for instance when $\mathscr{H}_t\supset\mathscr{G}_t$ or $\mathscr{H}_t\subset\mathscr{G}_t$; in Proposition 2.3.3, it is satisfied when $\mathscr{H}_t\supset\mathscr{G}_t$ or when $\mathscr{H}_t$ is a.s. trivial. We are now going to get the continuity of the stochastic integrals with respect to the driving semimartingales; with reference to the discussion following Corollary 2.2.5, we consider two families of semimartingales with respect to two families of filtrations; assuming that the semimartingales are contiguous as $\varepsilon\to\varepsilon_0$, we prove that the stochastic integrals are also contiguous.

Proposition 2.3.4. *Let $\mathscr{F}_t$, $\mathscr{G}_t$ and $\mathscr{H}_t$ be families of filtrations such that $\mathscr{G}_t\subset\mathscr{F}_t$. Let z_t be a family of $\mathscr{G}_t$ predictable $\mathscr{H}_t$ Riemann processes which are bounded in probability and let W_t and w_t be families of respectively $\mathscr{F}_t$ and $\mathscr{G}_t$ semimartingales* *which are $\mathscr{H}_t$ dominated. If W and w are contiguous in $L^0(\mathscr{H})$ then $\int_0^t z_s\,dW_s$ and $\int_0^t z_s\,dw_s$ are also contiguous in $L^0(\mathscr{H})$. If $\mathscr{G}_t$ and w_t do not depend on ε, the condition 'w is $\mathscr{H}_t$ dominated' can be removed.*

Remark. One must notice that the same expression '$\mathscr{H}_t$ dominated' is actually used for two different classes: a class of $\mathscr{F}_t$ semimartingales and a class of $\mathscr{G}_t$ semimartingales; in particular, the localization times involved in Definition 2.2.2 are respectively $\mathscr{F}_t$ and $\mathscr{G}_t$ stopping times.

Proof. Since the processes z are predictable and bounded in probability, by stopping them, we can assume that they are uniformly bounded by some $K>0$. For any ε and $\delta>0$, let $\bar{z}_t$ be an approximation of z_t of the form (2.3.1); we can also choose it bounded by K. From the form of $\bar{z}$, if τ is a family of $\mathscr{T}(\mathscr{H}, T)$ (indexed by ε),

$$\left|\int_0^\tau \bar{z}_s\,dW_s-\int_0^\tau \bar{z}_s\,dw_s\right| \leqq K(N+1)\,|W_\tau-w_\tau|$$
$$+K\sum_{i=0}^{N}|(W_{\tau_i}-w_{\tau_i})1_{\{\tau_i\leqq\tau\}}|+K\sum_{i=0}^{N}|\Delta(W_{\tau_i'}-w_{\tau_i'})1_{\{\tau_i'\leqq\tau\}}|. \quad (2.3.4)$$

Let us fix δ; then the right-hand side of (2.3.4) converges in probability to 0 (the last term is estimated with Lemma 2.1.4). Thus $\int_0^t \bar{z}_s\,dW_s$ and $\int_0^t \bar{z}_s\,dw_s$ are contiguous in $L^0(\mathscr{H}, T)$ as $\varepsilon\to\varepsilon_0$ for δ fixed. On the other hand, from Proposition 2.2.4 (c), the variable

$$\sup_{t\leqq T}\left|\int_0^t z_s\,dW_s-\int_0^t \bar{z}_s\,dW_s\right|$$

converges in probability to 0 as $(\varepsilon, \delta)\to(\varepsilon_0, 0)$; if w is $\mathscr{H}_t$ dominated, the same method can be applied to integrals with respect to w, and if w does not depend on ε, the convergence holds from the stochastic dominated convergence theorem (Theorem VIII.14 of [6]). We can deduce the proposition from these three convergences. □

2.4 The Main Theorems

This subsection is devoted to the stability theorems concerning the regularly perturbed system (1.6). We will state convergence results in $L^0(\mathscr{H})$. We first make precise the assumptions about our model.

Definition 2.4.1. Let $\mathscr{F}_t$ be a family of filtrations and let $F(t, w, x)$ be a family of functions defined on $[0, \infty)\times\Omega\times\mathbb{R}^{d_1}$ with values in $\mathbb{R}^{d_1}\otimes\mathbb{R}^{d_2}$, measurable with respect to $\mathscr{P}(\mathscr{F})\otimes\mathscr{B}(\mathbb{R}^{d_1})$.

(a) We will say that F satisfies $\mathbb{H}_0$ if

$$|F(t, \omega, x)|\leqq\rho_t(\omega)(1+|x|) \tag{2.4.1}$$

for a family ρ_t of $\mathscr{F}_t$ predictable processes which are bounded in probability, and

$$|F(t, \omega, x)-F(t, \omega, y)|\leqq\rho'_t(\omega, K, |x-y|) \tag{2.4.2}$$

for $|x|\leqq K$, $|y|\leqq K$ and $\rho'_t(K, \delta)$ a family of processes which are nondecreasing in t and δ and such that for any t and K, $\rho'_t(K, \delta)$ converges in probability to 0 as $(\delta, \varepsilon)\to(0, \varepsilon_0)$.

(b) We will say that F satisfies $\mathbb{H}'_0$ if it satisfies (2.4.1) and

$$|F(t, \omega, x)-F(t, w, y)|\leqq\rho_t(\omega)\,|x-y| \tag{2.4.3}$$

for $\mathscr{F}_t$ predictable bounded in probability processes ρ_t.

(c) If W_t is a family of $\mathscr{F}_t$ semimartingales, we will say that (F, W) is asymptotically monotone if for any x, y,

$$\begin{aligned}2(x-y)^*(F(t,x)-F(t,y))\,dV_t&+\operatorname{Trace}((F(t,x)-F(t,y))^*(F(t,x)\\ &-F(t,y))\,d\langle W, W\rangle_t)\leqq|x-y|^2\,dA_t+d\bar{A}_t^{(x,y)}\end{aligned} \tag{2.4.4}$$

where V is the predictable finite variation part of W and A, $\bar{A}^{(x,y)}$ are families of nondecreasing right-continuous bounded in probability $\mathscr{F}_t$ predictable processes such that $A_0=0$, $\bar{A}_0^{(x,y)}=0$ and $\bar{A}_t^{(x,y)}$ converges in probability to 0 as $\varepsilon\to\varepsilon_0$ for any t.

It is of course clear that the Lipschitz assumption $\mathbb{H}'_0$ is stronger than $\mathbb{H}_0$ and if moreover $|W|_t$ and $\langle\!\langle W, W\rangle\!\rangle_t$ are bounded in probability, it also implies (2.4.4) with $\bar{A}_t^{(x,y)}=0$. More generally, (2.4.4) with $\bar{A}_t^{(x,y)}=0$ means that the equations that we are going to consider are monotone in the sense of [12] uniformly in ε; it is assumed in [3]. However, adding a small term $\bar{A}_t^{(x,y)}$ may be useful (see the application to the Euler discretization scheme in Sect. 2.5). Here is the framework of the theorems of this subsection.

Assumption 2.4.2. We are given three families of filtrations $\mathscr{F}_t$, $\mathscr{G}_t$ and $\mathscr{H}_t$ such that $\mathscr{G}_t\subset\mathscr{F}_t$; we suppose that the $\mathscr{H}_t$ optional projection of a $\mathscr{G}_t$ optional process is $\mathscr{G}_t$ optional; we consider families W_t and w_t of respectively $\mathscr{F}_t$ and $\mathscr{G}_t$ $\mathbb{R}^{d_2}$ valued semimartingales, families R_t and r_t of respectively $\mathscr{F}_t$ and $\mathscr{G}_t$ adapted càdlàg $\mathbb{R}^{d_1}$ valued processes, families of functions $F(t, \omega, x)$ and $f(t, \omega, x)$ measurable respectively with respect to $\mathscr{P}(\mathscr{F})\otimes\mathscr{B}(\mathbb{R}^{d_1})$ and $\mathscr{P}(\mathscr{G})\otimes\mathscr{B}(\mathbb{R}^{d_1})$; we assume that F and f satisfy $\mathbb{H}_0$ and that (F, W) is asymptotically monotone. Finally we suppose that (X, x) is solution of

$$\begin{aligned} X_t(\omega)&=R_t(\omega)+\int_0^t F(s,\omega,X_{s-}(\omega))\,dW_s(\omega),\\ x_t(\omega)&=r_t(\omega)+\int_0^t f(s,\omega,x_{s-}(\omega))\,dw_s(\omega). \end{aligned} \tag{2.4.5}$$

Remark. Since we suppose that (X, x) is a solution of (2.4.5), we implicitly assume the existence of a solution; however, the assumptions of [12] are generally not satisfied for ε fixed, so we do not know anything about the uniqueness.

We now state the two main stability theorems which will be proved in Sect. 3.4: the first one is concerned with the contiguity, and the second one with the convergence (i.e., the case where the equation for x does not depend on ε, which is also considered in [3]). Then we will deduce a result which is particular to the case $\mathscr{H}_t=\mathscr{F}_t$.

Theorem 2.4.3. *Under assumption* 2.4.2, *suppose that W, w and A (involved in* (2.4.4)) *are $\mathscr{H}_t$ dominated, that R_t and r_t are bounded in probability, that r_{t-} is $\mathscr{H}_t$ Riemann and that for every fixed z, $f(t, z)$ is $\mathscr{H}_t$ Riemann; suppose also that firstly R and r, secondly W and w are contiguous in $L^0(\mathscr{H})$ and that $F(\cdot, z)$ and $f(\cdot, z)$ are contiguous in $\mathbb{L}^0(\mathscr{H})$. Then X and x are contiguous in $L^0(\mathscr{H})$. For $1\leqq j\leqq d_2$, if $W^j=w^j$, the '$\mathscr{H}_t$ Riemann' assumption on the jth column of f can be removed; if $W=w$ and $F=f$, both '$\mathscr{H}_t$ Riemann' assumptions on r and f can be removed.*

Theorem 2.4.4. *Under assumption* 2.4.2, *suppose that r, f, w do not depend on ε, that W_t and A_t (involved in* (2.4.4)) *are $\mathscr{H}_t$ dominated, that R_t is bounded in probability and that f is làdlàg in t; suppose also that R and W converge respectively to r and w in $L^0(\mathscr{H})$ and that $F(\cdot, z)$ converges to $f(\cdot, z)$ in $\mathbb{L}^0(\mathscr{H})$. Then X converges to x in $L^0(\mathscr{H})$.*

Up to now, we have assumed that F and f have at most linear growth as $|x|\to\infty$, so that X_t and x_t do not explode as $\varepsilon\to\varepsilon_0$. However, in the case $\mathscr{H}_t=\mathscr{F}_t$, it is sufficient to assume that one of the two processes does not explode; such a condition is satisfied for instance in the convergence case, so that we can weaken conditions (2.4.1) and (2.4.4).

Theorem 2.4.5. *Assume that the conditions of Theorem* 2.4.4 *are satisfied with $\mathscr{H}_t=\mathscr{F}_t$, except* (2.4.1) *for F and f and* (2.4.4) *for (F, W). These conditions are replaced by:*

(i) *for any $k>0$, the supremum in $|x|\leqq k$ of $|f(t,\omega,x)|$ and $|F(t,w,x)|$ is bounded by some predictable processes $\rho_t^{(k)}$ which are bounded in probability;*

(ii) *condition* (2.4.4) *holds for $|x|$ and $|y|\leqq k$ and a process A_t depending on k. Then $\sup_{t\leqq T}|X_t-x_t|$ converges in probability to* 0.

Sketch of the proof. Consider some positive k and let τ_k be the first exit time of X or x outside the ball of center 0 and radius k; then one can verify with the technique developed for previous theorems that the processes X and x stopped at time τ_k are contiguous in $L^0(\mathscr{F})$, so $\sup_{t \leqq \tau_k \wedge T} |X_t - x_t|$ converges in probability to 0 (one can also apply the truncation technique of Lemma 4 of [10] and use Theorem 2.4.4). Thus, to obtain the proposition, it is sufficient to prove

$$\lim_{k \to \infty} \limsup_{\varepsilon \to \varepsilon_0} \mathbb{P}[\tau_k < T] = 0. \tag{2.4.6}$$

On the set $\{\tau_k < T\}$, we have either $|x_{\tau_k}| \geqq k$, or

$$k \leqq |X_{\tau_k}| \leqq \sup_{t \leqq \tau_k} |X_t - x_t| + |x_{\tau_k}|. \tag{2.4.7}$$

Thus

$$\mathbb{P}[\tau_k < T] \leqq \mathbb{P}\left[\sup_{t \leqq T} |x_t| \geqq \frac{k}{2}\right] + \mathbb{P}\left[\sup_{t \leqq \tau_k \wedge T} |X_t - x_t| \geqq \frac{k}{2}\right]. \tag{2.4.8}$$

The first term does not depend on ε and converges to 0 as $k \to \infty$ and the second one converges to 0 as $\varepsilon \to \varepsilon_0$ for every k, so we have proved (2.4.6). □

2.5 Time Discretization of a Single Equation

In this subsection, we apply our results in order to obtain a convergence theorem for the Euler discretization scheme for stochastic equations driven by semimartingales which are not necessarily continuous and with coefficients which are monotone. Let us first define our framework. We are given a filtration $\mathscr{G}_t$, a càdlàg $\mathscr{G}_t$ adapted process r_t, a $\mathscr{G}_t$ semimartingale w_t and a $\mathscr{P}(\mathscr{G}) \otimes \mathscr{B}(\mathbb{R}^{d_1})$ measurable function f which satisfies $\mathbb{H}_0$. If v_t is the predictable finite variation part of w, we assume also that

$$\begin{aligned} &2(x-y)^*(f(t,x)-f(t,y))\,dv_t + \operatorname{Trace}((f(t,x)-f(t,y))^*(f(t,x)-f(t,y))\,d\langle w, w\rangle_t) \\ &\quad \leqq |x-y|^2\, da_t \end{aligned} \tag{2.5.1}$$

for a nondecreasing right-continuous $\mathscr{G}_t$ predictable process a_t such that $a_0 = 0$. Let x_t be the solution of

$$x_t = r_t + \int_0^t f(s, x_{s-})\, dw_s. \tag{2.5.2}$$

Then for each ε, let $(t_k, k \geqq 0)$ a stochastic subdivision of $[0, \infty)$ consisting of $\mathscr{G}_t$ stopping times, such that $t_0 = 0$, $t_k < t_{k+1}$, $\lim t_k = \infty$ and $\sup_k \{t_{k+1} - t_k, t_k \leqq T\}$ converges in probability to 0 as $\varepsilon \to \varepsilon_0$; define the variables $X(0) = r_0$ and for $k \geqq 1$,

$$X(k+1) = X(k) + r_{t_{k+1}} - r_{t_k} + f(t_k, X(k))(w_{t_{k+1}} - w_{t_k}). \tag{2.5.3}$$

Theorem 2.5.1. *Suppose that f is left continuous in t; then the variable*

$$\sup\{|X(k)-x_{t_k}|;\, t_k \leqq T\}$$

converges in probability to 0.

Such a result was first obtained in a particular case in [25]; it is generalized in [2]; here, we use more general discretization times t_k and our scheme is slightly different (so that we need more regularity on f with respect to t). In order to prove this theorem, it is convenient to introduce a family of filtrations $\mathscr{F}_t$ to which we will apply our theory; this is the aim of the following lemma.

Lemma 2.5.2. *For every time t, put*

$$\mathscr{F}_t=\{B;\forall k, B\cap\{t<t_k\}\in\mathscr{G}_{t_k}\}. \tag{2.5.4}$$

Then this defines a filtration which satisfies the usual conditions; moreover $\mathscr{G}_t\subset\mathscr{F}_t$ and for any $k\geqq 1$,

$$\mathscr{G}_{t_k}=\mathscr{F}_{t_k-}=\mathscr{F}_{t_{k-1}}. \tag{2.5.5}$$

In particular, t_k is a $\mathscr{F}_t$ predictable stopping time.

Proof. Since $\mathscr{F}_t\supset\mathscr{G}_0$, it contains all the negligible events, and it is clear from its definition that it is right continuous. The inclusion $\mathscr{G}_t\subset\mathscr{F}_t$ is also easy (apply the definition of $\mathscr{G}_{t_k}$), so we now prove (2.5.5); we will check $\mathscr{F}_{t_{k-1}}\subset\mathscr{F}_{t_k-}\subset\mathscr{G}_{t_k}\subset\mathscr{F}_{t_{k-1}}$. The first inclusion is trivial, the second one comes from the definition of $\mathscr{F}_{t_k-}$ (it is generated by the events $B\cap\{t<t_k\}$ with $B\in\mathscr{F}_t$, which are in $\mathscr{G}_{t_k}$). Finally, to prove the third one, let $B\in\mathscr{G}_{t_k}$; then for every $j\geqq k$, the events B, $\{t_{k-1}\leqq t\}$ and $\{t<t_j\}$ are all in $\mathscr{G}_{t_j}$, so $B\cap\{t_{k-1}\leqq t<t_j\}$ is in $\mathscr{G}_{t_j}$; if $j<k$, this set is empty, so it is also in $\mathscr{G}_{t_j}$; this implies that $B\cap\{t_{k-1}\leqq t\}$ is in $\mathscr{F}_t$, so B is in $\mathscr{F}_{t_{k-1}}$. The proof of (2.5.5) is therefore complete. In particular, t_k is $\mathscr{F}_{t_{k-1}}$ measurable, so it is predictable. □

Proof of Theorem 2.5.1. First note that the proof can be reduced to the case $w_t=w_0$ for $t\leqq t_1$: if this property does not hold, one can consider a family $\alpha(\varepsilon)$ of positive numbers converging to 0, replace all the processes z_t by $\bar{z}_t=z_0$ if $t\leqq\alpha$, $\bar{z}_t=z_{t-\alpha}$ if $t\geqq\alpha$ and use the family of discretization times 0, α, $t_k+\alpha$; the conclusion of Theorem 2.5.1 for these new processes will imply the theorem for our original processes. If z_t is some càdlàg process, we will call the right discretization of z the process Z_t defined by

$$Z_t=\sum_{k=0}^{\infty} z_{t_{k+1}}1_{[t_k,t_{k+1})}(t). \tag{2.5.6}$$

If z_t is $\mathscr{G}_t$ adapted, then its right discretization is adapted to $\mathscr{F}_t$. Let W_t, R_t and $\bar{x}_t$ be the right discretizations of w_t, r_t and x_t; define

$$X_t=\sum_{k=0}^{\infty} X(k+1)\,1_{[t_k,t_{k+1})}(t) \tag{2.5.7}$$

and

$$\bar{r}_t=\sum_{k=0}^{\infty} 1_{[t_k,\infty)}(t)\int_{t_k}^{t_{k+1}}(f(s,x_{s-})-f(t_k,x_{t_k}))\,dw_s \tag{2.5.8}$$

where the integral from t_k to t_{k+1} has to be understood as an integral on $(t_k, t_{k+1}]$. All these processes are $\mathscr{F}_t$ adapted and to get the theorem, it is sufficient from Lemma 2.1.2 to prove that X and $\bar{x}$ are contiguous in $L^0(\mathscr{F})$. Using our assumption $w_{t_1}=w_0$, one easily verifies that $(X, \bar{x})$ is solution of

$$\begin{aligned} X_t &= R_t + \int_0^t f(s, X_{s-})\, dW_s \\ \bar{x}_t &= R_t + \bar{r}_t + \int_0^t f(s, \bar{x}_{s-})\, dW_s. \end{aligned} \tag{2.5.9}$$

Thus to apply Theorem 2.4.3 (with $F=f$, $W=w$ and $\mathscr{H}_t=\mathscr{F}_t$), we have to check that W is a $\mathscr{F}_t$ semimartingale such that $|W|_t$ and $\langle\!\langle W, W\rangle\!\rangle_t$ are bounded in probability, that $\sup\limits_{t \leqq T} |\bar{r}_t|$ converges in probability to 0 and that (f, W) is asymptotically monotone. By localization, the proof can be reduced to the case of bounded $|w|_t$, $\langle\!\langle w, w\rangle\!\rangle_t$, a_t and ρ_t (involved in $\mathbb{H}_0$). Since t_k is $\mathscr{F}_t$ predictable, it is evident that the step process W_t is a $\mathscr{F}_t$ special semimartingale with decomposition $W_t = V_t + M_t$ given by $V_t = V_{t_k}$ for $t_k \leqq t < t_{k+1}$ and

$$\begin{aligned} V_{t_N} &= \sum_{k=1}^N \mathbb{E}[\Delta W_{t_k} | \mathscr{F}_{t_k-}] = \sum_{k=1}^N \mathbb{E}[w_{t_{k+1}} - w_{t_k} | \mathscr{G}_{t_k}] \\ &= \sum_{k=1}^N \mathbb{E}[v_{t_{k+1}} - v_{t_k} | \mathscr{G}_{t_k}]. \end{aligned} \tag{2.5.10}$$

This implies

$$|W|_{t_N} = \sum_{k=1}^N |\mathbb{E}[v_{t_{k+1}} - v_{t_k} | \mathscr{G}_{t_k}]| \leqq \sum_{k=1}^N \mathbb{E}[|w|_{t_{k+1}} - |w|_{t_k} | \mathscr{G}_{t_k}]. \tag{2.5.11}$$

Thus the mean of $|W|_t$ is uniformly bounded by $\mathbb{E}|w|_\infty$, so it is bounded in probability. Similarly

$$\langle W, W\rangle_{t_N} = \sum_{k=1}^N \mathbb{E}[(w_{t_{k+1}} - w_{t_k})(w_{t_{k+1}} - w_{t_k})^* | \mathscr{G}_{t_k}], \tag{2.5.12}$$

so

$$\langle\!\langle W, W\rangle\!\rangle_{t_N} = \sum_{k=1}^N \mathbb{E}\left[2 \int_{t_k}^{t_{k+1}} (w_{s-} - w_{t_k})^* \, dv_s + \langle\!\langle w, w\rangle\!\rangle_{t_{k+1}} - \langle\!\langle w, w\rangle\!\rangle_{t_k} | \mathscr{G}_{t_k}\right] \tag{2.5.13}$$

and therefore

$$\mathbb{E}\langle\!\langle W, W\rangle\!\rangle_t \leqq \mathbb{E}[\langle\!\langle w, w\rangle\!\rangle_\infty + 4|w|_\infty \sup_s |w_s|]. \tag{2.5.14}$$

Thus it is also bounded in probability, so that W is $\mathscr{F}_t$ dominated from Proposition 2.2.3 (a). Let us now prove (2.4.4). Denoting

$$f_1(t,x,y)=2(x-y)^*(f(t,x)-f(t,y))$$

and

$$f_2(t,x,y)=(f(t,x)-f(t,y))^*(f(t,x)-f(t,y)), \tag{2.5.15}$$

we have from (2.5.1), (2.5.10) and (2.5.12)

$$\begin{aligned} & f_1(t_k,x,y)\,\Delta V_{t_k}+\operatorname{Trace}(f_2(t_k,x,y)\,\Delta\langle W,W\rangle_{t_k}) \\ & \leqq \mathbb{E}\Bigg[|x-y|^2(a_{t_{k+1}}-a_{t_k})+\int_{t_k}^{t_{k+1}}(f_1(t_k,x,y)-f_1(t,x,y))\,dv_t \\ & \quad +\operatorname{Trace}\Bigg(f_2(t_k,x,y)\Bigg(\int_{t_k}^{t_{k+1}}(w_{t-}-w_{t_k})\,dv_t^*+\int_{t_k}^{t_{k+1}}dv_t(w_{t-}-w_{t_k})^*\Bigg)\Bigg) \\ & \quad +\operatorname{Trace}\int_{t_k}^{t_{k+1}}(f_2(t_k,x,y)-f_2(t,x,y))\,d\langle W,W\rangle_t\,|\,\mathscr{G}_{t_k}\Bigg]. \end{aligned} \tag{2.5.16}$$

Thus if A_t and $\bar{A}_t^{(x,y)}$ are the step processes which take on $[t_N, t_{N+1})$ the values

$$A_{t_N}=\sum_{k=1}^{N}\mathbb{E}[a_{t_{k+1}}-a_{t_k}\,|\,\mathscr{G}_{t_k}] \tag{2.5.17}$$

and

$$\begin{aligned} \bar{A}_{t_N}^{(x,y)}=\sum_{k=1}^{N}\mathbb{E}\Bigg[& \int_{t_k}^{t_{k+1}}|f_1(t,x,y)-f_1(t_k,x,y)|\,d|w|_t \\ & +2\,|f_2(t_k,x,y)|\int_{t_k}^{t_{k+1}}|w_{t-}-w_{t_k}|\,d|w|_t \\ & +\int_{t_k}^{t_{k+1}}|f_2(t,x,y)-f_2(t_k,x,y)|\,d\langle\!\langle w,w\rangle\!\rangle_t\,|\,\mathscr{G}_{t_k}\Bigg] \end{aligned} \tag{2.5.18}$$

then the estimate (2.4.4) is valid. It is easy to check that A_t is bounded in probability, so let us prove that $\bar{A}_t^{(x,y)}$ converges in probability to 0; for any time t, let $\tau(t)$ be equal to t_k for $t_k<t\leqq t_{k+1}$; then

$$\begin{aligned} \mathbb{E}\bar{A}_\infty^{(x,y)}\leqq\mathbb{E}\Bigg[& \int_0^\infty|f_1(t,x,y)-f_1(\tau(t),x,y)|\,d|w|_t \\ & +2\int_0^\infty|f_2(\tau(t),x,y)|\,|w_{t-}-w_{\tau(t)}|\,d|w|_t \\ & +\int_0^\infty|f_2(t,x,y)-f_2(\tau(t),x,y)|\,d\langle\!\langle w,w\rangle\!\rangle_t\Bigg]. \end{aligned} \tag{2.5.19}$$

As $\varepsilon \to \varepsilon_0$, for any t, $\tau(t)$ converges to t on the left, and f, and therefore also f_1 and f_2 are left continuous, so the convergence to 0 follows from the Lebesgue dominated convergence theorem. In order to prove the convergence of $\bar{r}_t$ to 0, we apply the classical stochastic dominated convergence theorem [6]; it is sufficient to prove that $f(\tau(t), x_{\tau(t)})$ converges to $f(t, x_{t-})$ for every t. But from the left continuity in t of f and (2.4.2), we obtain that $f(t, x)$ is (almost surely) jointly left continuous in t and continuous in x, so that the limit as $s\uparrow\uparrow t$ of $f(s, x_s)$ is $f(t, x_{t-})$; the proof of the theorem is now complete. □

Corollary 2.5.3. *With the assumption of Theorem* 2.5.1, *consider the processes X_t and $\tilde{X}_t$ defined by* (2.5.7) *and*

$$\tilde{X}_t = \sum_{k=0}^{\infty} X(k)\, 1_{[t_k, t_{k+1})}(t). \tag{2.5.20}$$

Then X and $\tilde{X}$ converge in probability for the Skorohod topology to x.

Sketch of the Proof. Let us prove the result for X: the convergence of $\tilde{X}$ is checked in a similar way, or one can prove the contiguity of X and $\tilde{X}$ for some compatible metric. From Theorem 2.5.1, we only have to prove that $\bar{x}$ converges to x for the Skorohod topology; since the convergence holds for fixed times t, from Lemma 2.1.3(a), it is sufficient to verify that the laws of $\bar{x}$ are tight. But the tightness of $\bar{x}$ on $D([0, T])$, where T is a continuity point of x, can be deduced from Theorem 15.3 of [4] and the remark

$$\begin{aligned}&\sup\{|\bar{x}_s - \bar{x}_{s_1}| \wedge |\bar{x}_s - \bar{x}_{s_2}|;\ s_1 \leqq s \leqq s_2,\ s_2 - s_1 \leqq \delta\}\\ &\quad \leqq \sup\{|x_s - x_{s_1}| \wedge |x_s - x_{s_2}|;\ s_1 \leqq s \leqq s_2,\ s_2 - s_1 \leqq \delta + \sup_k (t_{k+1} - t_k)\}. \quad \square\end{aligned} \tag{2.5.21}$$

With a prelocalization argument, it is not hard to prove Theorem 2.5.1 when the semimartingale w_t is not special. More complicated discretization operations can also be performed with this method; for instance, the equation

$$x_t = r_t + \int_0^t f_1(s, x_{s-})\, dw_s + \sum_{i,j} \int_0^t f_2^{ij}(s, x_{s-})\, d[w^i, w^j]_s \tag{2.5.22}$$

may be discretized with the scheme

$$\begin{aligned} X(k+1) = X(k) &+ r_{t_{k+1}} - r_{t_k} + f_1(t_k, X(k))(w_{t_{k+1}} - w_{t_k})\\ &+ \sum_{i,j} f_2^{ij}(t_k, X(k))(w^i_{t_{k+1}} - w^i_{t_k})(w^j_{t_{k+1}} - w^j_{t_k}). \end{aligned} \tag{2.5.23}$$

This example uses the convergence of the discretized quadratic variation to $[w, w]_t$ (Theorem I.4.47 of [13]).

2.6 Time Discretization of a Family of Equations

In Sect. 2.5, we have studied time discretization of a single equation, and have obtained a convergence result. We would now like to prove the same result, but for equations which depend on ε; the previous proof cannot be directly applied: for instance, one cannot use the Lebesgue dominated convergence theorem to estimate $\bar{r}_t$. However, such an extension may be useful; suppose for instance that x_t^α are solutions of stochastic integral equations depending on some parameter α and let $X_t^{\alpha,\delta}$ be the result of the Euler scheme with discretization step δ. Then the next theorem will imply the contiguity of $x_t^{\alpha(\delta)}$ and $X_t^{\alpha(\delta),\delta}$ as $\delta \to 0$ for any family $\alpha(\delta)$; thus we will obtain the contiguity of x_t^α and $X_t^{\alpha,\delta}$ as $\delta \to 0$ uniformly in α. Such a result can for instance be applied to some weak convergence results (see the end of this subsection).

Theorem 2.4.3 is still convenient for this problem but we will need more stringent assumptions on the equations. With reference to [2], our assumptions concerning the coefficients are stronger, but more general discretization times are allowed. We will limit ourselves to the case of Lipschitz functions f. We will use the notion of asymptotic uniform quasi-left continuity (AUQL): we will say that a family of $\mathscr{G}_t$ adapted càdlàg processes z_t is AUQL if for any families $\tau \leqq \tau'$ of times of $\mathscr{T}(\mathscr{G}, T)$ such that $\tau' - \tau \to 0$ in probability as $\varepsilon \to \varepsilon_0$, $z_{\tau'} - z_\tau$ converges in probability to 0; this notion is often used in order to prove the tightness of processes for the Skorohod topology (see [1]) and when z_t is bounded in probability, our definition can be proved to be equivalent to the classical one where $\tau' - \tau$ is assumed to be deterministic.

Theorem 2.6.1. *Let $\mathscr{G}_t$ be a family of filtrations, and suppose that we are given a family of $\mathscr{G}_t$ semimartingales w_t, a family of càdlàg $\mathscr{G}_t$ adapted processes r_t and a family of $\mathscr{P}(\mathscr{G}) \otimes \mathscr{B}(\mathbb{R}^{d_1})$ measurable functions f; we suppose that $|w|_t$, $\langle\langle w, w \rangle\rangle_t$ and r_t are bounded in probability, that w_t and r_t are* AUQL, *that f satisfies $\mathbb{H}_0'$ and that $f(t,x)$ is $\mathscr{G}_t$ Riemann and right continuous for every x. Then let (t_k) be a family of subdivisions consisting of $\mathscr{G}_t$ stopping times such that $t_0 = 0$, $t_k < t_{k+1}$, $\lim t_k = \infty$ and $\sup_k \{t_{k+1} - t_k, t_k \leqq T\}$ converges in probability to 0. If x is solution of* (2.5.2.) *and $X(k)$ is the family defined by* (2.5.3), *then the variable* $\sup\{|X(k) - x_{t_k}|, t_k \leqq T\}$ *converges in probability to* 0.

Sketch of the Proof. The proof is divided into two steps. A particular case is dealt with in step 1, and the general case is deduced in step 2.

Step 1. In this step, we suppose moreover that t_k is $\mathscr{G}_t$ predictable and that the processes r and w do not jump at t_k. As in Sect. 2.5, we reduce ourselves to the case $w_t = w_0$ for $t \leqq t_1$. Consider the σ-algebra

$$\tilde{\mathscr{F}}_t = \{B; \forall k, B \cap \{t < t_k\} \in \mathscr{G}_{t_k -}\}. \tag{2.6.1}$$

One can prove like in Lemma 2.5.2 that this defines a filtration satisfying the usual conditions, that $\mathscr{G}_t \subset \tilde{\mathscr{F}}_t$ and $\mathscr{G}_{t_k -} = \tilde{\mathscr{F}}_{t_k -} = \tilde{\mathscr{F}}_{t_{k-1}}$. As in Sect. 2.5, let X_t be given by (2.5.7) and let W_t and R_t be the right discretizations of w_t and r_t;

since we have assumed that r and w are left continuous at times t_k, the processes R_t and W_t are $\tilde{\mathscr{F}}_t$ adapted. We are going to apply Theorem 2.4.3 to the system

$$\begin{aligned} X_t &= R_t + \int_0^t f(s, X_{s-})\, dW_s, \\ x_t &= r_t + \int_0^t f(s, x_{s-})\, dw_s \end{aligned} \tag{2.6.2}$$

where these two equations are respectively relative to the filtrations $\tilde{\mathscr{F}}_t$ and $\mathscr{G}_t$. The process W_t is indeed a $\tilde{\mathscr{F}}_t$ semimartingale and $|W|_t$, $\langle W, W\rangle_t$ are defined by formulas of type (2.5.11) and (2.5.12), but with $\mathscr{G}_{t_k}$ replaced by $\mathscr{G}_{t_k-}$; in particular, they are $\mathscr{G}_t$ predictable. They are also bounded in probability as in Theorem 2.5.1, so W_t is $\mathscr{G}_t$ dominated and (f, W) is monotone. The contiguity in $L^0(\mathscr{G})$ of r, R and of w, W directly follows from our asymptotic uniform quasi-left continuity. Finally, since r_t is AUQL and bounded in probability, the family of its laws is tight for the Skorohod topology from [1], so r_{t-} is $\mathscr{G}_t$ Riemann from Proposition 2.3.2. Thus we can apply Theorem 2.4.3 and obtain the contiguity of X and x in $L^0(\mathscr{G})$. Then consider the right discretization $\bar{x}$ of x; for any family of $\tilde{\mathscr{F}}_t$ stopping times τ, let τ' be equal to t_{k+1} on $t_k \leqq \tau < t_{k+1}$; then

$$1_{[\tau', \infty)}(t) = \sum_k 1_{\{t_k \leqq \tau < t_{k+1}\}} 1_{[t_{k+1}, \infty)}(t) \tag{2.6.3}$$

and since $\{t_k \leqq \tau < t_{k+1}\}$ is $\tilde{\mathscr{F}}_{t_{k+1}-} = \mathscr{G}_{t_{k+1}-}$ measurable, $1_{[\tau', \infty)}(t)$ is $\mathscr{G}_t$ predictable, so τ' is $\mathscr{G}_t$ predictable; on the other hand

$$X_\tau - \bar{x}_\tau = X_{\tau'-} - x_{\tau'-} \tag{2.6.4}$$

and since X and x are contiguous in $L^0(\mathscr{G})$, this expression converges to 0 from Lemma 2.1.4, so X and $\bar{x}$ are contiguous in $L^0(\tilde{\mathscr{F}})$; but they are $\tilde{\mathscr{F}}_t$ adapted, so $\sup_{t \leqq T} |X_t - \bar{x}_t|$ converges to 0; by restricting to the times $t = t_k$, we get the statement of the theorem.

Step 2. Now consider the general case. For $\alpha > 0$, put $t_k^\alpha = t_k + \alpha$ for $k \geqq 1$ and let α describe the set of positive numbers such that with probability 1, the processes r and w do not jump at times t_k^α; consider any family $\alpha(\varepsilon)$ of such numbers converging to 0 as $\varepsilon \to \varepsilon_0$; then we can apply the result of step 1 to the discretization times $t_k^{\alpha(\varepsilon)}$. This means that (with evident notation) $\sup |X^\alpha(k) - x_{t_k^\alpha}|$ converges to 0 as $(\alpha, \varepsilon) \to (0, \varepsilon_0)$. But for ε fixed, from the right continuity of f, r and w, $\sup |X^\alpha(k) - X(k)|$ is easily shown to converge to 0 as $\alpha \to 0$, and $\sup |x_{t_k^\alpha} - x_{t_k}|$ also converges to 0; these three convergences imply the theorem. □

Other sets of assumptions are possible: if $\mathscr{H}_t$ is the almost surely trivial filtration, if w is $\mathscr{H}_t$ dominated, if f is $\mathscr{H}_t$ Riemann and if the times t_k of the subdivisions are deterministic, the asymptotic uniform quasi-left continuity can be replaced by an asymptotic uniform continuity on deterministic times.

These results can be applied to get an approach to some weak convergence problems. Let us assume that $w^{(n)}$ and $r^{(n)}$ are sequences of processes such that their finite dimensional distributions – the distributions of $(w_{s_1}^{(n)}, r_{s_1}^{(n)}, \ldots, w_{s_k}^{(n)}, r_{s_k}^{(n)})$ – converge weakly to the finite distributions of $w^{(\infty)}$, $r^{(\infty)}$ and let $x^{(n)}$, $x^{(\infty)}$ be the solutions of the corresponding equations. For any $\delta > 0$, consider a subdivision (t_k) with step δ, and let $X^{(n,\delta)}$ be the solution of the discretized equation. If the assumptions of Theorem 2.6.1 are satisfied, by using the theorem as explained in the beginning of this subsection, one can deduce that for any t, the variable $x_t^{(n)} - X_t^{(n,\delta)}$ converges to 0 in probability as $\delta \to 0$, uniformly in n; on the other hand, for any fixed δ, the finite distributions of $X^{(n,\delta)}$ converge weakly to $X^{(\infty,\delta)}$, so that one can deduce that the finite distributions of $x^{(n)}$ converge weakly to those of $x^{(\infty)}$. Of course, classical weak convergence methods can often be also applied for this problem.

2.7 Convergence of Moments

In Sect. 2.4, we have stated theorems concerning convergence in probability; if now one looks for convergence results in L^q, one has to study uniform integrability of the solutions (X, x) of (2.4.5). This is the aim of this subsection.

Definition 2.7.1. Let $\mathscr{F}_t$ be a family of filtrations, let R_t be $\mathscr{F}_t$ adapted processes, let W_t be $\mathscr{F}_t$ semimartingales and let $F(t, \omega, x)$ be $\mathscr{P}(\mathscr{F}) \otimes \mathscr{B}(\mathbb{R}^{d_1})$ measurable functions. For $q \geqq 2$, we will say that the family (R, F, W) satisfies $\mathbb{H}_q$ if for some $q'' \geqq q$, the following properties hold;

(i) the variable $\sup_{t \leqq T} |R_t|$ is bounded in $L^{q''}$ (uniformly in ε);

(ii) the semimartingale $\int_0^t F(s, x)\, dW_s$ admits the decomposition

$$F(t, x)\, dW_t = F_1(t, x)\, dW_t^1 + F_2(t, x)\, dW_t^2 \tag{2.7.1}$$

where F_1 and F_2 are $\mathscr{P}(\mathscr{F}) \otimes \mathscr{B}(\mathbb{R}^{d_1})$ measurable, satisfy

$$|F_1(t, x)| \leqq 1, \qquad |F_2(t, x)| \leqq (1 + |x|) \tag{2.7.2}$$

and W_t^1, W_t^2 are $\mathscr{F}_t$ semimartingales;

(iii) the variables $|W^1|_T$, $\langle\!\langle W^1, W^1 \rangle\!\rangle_T$ and $\langle\!\langle W^1 \rangle\!\rangle_T^{(q'')}$ are respectively bounded in $L^{q''}$, $L^{q''/2}$ and L^1;

(iv) either the exponential moments of $|W^2|_T$, $\langle\!\langle W^2, W^2 \rangle\!\rangle_T$ and $\langle\!\langle W^2 \rangle\!\rangle_T^{(q)}$ are bounded and $q'' > q$, or these variables are uniformly bounded by some constant number and $q'' = q$.

Recall that the definition of $\langle\!\langle W \rangle\!\rangle^{(q)}$ was given in Sect. 2.2 and that when we refer to it, we implicitly assume that $\sum |\Delta W_s|^q$ is locally integrable.

Proposition 2.7.2. *For $q \geqq 2$, assume that (R, F, W) satisfies $\mathbb{H}_q$ and suppose that X is a process satisfying*

$$X_t = R_t + \int_0^t F(s, X_{s-})\, dW_s. \tag{2.7.3}$$

Then for any $q' < q$, $\sup_{t \leqq T} |X_t|^{q'}$ is uniformly integrable as $\varepsilon \to \varepsilon_0$.

By using jointly theorems of Sect. 2.4 and Proposition 2.7.2, we obtain contiguity results in $L^{q'}(\mathscr{H})$: in Theorem 2.4.3, suppose that (R, F, W) and (r, f, w) satisfy $\mathbb{H}_q$; in Theorem 2.4.4, suppose only that (R, F, W) satisfies $\mathbb{H}_q$. Proposition 2.7.2 will be proved in Sect. 3.3.

3. Technical Proofs

In this section, we prove the theorems which were only stated in Sect. 2. We first study the examples of $\mathscr{H}_t$ Riemann functions of Sect. 2.3, then obtain the generalized stochastic dominated convergence theorem of Sect. 2.2, the integrability results of Sect. 2.7 and the contiguity results of Sect. 2.4.

3.1 Examples of Riemann Functions

In this subsection, we prove Propositions 2.3.2 and 2.3.3; we will use the notation $\mathbb{E}[\cdot \mid \mathscr{H}_t]$ for the optional projection on $\mathscr{H}_t$.

Proof of Proposition 2.3.2. First note that if there exist admissible localization times v_k such that for each fixed k, $z_{t \wedge v_k}$ is $\mathscr{H}_t$ Riemann, then z_t is $\mathscr{H}_t$ Riemann; one can indeed choose k such that $v_k > T$ with probability $1 - \delta/2$ for every ε; then if $\bar{z}_t$ is a step process satisfying

$$\mathbb{E}[|z_{\tau \wedge v_k} - \bar{z}_\tau| \wedge 1] \leqq \delta/2 \tag{3.1.1}$$

for $\tau \in \mathscr{T}_p(\mathscr{H}, T)$, then (2.3.2) is satisfied. Thus in order to prove Proposition 2.3.2, we can localize z_t, or equivalently, prelocalize x_t; since x_t is assumed to be bounded in probability, we can assume that it remains in some bounded interval $[-L, L]$. Fix some $\delta > 0$. The process $\bar{z}_t$ of Definition 2.3.1 will be taken of the form $\bar{x}_{t-}$ with

$$\bar{x}_t = \sum_{i=0}^{N-1} 1_{[\tau_i, \tau_{i+1})}(t)\, x_{\tau_i}. \tag{3.1.2}$$

In this expression, τ_i is defined by induction by $\tau_0 = 0$ and

$$\tau_{i+1} = \inf\{t \geqq \tau_i;\ \mathbb{E}[|x_t - x_{\tau_i}| \wedge 1 \mid \mathscr{H}_t] > \delta/2\} \tag{3.1.3}$$

and

$$N = \inf\{i;\ \mathbb{P}[\tau_i \leqq T] \leqq \delta/2\}. \tag{3.1.4}$$

We will actually prove that N is finite. One checks by induction on i that the $\mathscr{H}_t$ optional projection involved in this definition is also $\mathscr{G}_t$ optional, is càdlàg (because the optional projection of a càdlàg process is càdlàg: Theorem VI.47 of [6]) and that τ_i is a $\mathscr{G}_t \cap \mathscr{H}_t$ stopping time. Then

$$|x_t - \bar{x}_t| \wedge 1 = \sum_{i=0}^{N-1} 1_{[\tau_i, \tau_{i+1})}(t)(|x_t - x_{\tau_i}| \wedge 1) + 1_{[\tau_N, \infty)}(t)(|x_t| \wedge 1). \tag{3.1.5}$$

From the Definition (3.1.3), the $\mathscr{H}_t$ optional projection of this process is dominated by $\delta/2$ on $\{t < \tau_N\}$ so if $\tau \in \mathscr{T}(\mathscr{H}, T)$,

$$\mathbb{E}[|x_\tau - \bar{x}_\tau| \wedge 1] \leqq \delta/2 + \mathbb{P}[\tau \geqq \tau_N] \leqq \delta. \tag{3.1.6}$$

The estimate (2.3.2) then follows from Lemma 2.1.4. The only thing which is still to be proved is that N is bounded uniformly in ε. Since the optional projection in (3.1.3) is right continuous, we deduce that

$$\mathbb{E}[|x_{\tau_{i+1}} - x_{\tau_i}| \wedge 1 \,|\, \mathscr{H}_{\tau_{i+1}}] \geqq \delta/2 \tag{3.1.7}$$

on $\{\tau_{i+1} < \infty\}$, and therefore on $\{\tau_{i+1} \leqq T\}$; thus, by taking the expectation,

$$\mathbb{E}[|x_{\tau_{i+1} \wedge T} - x_{\tau_i \wedge T}| \wedge 1] \geqq \mathbb{P}[\tau_{i+1} \leqq T]\, \delta/2 \geqq \delta^2/4 \tag{3.1.8}$$

for $i+1 < N$ (recall (3.1.4)). We remark

$$\mathbb{E}[|x_{\tau_{i+1} \wedge T} - x_{\tau_i \wedge T}| \wedge 1] \leqq \mathbb{P}\left[|x_{\tau_{i+1} \wedge T} - x_{\tau_i \wedge T}| \geqq \frac{\delta^2}{8}\right] + \frac{\delta^2}{8}, \tag{3.1.9}$$

and deduce

$$\mathbb{P}\left[|x_{\tau_{i+1} \wedge T} - x_{\tau_i \wedge T}| \geqq \frac{\delta^2}{8}\right] \geqq \frac{\delta^2}{8} \tag{3.1.10}$$

for $i+1 < N$. Defining

$$\xi := \#\left\{i < N-1;\, |x_{\tau_{i+1} \wedge T} - x_{\tau_i \wedge T}| \geqq \frac{\delta^2}{8}\right\}, \tag{3.1.11}$$

we have for any integer N_0

$$(N-1)\frac{\delta^2}{8} \leqq \mathbb{E}\xi \leqq N_0 + (N-1)\,\mathbb{P}[\xi > N_0]. \tag{3.1.12}$$

On the other hand,

$$\mathbb{P}[\xi > N_0] \leqq \mathbb{P}[\exists 0 \leqq t_1 < t'_1 \leqq t_2 < t'_2 \leqq \ldots < t'_{N_0} \leqq T, \forall i, |x(t'_i) - x(t_i)| \geqq \delta^2/8]. \tag{3.1.13}$$

Now consider a discretization of the space $[-L, L]$ with a grid (ζ_k) of step size $\delta^2/16$; on the event described in the right side of (3.1.13), the sum over k of the numbers of downcrossings or upcrossings of x from ζ_k to ζ_{k+1} or vice versa is at least equal to N_0. But the assumption (ii) implies that these numbers are uniformly bounded in probability. Thus the right-hand side of (3.1.13) converges to 0 as $N_0 \uparrow \infty$ uniformly in ε, so we can choose N_0 such that its value is less than $\delta^2/8$; thus N is bounded from (3.1.12) and we get the result of the proposition if x_t satisfies (i) and (ii). If x_t are $\mathscr{G}_t$ semimartingales such that x_t and $|x|_t$ are bounded in probability, (ii) can be proved by applying the results of [27] and if the laws of x_t are tight for the Skorohod topology, it follows easily from Theorem 15.2 of [4]. □

Proof of Proposition 2.3.3. Fix δ and put

$$\bar{z}_t = \sum_{i=0}^{N-1} 1_{[\tau_i]}(t)\, z_{\tau_i} \tag{3.1.14}$$

where τ_i is defined by induction with $\tau_0 = 0$ and

$$\tau_{i+1} = \inf\{t \geqq \tau_i;\ \mathbb{E}[\sup_{\tau_i < s \leqq t} |z_s| \wedge 1 \,|\, \mathscr{H}_t] \geqq \delta/2\}, \tag{3.1.15}$$

and N is defined by (3.1.4). The term $\sup_{\tau_i < s \leqq t} |z_s|$ is càdlàg – the only non trivial point is the right continuity at τ_i, which holds because the right limit of z_s at any time is 0 – so its optional projection is also càdlàg and we prove as in Proposition 2.3.2 that τ_i is a $\mathscr{G}_t \cap \mathscr{H}_t$ stopping time; moreover, since $\sup_{\tau_i < s \leqq t} |z_s|$ is $\mathscr{G}_t$ predictable, the $\mathscr{H}_t$ optional projection is from our assumptions $\mathscr{G}_t$ and $\mathscr{H}_t$ predictable, so τ_i is $\mathscr{G}_t$ and $\mathscr{H}_t$ predictable: it is indeed the beginning of a right continuous predictable set. We also have

$$|z_t - \bar{z}_t| \wedge 1 = \sum_{i=0}^{N-1} 1_{(\tau_i, \tau_{i+1})}(t)(|z_t| \wedge 1) + 1_{(\tau_N, \infty)}(t)(|z_t| \wedge 1) \tag{3.1.16}$$

so by taking the expectation at any time τ of $\mathscr{T}(\mathscr{H}, T)$, it follows from the definitions of τ_i and N that (2.3.2) is satisfied. Let us now prove that N is bounded; on $\{\tau_{i+1} < \infty\}$,

$$\mathbb{E}[\sup_{\tau_i < s \leqq \tau_{i+1}} |z_s| \wedge 1 \,|\, \mathscr{H}_{\tau_{i+1}}] \geqq \frac{\delta}{2}, \tag{3.1.17}$$

so we can prove like previously that if

$$\xi = \#\left\{i < N-1;\ \sup_{\tau_i < s \leqq \tau_{i+1}} |z_s| \geqq \frac{\delta^2}{8},\ \tau_{i+1} \leqq T\right\}, \tag{3.1.18}$$

then (3.1.12) holds; from our assumptions, $\mathbb{P}[\xi > N_0]$ converges uniformly to 0 as $N_0 \uparrow \infty$, so we can conclude. □

3.2 Proof of the Stochastic Dominated Convergence Theorem

The aim of this subsection is to prove Proposition 2.2.4. We will study separately the case of nondecreasing processes and the case of martingales. The following lemma is basic for all subsequent results.

Lemma 3.2.1. *Let $\mathscr{H}_t$ be a family of filtrations. Consider $1 \leqq q < p \leqq \infty$, ψ_t a family of positive measurable processes and A_t a family of nondecreasing processes such that*

$$\sup_{\varepsilon} \mathbb{E} \int_0^T \left| \frac{dA_t}{dL_t} \right|^p dL_t < \infty \tag{3.2.1}$$

for some admissible $\mathscr{H}_t$ dominating processes L_t. Then for $T > 0$, there exists C which does not depend on ψ such that

$$\left\| \int_0^T \psi_t \, dA_t \right\|_q \leqq C \|\psi\|_{\mathscr{H}, q', T} \tag{3.2.2}$$

with $1/q = 1/p + 1/q'$.

Proof. Put $\phi_t = dA_t/dL_t$ in (3.2.1); then

$$\mathbb{E} \left| \int_0^T \psi_s \, dA_s \right|^q \leqq \mathbb{E} \int_0^T \psi_s^q \phi_s^q \, dL_s. \tag{3.2.3}$$

On the other hand, consider the change of time

$$\tau_t = \inf\{s;\, L_s \geqq t\}. \tag{3.2.4}$$

Then the τ_t are $\mathscr{H}_t$ predictable stopping times and

$$\mathbb{E} \int_0^T \psi_s^q \phi_s^q \, dL_s = \mathbb{E} \int_0^1 \psi_{\tau_t}^q \phi_{\tau_t}^q 1_{\{\tau_t \leqq T\}} \, dt. \tag{3.2.5}$$

If ϕ is bounded, we can immediately deduce (3.2.2) with $q = q'$ and if $p < \infty$, we apply Hölder's inequality and notice that

$$\mathbb{E} \int_0^1 \phi_{\tau_t}^p 1_{\{\tau_t \leqq T\}} \, dt = \mathbb{E} \int_0^T \phi_s^p \, dL_s \tag{3.2.6}$$

is uniformly bounded. □

The following result says that in a decomposition $W_t = V_t + M_t$, the predictable characteristics of the local martingale M_t are dominated by the characteristics of W_t.

Lemma 3.2.2. *Let $q \geqq 2$. If the semimartingales W_t admit the canonical decomposition $W_t = V_t + M_t$, then the increments of $\langle\!\langle M, M \rangle\!\rangle_t$ and $\langle\!\langle M \rangle\!\rangle_t^{(q)}$ are dominated by the corresponding increments of $\langle\!\langle W, W \rangle\!\rangle_t$ and $\langle\!\langle W \rangle\!\rangle_t^{(q)}$ (with a multiplicative positive constant number depending only on q).*

Proof. It is sufficient to study the increments of $\langle\!\langle M \rangle\!\rangle_t^{(q)}$ *and* $\langle\!\langle W \rangle\!\rangle_t^{(q)}$ for $q \geqq 2$; this will indeed imply that the increments of $\langle\!\langle M, M \rangle\!\rangle_t$ are dominated by the increments of $\langle\!\langle W, W \rangle\!\rangle_t$ because

$$\langle\!\langle W, W \rangle\!\rangle_t - \langle\!\langle M, M \rangle\!\rangle_t = \langle\!\langle W \rangle\!\rangle_t^{(2)} - \langle\!\langle M \rangle\!\rangle_t^{(2)}. \tag{3.2.7}$$

We can decompose the two processes $\langle\!\langle M \rangle\!\rangle_t^{(q)}$ and $\langle\!\langle W \rangle\!\rangle_t^{(q)}$ into a continuous part and a countable sum of jumps at predictable times. The continuous parts come from the totally inaccessible jumps of M_t and W_t, so since $W_t - M_t$ is predictable, they coincide; thus we have to compare $\Delta \langle\!\langle M \rangle\!\rangle_\tau^{(q)}$ and $\Delta \langle\!\langle W \rangle\!\rangle_\tau^{(q)}$ for τ $\mathscr{F}_t$ predictable. By localization, we can suppose that $\sum_s |\Delta W_s|^q$ is integrable. Then

$$\Delta M_\tau = \Delta W_\tau - \mathbb{E}[\Delta W_\tau | \mathscr{F}_{\tau-}] \tag{3.2.8}$$

so

$$\Delta \langle\!\langle M \rangle\!\rangle_\tau^{(q)} = \mathbb{E}[|\Delta M_\tau|^q | \mathscr{F}_{\tau-}] \leqq C\,\mathbb{E}[|\Delta W_\tau|^q | \mathscr{F}_{\tau-}] = C\,\Delta \langle\!\langle W \rangle\!\rangle_\tau^{(q)}. \quad \square \tag{3.2.9}$$

If M is a local martingale and τ is a stopping time, classical Burkholder-Davis-Gundy estimates relate the integrability of $\sup_{t \leqq \tau} |M_t|$ to $[\![M, M]\!]_\tau$; the next result explains how it can also be related to the predictable processes $\langle\!\langle M, M \rangle\!\rangle_t$ and $\langle\!\langle M \rangle\!\rangle_t^{(q)}$.

Lemma 3.2.3. *Let $q \geqq 2$ and let M be a locally square integrable $\mathscr{F}_t$ martingale such that $M_0 = 0$ and $\sum |\Delta M_s|^q$ is locally integrable. Then there exists a constant C which depends only on q such that for any $\mathscr{F}_t$ stopping time τ,*

$$\mathbb{E}[\sup_{t \leqq \tau} |M_t|^q] \leqq C\,\mathbb{E}[\langle\!\langle M, M \rangle\!\rangle_\tau^{q/2} + \langle\!\langle M \rangle\!\rangle_\tau^{(q)}]. \tag{3.2.10}$$

Proof. It is sufficient to prove the result for $\tau = \infty$. From the above discussion, in order to get (3.2.10), the only thing we have to prove is

$$\mathbb{E}[\![M, M]\!]_\infty^{q/2} \leqq C\,\mathbb{E}[\langle\!\langle M, M \rangle\!\rangle_\infty^{q/2} + \langle\!\langle M \rangle\!\rangle_\infty^{(q)}]. \tag{3.2.11}$$

The continuous part of $[\![M, M]\!]$ is easily estimated, and for the purely discontinuous part, we are reduced to prove the following result: if D_t is an adapted, nondecreasing, purely discontinuous, locally integrable process with $D_0 = 0$, and if for $q' \geqq 1$, $D_t^{(q')}$ is the predictable compensator of $\sum |\Delta D_s|^{q'}$, then

$$\|D_\infty\|_{q'} \leqq C\,\|D_\infty^{(1)}\|_{q'} + C\,\|D_\infty^{(q')}\|_1^{1/q'}. \tag{3.2.12}$$

One indeed uses this result for $D = \sum |\Delta M_s|^2$ and $q' = q/2$. We are now going to check that (3.2.12) is a corollary of Theorem VI.99 of [6]; first, by a localization argument and Fatou's lemma, we can reduce ourselves to the case of an integrable process D; we recall that the left potential of D is defined by

$$\begin{aligned} Z_t &= \mathbb{E}[D_\infty \mid \mathscr{F}_t] - D_{t-} \\ &= \mathbb{E}[D_\infty - D_t \mid \mathscr{F}_t] + \Delta D_t \\ &= \mathbb{E}[D_\infty^{(1)} - D_t^{(1)} \mid \mathscr{F}_t] + \Delta D_t \\ &\leqq \mathbb{E}[D_\infty^{(1)} + \sup_s |\Delta D_s| \mid \mathscr{F}_t]. \end{aligned} \tag{3.2.13}$$

Thus from [6],

$$\|D_\infty\|_{q'} \leqq C \|D_\infty^{(1)}\|_{q'} + C \|\sup_s |\Delta D_s|\|_{q'} \tag{3.2.14}$$

and this implies (3.2.12). $\square$

Proof of Proposition 2.2.4. We get from Lemma 3.2.3

$$\begin{aligned} \mathbb{E} \sup_{t \leqq T} |W_t|^q &\leqq C \mathbb{E} |W|_T^q + C \mathbb{E} \sup_{t \leqq T} |M_t|^q \\ &\leqq C \mathbb{E} |W|_T^q + C \mathbb{E} \langle\!\langle M, M \rangle\!\rangle_T^{q/2} + C \mathbb{E} \langle\!\langle M \rangle\!\rangle_T^{(q)}. \end{aligned} \tag{3.2.15}$$

By means of Lemma 3.2.2, we can estimate the characteristics of M with those of W and obtain (a). The result (b) follows from (a) applied to the stochastic integral and Lemma 3.2.1. Finally, let us prove (c). Since the result that we want to prove is local, we can suppose that the localization times τ_k involved in Definition 2.2.2 are $+\infty$; since ψ_t is bounded in probability and predictable, we can also reduce ourselves to the case where it is uniformly bounded; in particular, its convergence in $\mathbb{L}^0(\mathscr{H})$ will imply therefore its convergence in $\mathbb{L}^q(\mathscr{H})$ for any q. We decompose the integral $\int_0^t \psi_s \, dW_s$; from Lemma 3.2.1, if we define q_1 by $1 = 1/p + 1/q_1$, the L^1 norm of $\int_0^T |\psi_s| \, d|W|_s$ is dominated by $\|\psi\|_{\mathscr{H}, q_1, T}$ so the finite variation part of the integral converges to 0; on the other hand from the part (b) and Lemma 3.2.1, if we define $q_2 = 2q_1$, the L^2 norm of $\sup_{t \leqq T} \left| \int_0^t \psi_s \, dM_s \right|$ is dominated by $\|\psi\|_{\mathscr{H}, q_2, T}$ so the martingale part also converges to 0. $\square$

3.3 Estimation for Stochastic Integral Equations

We now derive lemmas which will be used with Proposition 2.2.4 to prove the results of Sect. 2.4; as a corollary, we also prove Proposition 2.7.2. Suppose that we want to estimate a family of stochastic integrals

$$U_t = \int_0^t \mu_s \, dW_s \tag{3.3.1}$$

such that μ_t is dominated by some quantity depending on U_{t-}; for instance, U_t may be the solution of a stochastic integral equation; then Proposition 2.2.4 is no more sufficient to estimate it, but we rather need Gronwall-type results. The proofs of the following lemmas are partly based on [10].

Lemma 3.3.1. *Let W_t be a family of $\mathbb{R}^{d_2}$ valued $\mathscr{F}_t$ semimartingales, Z_t and μ_t families of $\mathscr{F}_t$ predictable locally bounded processes which take their values respectively in* $(0, \infty)$ *and* $\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2}$, *and U_t the processes defined by* (3.3.1). *We suppose*

$$|\mu_t| \leqq C_0(|U_{t-}| + Z_t) \tag{3.3.2}$$

for some constant C_0. Fix $q \geqq 2$, $1 < p \leqq \infty$, $1 \leqq q' < q$, $T > 0$, define q'' by $1 = q/q'' + 1/p$ and put

$$A_t = |W|_t + \langle\!\langle W, W \rangle\!\rangle_t + \langle\!\langle W \rangle\!\rangle_t^{(q)}. \tag{3.3.3}$$

We suppose that for some admissible $\mathscr{H}_t$ dominating processes L_t, relation (3.2.1) *holds. There exists a $K > 0$ depending only on C_0, q, q' such that if the exponential moment of order K of A_T is bounded, then one has*

$$\|\sup_{t \leqq T} |U_t|\,\|_{q'} \leqq C_1 \|Z\|_{\mathscr{H}, q'', T}, \tag{3.3.4}$$

for some constant C_1 depending neither on ε nor on the family Z. If $p = \infty$ and $\mathscr{H}_t \subset \mathscr{F}_t$, one can take $q' = q = q''$.

Proof. For some fixed $\lambda > 0$, let E_t be the solution of

$$E_t = 1 + \lambda \int_0^t E_{s-}\, dA_s. \tag{3.3.5}$$

Then E_t is a $\mathscr{F}_t$ predictable process and $\bar{E}_t := E_t^{-1}$ is solution of

$$\bar{E}_t = 1 - \lambda \int_0^t \bar{E}_s\, dA_s. \tag{3.3.6}$$

We write the integration by parts formula

$$|U_t|^q \bar{E}_t = \int_0^t |U_{s-}|^q\, d\bar{E}_s + \int_0^t \bar{E}_s\, d(|U_s|^q), \tag{3.3.7}$$

the differential of $\bar{E}_t$ is given by (3.3.6) and that of $|U_t|^q$ can be deduced from Itô's formula; after some calculation, we obtain

$$\begin{aligned} |U_t|^q \bar{E}_t = {} & -\lambda \int_0^t \bar{E}_s |U_{s-}|^q\, dA_s + q \int_0^t \bar{E}_s |U_{s-}|^{q-2} U_{s-}^* \mu_s\, dW_s \\ & + \frac{q}{2} \operatorname{Trace} \int_0^t \bar{E}_s |U_{s-}|^{q-2} \mu_s^* \mu_s\, d\langle W, W \rangle_s^c \\ & + \frac{q(q-2)}{2} \operatorname{Trace} \int_0^t \bar{E}_s |U_{s-}|^{q-4} \mu_s^* U_{s-} U_{s-}^* \mu_s\, d\langle W, W \rangle_s^c \\ & + \sum_{s \leqq t} \bar{E}_s [\Delta(|U_s|^q) - q |U_{s-}|^{q-2} U_{s-}^* \Delta U_s]. \end{aligned} \tag{3.3.8}$$

Let us estimate the terms of the right-hand side. Using the canonical decomposition $W=V+M$, the integral with respect to W can be decomposed into two parts; the 'finite variation' part can be estimated by means of

$$|U_{s-}|^{q-1}|\mu_s| \leqq C(|U_{s-}|^q + Z_s^q) \tag{3.3.9}$$

which is a consequence of (3.3.2); the two integrals with respect to $\langle W, W\rangle^c$ can be estimated in a similar way. For the jumps, we have

$$\begin{aligned}|\Delta(|U_s|^q) - q|U_{s-}|^{q-2} U_{s-}^* \Delta U_s| &\leqq C(|\Delta U_s|^2 |U_{s-}|^{q-2} + |\Delta U_s|^q)\\ &\leqq C(|U_{s-}|^q + Z_s^q)(|\Delta W_s|^2 + |\Delta W_s|^q).\end{aligned} \tag{3.3.10}$$

From all these estimates we get

$$\begin{aligned}|U_t|^q \bar{E}_t \leqq &-\lambda \int_0^t \bar{E}_s |U_{s-}|^q dA_s + C\int_0^t \bar{E}_s(|U_{s-}|^q + Z_s^q)\, dA_s\\ &+ q\int_0^t \bar{E}_s |U_{s-}|^{q-2} U_{s-}^* \mu_s\, dM_s\\ &+ C\Bigg(\sum_{s\leqq t} \bar{E}_s(|U_{s-}|^q + Z_s^q)(|\Delta W_s|^2 + |\Delta W_s|^q)\\ &- \int_0^t \bar{E}_s(|U_{s-}|^q + Z_s^q)(d\langle\!\langle W\rangle\!\rangle_s^{(2)} + d\langle\!\langle W\rangle\!\rangle_s^{(q)})\Bigg).\end{aligned} \tag{3.3.11}$$

Note that in all the calculation, the constant numbers C do not depend on λ so we can now choose it greater than C. Let us take the mean at time $t=\tau$ for a $\mathscr{F}_t$ stopping time $\tau \leqq T$. By choosing stopping times τ_k which reduce the local martingales involved in this formula, by writing the formula at time $\tau \wedge \tau_k$ and using Fatou's lemma in the limit, we can check that the martingale parts of W_t, $\sum |\Delta W_s|^2$ and $\sum |\Delta W_s|^q$ can be neglected. Thus we obtain

$$\mathbb{E}[|U_\tau|^q \bar{E}_\tau] \leqq C\mathbb{E}\left[\int_0^\tau \bar{E}_s Z_s^q\, dA_s\right]. \tag{3.3.12}$$

Now notice that $\bar{E}_t \leqq 1$ and estimate the integral with respect to A with Lemma 3.2.1; this yields

$$\mathbb{E}[|U_\tau|^q \bar{E}_\tau] \leqq C\, \|Z\|^q_{\mathscr{H}, q'', T}. \tag{3.3.13}$$

If $q'<q$, choose $q'<q_1<q$ and define q_2 by $1/q_1 = 1/q + 1/q_2$; we derive from Hölder's inequality

$$\|U_\tau\|_{q_1} \leqq \|U_\tau \bar{E}_\tau^{1/q}\|_q \, \|E_\tau^{1/q}\|_{q_2} \leqq C\mathbb{E}[E_T^{q_2/q}]^{1/q_2} \|Z\|_{\mathscr{H}, q'', T}. \tag{3.3.14}$$

On the other hand,

$$E_T = \exp \lambda A_T \prod_{s\leqq T} (1+\lambda \Delta A_s) e^{-\lambda \Delta A_s} \leqq \exp \lambda A_T \tag{3.3.15}$$

so by choosing $K=\lambda q_2/q$, the exponential integrability of order K of A_T implies

$$\|U_t\|_{q_1} \leqq C\,\|Z\|_{\mathscr{H},q'',T}. \tag{3.3.16}$$

Thus we can deduce (3.3.4) from Lemma 2.1.2(a). Now suppose that $p=\infty$ and $\mathscr{H}_t\subset\mathscr{F}_t$. Then $q''=q$ and E_T is uniformly bounded so we deduce from (3.3.13) that

$$|U|_{\mathscr{F},q,T} \leqq C\,\|Z\|_{\mathscr{H},q,T}. \tag{3.3.17}$$

On the other hand, from Proposition 2.2.4(b) applied to the stochastic integral (3.3.1), we have

$$\|\sup_{t\leqq T}|U_t|\,\|_q \leqq C\,\|\mu\|_{\mathscr{H},q,T} \leqq C\,\|U_-\|_{\mathscr{H},q,T} + C\,\|Z\|_{\mathscr{H},q,T}. \tag{3.3.18}$$

From Lemma 2.1.4, $\|U_-\|_{\mathscr{H},q,T}$ is dominated by $|U|_{\mathscr{H},q,T}$ and this last quantity is estimated by means of (3.3.17), so that we get (3.3.4) with $q'=q=q''$. □

Proof of Proposition 2.7.2. Consider the processes

$$\begin{aligned} U_t &= X_t - R_t - \int_0^t F_1(s, X_{s-})\,dW_s^1, \\ \mu_t &= F_2(t, X_{t-}). \end{aligned} \tag{3.3.19}$$

Consider also the number q'' of Definition 2.7.1, some $q'<q_1<q$ and the number p defined by $1=q/q''+1/p$. Then it follows from the assumptions that we can apply Lemma 3.3.1 with $\mathscr{H}_t=\mathscr{F}_t$, $W_t=W_t^2$ and

$$Z_t = 1 + |R_{t-}| + \left|\int_0^{t-} F_1(s, X_{s-})\,dW_s^1\right|. \tag{3.3.20}$$

We deduce that

$$\|\sup_{t\leqq T}|U_t|\,\|_{q_1} \leqq C\left(1 + \|\sup_{t\leqq T}|R_t|\,\| + \left\|\sup_{t\leqq T}\left|\int_0^t F_1(s, X_{s-})\,dW_s^1\right|\right\|_{q''}\right). \tag{3.3.21}$$

From (3.3.19), the L^{q_1} norm of $\sup|X_t|$ is also dominated by the right-hand side of (3.3.21); moreover we have assumed that the $L^{q''}$ norm of $\sup_t|R_t|$ is bounded and we easily deduce from Proposition 2.2.4(a) that the $L^{q''}$ norm of the stochastic integral with respect to W^1 is also bounded. The family of variables $\sup_{t\leqq T}|X_t|$ is therefore bounded in L^{q_1} so its q'th power is uniformly integrable. □

We now describe, in the case $q=2$, a modification of condition (3.3.2) which implies the same type of result; when applied to stochastic integral equations, this will correspond to the case of monotone coefficients as considered in [12] or [10].

Lemma 3.3.2. *Let W_t be a family of $\mathbb{R}^{d_2}$ valued $\mathscr{F}_t$ semimartingales, Z_t and μ_t families of non-negative and $\mathbb{R}^{d_1}\otimes\mathbb{R}^{d_2}$ valued $\mathscr{F}_t$ predictable locally bounded processes, and U_t the processes defined by* (3.3.1). *We suppose that*

$$2\,U_{t-}^{*}\,\mu_t\,d V_t+\operatorname{Trace}(\mu_t^{*}\,\mu_t\,d\langle W, W\rangle_t)\leqq |U_{t-}|^2\,dA'_t+Z_t\,dA''_t+dA'''_t \quad (3.3.22)$$

where V_t is the predictable finite variation part of W and A'_t, A''_t, A'''_t are families of nondecreasing $\mathscr{F}_t$ predictable processes. Let $1<p\leqq\infty$; we suppose that the exponential moments of A'_T are bounded and that A''_t satisfies (3.2.1) *for some admissible $\mathscr{H}_t$ dominating processes L_t. Then for $1\leqq q'<2$, there exists C such that*

$$\|\sup_{t\leqq T}|U_t|\,\|_{q'}^2\leqq C\,\|Z\|_{\mathscr{H},q'',T}+C\,\|A'''_T\|_1 \quad (3.3.23)$$

holds for $1=1/p+1/q''$.

Proof. Consider processes E' and $\bar{E}'$ satisfying (3.3.5) and (3.3.6) for $\lambda=1$, but with A replaced by A'. The Itô formula (3.3.8) can be written in this case as

$$\begin{aligned}|U_t|^2\,\bar{E}'_t &= -\int_0^t \bar{E}'_s\,|U_{s-}|^2\,dA'_s+2\int_0^t \bar{E}'_s\,U_{s-}^{*}\,\mu_s\,dW_s\\ &\quad+\operatorname{Trace}\int_0^t \bar{E}'_s\mu_s^{*}\,\mu_s\,d\langle W, W\rangle_s^c+\sum_{s\leqq t}\bar{E}'_s\,|\Delta U_s|^2\\ &= -\int_0^t \bar{E}'_s\,|U_{s-}|^2\,dA'_s+2\int_0^t \bar{E}'_s\,U_{s-}^{*}\,\mu_s\,dV_s\\ &\quad+\operatorname{Trace}\int_0^t \bar{E}'_s\mu_s^{*}\,\mu_s\,d\langle W, W\rangle_s+\{\text{local martingale}\}\\ &\leqq \int_0^t \bar{E}'_s Z_s\,dA''_s+\int_0^t \bar{E}'_s\,dA'''_s+\{\text{local martingale}\}\end{aligned} \quad (3.3.24)$$

from (3.3.22). The end of the proof is similar to previous lemma: we take the mean at a $\mathscr{F}_t$ stopping time $\tau\in\mathscr{T}(\mathscr{F}, T)$; since the local martingale can be neglected, we deduce that if $q'<q_1<2$,

$$\|U_\tau\|_{q_1}^2\leqq C\,\mathbb{E}[|U_\tau|^2\,\bar{E}'_\tau]\leqq C\,\mathbb{E}\left(\int_0^\tau Z_s\,dA''_s+A'''_\tau\right) \quad (3.3.25)$$

so

$$|U|^2_{\mathscr{F},q_1,T}\leqq C\,\|Z\|_{\mathscr{H},q'',T}+C\,\|A'''_T\|_1, \quad (3.3.26)$$

and we use Lemma 2.1.2 to prove (3.3.23). □

3.4 Proof of the Main Theorems

Theorems 2.4.3 and 2.4.4 will be easily proved from the lemmas of this subsection. First, we explain how the process $\bar{A}^{(x,y)}$ in Definition 2.4.1 (c) can be chosen uniformly dominated for (x, y) in a bounded subset of $\mathbb{R}^{d_1} \times \mathbb{R}^{d_1}$.

Lemma 3.4.1. *Let $F(t, \omega, x)$ be a family of $\mathscr{P}(\mathscr{F}) \otimes \mathscr{B}(\mathbb{R}^{d_1})$ measurable functions and W_t a family of $\mathscr{F}_t$ semimartingales such that F satisfies $\mathbb{H}_0$, $|W|_t$ and $\langle\langle W, W\rangle\rangle_t$ are bounded in probability and (F, W) is asymptotically monotone. Then for any $K>0$, there exists a nondecreasing right-continuous $\mathscr{F}_t$ predictable process $\bar{A}_t^{(K)}$ such that firstly $\bar{A}_t^{(K)}$ is bounded in probability and converges in probability to 0 as $\varepsilon \to \varepsilon_0$ and secondly, for $|x|$ and $|y| \leqq K$, (2.4.4) holds with $\bar{A}_t^{(x,y)} = \bar{A}_t^{(K)}$.*

Proof. It is sufficient to prove the lemma for $t \leqq T$, where T is any fixed number; fix also $K>0$, let B_K be the ball in $\mathbb{R}^{d_1}$ with center 0 and radius K and consider $G_k = \{\alpha_1, \ldots, \alpha_k\}$ where $\{\alpha_i, i \in \mathbb{N}\}$ is a dense subset of B_K; one can construct by induction on k a sequence N_k of neighbourhoods of ε_0 which decreases to $\{\varepsilon_0\}$ as $k \uparrow \infty$ and such that if one defines the family $G = G(\varepsilon) = G_k$ for $\varepsilon \in N_k \setminus N_{k+1}$, then $\sum_{(x,y) \in G \times G} \bar{A}_T^{(x,y)}$ converges in probability to 0 as $\varepsilon \to \varepsilon_0$. Now if one considers the left-hand side of (2.4.4) for x and y in B_K, one can first approach x and y by x' and y' in G and then estimate the result of this approximation with $\bar{A}_t^{(x',y')}$. The final result of this method involves the process

$$
\begin{aligned}
\bar{A}_t^{(K)} = & \sum_{(x,y) \in G \times G} \bar{A}_t^{(x,y)} + \sup_{(x,y) \in B_K \times B_K} \inf_{(x',y') \in G \times G} \left| |x-y|^2 - |x'-y'|^2 \right| A_t \\
& + 2 \sup_{(x,y) \in B_K \times B_K} \inf_{(x',y') \in G \times G} \sup_{s \leqq t} |(x-y)^* (F(s,x) - F(s,y)) \\
& - (x'-y')^* (F(s,x') - F(s,y'))| \, |W|_t \\
& + \sup_{(x,y) \in B_K \times B_K} \inf_{(x',y') \in G \times G} \sup_{s \leqq t} \left| |F(s,x) - F(s,y)|^2 \right. \\
& \left. - |F(s,x') - F(s,y')|^2 \right| \langle\langle W, W \rangle\rangle_t .
\end{aligned} \tag{3.4.1}
$$

It is then easy to deduce from our assumptions the convergence of $\bar{A}_t^{(K)}$ to 0. $\square$

As another lemma, we give an a priori estimate on the solution (X, x) of (2.4.5).

Lemma 3.4.2. *Under Assumption 2.4.2 (without (2.4.2) and (2.4.4)), if R_t, r_t and the predictable characteristics of W_t and w_t are bounded in probability, then the processes X_t and x_t are also bounded in probability.*

Proof. The two processes are estimated in a similar way so we will study X_t; the proof will follow from a localization argument and Lemma 3.3.1. First, there exist admissible localization $\mathscr{F}_t$ times τ_k' such that $|W|_{\tau_k'}$, $\langle\langle W, W\rangle\rangle_{\tau_k'}$ and $\rho_{t \wedge \tau_k'}$ are bounded (where ρ_t is the process involved in (2.4.1) which controls the growth of F); we define τ_k'' as the first time at which the process $|R|$ is above k and put $\tau_k=\tau_k'\wedge\tau_k''$; then τ_k are admissible localization times; next consider

$$W_t^{(k)}=W_{t\wedge\tau_k},\qquad R_t^{(k)}=h_k(R_t),\qquad F^{(k)}(t,x)=F(t,x)\,1_{\{t\leqq\tau_k\}} \tag{3.4.2}$$

where h_k is a bounded continuous function such that $h_k(x)=x$ for $|x|\leqq k$. Then the equation

$$X_t^{(k)}=R_t^{(k)}+\int_0^t F^{(k)}(s,X_{s-}^{(k)})\,dW_s^{(k)} \tag{3.4.3}$$

has a solution which coincides with X_t on $[0,\tau_k)$. Moreover by applying Lemma 3.3.1 to $U_t=X_t^{(k)}-R_t^{(k)}$, we check that $\sup_{t\leqq T}|X_t^{(k)}|$ is bounded in L^2, so since

$$\mathbb{P}[\sup_{t\leqq T}|X_t|>K]\leqq\mathbb{P}[\sup_{t\leqq T}|X_t^{(k)}|>K]+\mathbb{P}[\tau_k\leqq T], \tag{3.4.4}$$

we complete the proof by noticing that the '$\lim_{k\uparrow\infty}\lim_{K\uparrow\infty}\sup_{\varepsilon}$' of the right-hand side is 0. □

Lemma 3.4.3. *If $F=f$ and $W_t=w_t$, the statement of Theorem 2.4.3 holds without the '$\mathscr{H}_t$ Riemann' assumption on r_{t-} and f.*

Proof. This lemma will again be proved by means of a localization argument; so let $\bar{A}_t^{(k)}$ be the process constructed in Lemma 3.4.1 and in view of Definition 2.2.2, let τ_k' be admissible localization $\mathscr{F}_t$ stopping times such that for every k, $A_{t\wedge\tau_k'}$, $|W|_{t\wedge\tau_k'}$ satisfy (3.2.1) for some $p>1$; we can also choose τ_k' such that $A_{t\wedge\tau_k'}$, $\bar{A}_{t\wedge\tau_k'}^{(k)}$ and $\rho_{t\wedge\tau_k'}$ are bounded. On the other hand, let

$$\tau_k''=\inf\{t;\,|R_t|\vee|r_t|\vee|X_t|\vee|x_t|\geqq k\}. \tag{3.4.5}$$

From the assumptions and Lemma 3.4.2, τ_k'' are admissible localization times, so $\tau_k=\tau_k'\wedge\tau_k''$ are also admissible. Then define

$$W_t^{(k)}=W_{t\wedge\tau_k},\qquad R_t^{(k)}=h_k(R_t),\qquad r_t^{(k)}=h_k(r_t),\qquad F^{(k)}(t,x)=F(t,x)\,1_{\{t\leqq\tau_k\}} \tag{3.4.6}$$

where h_k is a bounded Lipschitz function such that $h_k(x)=x$ for $|x|\leqq k$. Consider now

$$\begin{aligned} X_t^{(k)}&=R_t^{(k)}+\int_0^t F^{(k)}(s,X_{s-}^{(k)})\,dW_s^{(k)},\\ x_t^{(k)}&=r_t^{(k)}+\int_0^t F^{(k)}(s,x_{s-}^{(k)})\,dW_s^{(k)}. \end{aligned} \tag{3.4.7}$$

This system has a solution $(X_t^{(k)},x_t^{(k)})$ which coincides with (X_t,x_t) on $[0,\tau_k)$; moreover

$$X_t^{(k)}-x_t^{(k)}=R_t^{(k)}-r_t^{(k)}+\int_0^t (F^{(k)}(s,X_{s-}^{(k)})-F^{(k)}(s,x_{s-}^{(k)}))\,dW_s^{(k)} \tag{3.4.8}$$

and it turns out that Lemma 3.3.2 can be applied to this equation: defining

$$\begin{aligned} U_t &= X_t^{(k)} - x_t^{(k)} - R_t^{(k)} + r_t^{(k)}, \\ \mu_t &= F^{(k)}(t, X_{t-}^{(k)}) - F^{(k)}(t, x_{t-}^{(k)}), \end{aligned} \tag{3.4.9}$$

the estimate (3.3.22) is indeed satisfied for

$$\begin{aligned} &A'_t = 2A_{t\wedge\tau_k}, \qquad A''_t = A_{t\wedge\tau_k} + |W|_{t\wedge\tau_k}, \qquad A'''_t = \bar{A}^{(k)}_{t\wedge\tau_k}, \\ &Z_t = 2|R_{t-}^{(k)} - r_{t-}^{(k)}|^2 + 4|R_{t-}^{(k)} - r_{t-}^{(k)}|(1 + \rho_{t\wedge\tau_k}). \end{aligned} \tag{3.4.10}$$

For k fixed, since $R^{(k)}$ and $r^{(k)}$ are contiguous in $L^q(\mathscr{H})$ for any q and since $\bar{A}^{(k)}_{T\wedge\tau_k}$ converges to 0 in L^1, one can deduce that $X^{(k)}$ and $x^{(k)}$ are contiguous in $L^1(\mathscr{H})$, and therefore in $L^0(\mathscr{H})$. But $(X_t^{(k)}, x_t^{(k)})$ and (X_t, x_t) are equal on $\{t<\tau_k\}$ so

$$\begin{aligned} &\sup\{\mathbb{P}[|X_\tau - x_\tau| > C];\ \tau\in\mathscr{T}(\mathscr{H}, T)\} \\ &\qquad \leqq \mathbb{P}[\tau_k \leqq T] + \sup\{\mathbb{P}[|X_\tau^{(k)} - x_\tau^{(k)}| > C];\ \tau\in\mathscr{T}(\mathscr{H}, T)\}. \end{aligned} \tag{3.4.11}$$

By taking '$\lim_{k\uparrow\infty} \limsup_{\varepsilon\to\varepsilon_0}$' on both sides, we obtain the contiguity in $L^0(\mathscr{H})$ of X and x. □

In Lemma 3.4.3, we have considered the case where only the driving processes r_t are perturbed, and we now want to deal with perturbations on the driving semimartingales and the coefficients; let us explain how the general situation can be reduced to this case. Consider the process

$$\bar{r}_t = r_t + \int_0^t f(s, x_{s-})\, dw_s - \int_0^t F(s, x_{s-})\, dW_s. \tag{3.4.12}$$

Then the system (2.4.5) can be written as

$$\begin{aligned} X_t &= R_t + \int_0^t F(s, X_{s-})\, dW_s \\ x_t &= \bar{r}_t + \int_0^t F(s, x_{s-})\, dW_s \end{aligned} \tag{3.4.13}$$

so that we are in the restricted framework of Lemma 3.4.3. In order to use this lemma, we still have to prove the

Lemma 3.4.4. *Under the assumptions of Theorems* 2.4.3 *or* 2.4.4, *the stochastic integrals*

$$\int_0^t F(s, x_{s-})\, dW_s \quad \textit{and} \quad \int_0^t f(s, x_{s-})\, dw_s$$

are bounded in probability and contiguous in $L^0(\mathscr{H})$.

Proof. We only give the proof under the main assumptions of Theorems 2.4.3 or 2.4.4 (the case of weakened assumptions given at the end of Theorem 2.4.3 is easily checked with the same method). First note that for each fixed x, the processes $\int_0^{\cdot} F(s,x)\,dW_s$ and $\int_0^{\cdot} f(s,x)\,dW_s$ are contiguous in $L^0(\mathscr{H})$ from Proposition 2.2.4(c), and that the processes $\int_0^{\cdot} f(s,x)\,dW_s$ and $\int_0^{\cdot} f(s,x)\,dw_s$ are contiguous from Proposition 2.3.4; thus $\int_0^{\cdot} F(s,x)\,dW_s$ and $\int_0^{\cdot} f(s,x)\,dw_s$ are contiguous. Now fix some terminal time T. Under the assumptions of Theorem 2.4.3, from Lemma 3.4.2, x_t is bounded in probability, so the semimartingale $I_t = \int_0^t f(s, x_{s-})\,dw_s$ and $|I|_t$ are also bounded in probability (use Proposition 2.2.4(c)); thus from Proposition 2.3.2, the process I_{t-} is $\mathscr{H}_t$ Riemann and therefore $z_t = x_{t-}$ is also $\mathscr{H}_t$ Riemann; this property also holds from Proposition 2.3.2 under the assumptions of Theorem 2.4.4. We want to prove that $\int F(s,z_s)\,dW_s$ and $\int f(s,z_s)\,dw_s$ are contiguous; by localization, we only have to consider the case where the processes z_t are uniformly bounded; denote by K a bound for it; for any $\delta > 0$, let $\bar{z}_t$ be a process satisfying (2.3.1) and (2.3.2) with $\phi'_i = 0$ on $\{\tau'_i > 0\}$; we can suppose that $|\bar{z}_t| \leqq K$ (otherwise replace it by its projection on the ball); we can also modify the subdivision τ_i so that $\tau_i \leqq \tau_{i+1}$ and we put $\tau_{N+1} = \infty$. Fix δ and let G be a finite subset of $\mathbb{R}^{d_1}$ such that there exists a measurable function ψ defined on $\mathbb{R}^{d_1}$, with values in G and such that $|\psi(x) - x| < \delta$ as soon as $|x| \leqq K$. By proceeding as in the proof of Proposition 2.3.4, we can reduce the contiguity of integrals of $f(s, z_s)$ and $F(s, z_s)$ as $\varepsilon \to \varepsilon_0$ to the contiguity of integrals of $f(s, \psi(\bar{z}_s))$ and $F(s, \psi(\bar{z}_s))$ as $\varepsilon \to \varepsilon_0$ for any fixed δ; if τ is a family of times of $\mathscr{T}(\mathscr{H}, T)$, we have

$$\left| \int_0^{\tau} F(s, \psi(\bar{z}_s))\,dW_s - \int_0^{\tau} f(s, \psi(\bar{z}_s))\,dw_s \right|$$

$$\leqq \sum_{i=0}^{N} \sum_{y \in G} \left| \int_{\tau_i \wedge \tau}^{\tau_{i+1} \wedge \tau} F(s,y)\,dW_s - \int_{\tau_i \wedge \tau}^{\tau_{i+1} \wedge \tau} f(s,y)\,dw_s \right|. \tag{3.4.14}$$

Since $\int_0^{\cdot} F(s,y)\,dW_s$ and $\int_0^{\cdot} f(s,y)\,dw_s$ are contiguous in $L^0(\mathscr{H})$, the right-hand side of (3.4.14) converges in probability to 0 as $\varepsilon \to \varepsilon_0$. □

The contiguity of X and x, as stated in Theorems 2.4.3 and 2.4.4, is now easily proved from Lemmas 3.4.3 and 3.4.4.

4. Singular Perturbations

We are now going to consider some cases where the Itô differential equations are no more stable, so that one has to add some corrective terms. We will consider equations of type (1.7) which contain both regularly and singularly perturbed semimartingales. We will first study the case where the coefficients G of the singularly perturbed semimartingales do not depend on time; we will explain how it can be reduced to a regular perturbation problem; as an example, we will study the polygonal interpolation (1.3) for general subdivisions and continuous semimartingales. Then we will consider some more general cases, more precisely the case of a nuclear coefficient $G(t, x)$ (when the processes are continuous). Finally, we will give some examples of singular perturbations of a Brownian motion. The basic idea in our method comes from [22]. These results have numerous applications; of course, they are interesting for numerical computations; moreover, in [20], this type of techniques was applied in order to obtain a generalization and a new proof of the Stroock-Varadhan theorem about support of diffusions processes; in [29], we also apply singular perturbations to prove that the classical sufficient condition of [7] for the existence of robust solutions of stochastic differential equations is actually also necessary.

4.1 Introduction

In the beginning of this section, we will be concerned with equations of the type

$$X_t = X_0 + \int_0^t F(s, X_{s-})\, dW_s + \int_0^t G(X_{s-})\, dY_s. \tag{4.1.1}$$

In this equation, F and W are families of stochastic functions and of $\mathscr{F}_t$ semimartingales which satisfy the assumptions of last section; G and Y will be families of $\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_3}$ valued deterministic functions and of $\mathbb{R}^{d_3}$ valued $\mathscr{F}_t$ semimartingales, but unlike W, the characteristics of Y will not be assumed to be dominated, so that our previous convergence results are no more valid. What can we do in this case? Suppose that G is of class C^2, so we can apply Itô's formula

$$\begin{aligned} G_{ij}(X_t) = G_{ij}(X_0) + \int_0^t G'_{ij}(X_{s-})\, dX_s + \tfrac{1}{2} \operatorname{Trace} \int_0^t G''_{ij}(X_{s-})\, d\langle X, X\rangle_s^c \\ + \sum_{s \leq t} (\Delta G_{ij}(X_s) - G'_{ij}(X_{s-}) \Delta X_s). \end{aligned} \tag{4.1.2}$$

In (4.1.2), G'_{ij} is the line vector of first derivatives of G_{ij}, and G''_{ij} is the Hessian matrix of second derivatives. Thus, with the usual summation convention, we can write $G(X_t)$ in the form

$$\begin{aligned} G(X_t) = G(X_0) + \int_0^t H_i(s, X_{s-})\, dS_s^i + \int_0^t \frac{\partial G}{\partial x_i} G_{ij}(X_{s-})\, dY_s^j \\ + \sum_{s \leq t} \left(\Delta G(X_s) - \frac{\partial G}{\partial x_i}(X_{s-}) \Delta X_s^i \right) \end{aligned} \tag{4.1.3}$$

where H_i are functions, the components of which are components of $G'_{ij}F$, $F^* G''_{ij}F$, $G^* G''_{ij}F$, $G^* G''_{ij}G$, and S_t is a semimartingale, the components of which are components of W, $\langle W, W\rangle^c$, $\langle W, Y\rangle^c$ and $\langle Y, Y\rangle^c$. On the other hand, assume that Y is contiguous to some family of semimartingales $\bar{Y}$; we can write

$$G(X_t)(\bar{Y}_t - Y_t) = G(X_0)(\bar{Y}_0 - Y_0) + \int_0^t G(X_{s-})\,d(\bar{Y}_s - Y_s) + \int_0^t dG(X_s)(\bar{Y}_{s-} - Y_{s-}) + [G(X), \bar{Y} - Y]_t \tag{4.1.4}$$

where $dG(X_s)$ is expressed by means of (4.1.3). Thus

$$X_t = G(X_t)(Y_t - \bar{Y}_t) - G(X_0)(Y_0 - \bar{Y}_0) + X_0 + \int_0^t F(s, X_{s-})\,dW_s + \int_0^t G(X_{s-})\,d\bar{Y}_s + \int_0^t dG(X_s)(\bar{Y}_{s-} - Y_{s-}) + [G(X), \bar{Y} - Y]_t. \tag{4.1.5}$$

Defining the processes

$$Z_t = \sum_{s \le t} \left(\Delta G(X_s) - \frac{\partial G}{\partial x_i}(X_{s-})\,\Delta X_s^i \right), \tag{4.1.6}$$

$$R_t = X_0 + G(X_t)(Y_t - \bar{Y}_t) - G(X_0)(Y_0 - \bar{Y}_0) + \int_0^t H_i(s, X_{s-})(\bar{Y}_{s-} - Y_{s-})\,dS_s^i + \int_0^t dZ_s(\bar{Y}_{s-} - Y_{s-}) + \sum_{s \le t} \Delta Z_s(\Delta \bar{Y}_s - \Delta Y_s), \tag{4.1.7}$$

$$\Gamma_t = \int_0^t dY_s(\bar{Y}_{s-} - Y_{s-})^* + [Y, \bar{Y} - Y]_t \tag{4.1.8}$$

and

$$\Lambda_t = [W, \bar{Y} - Y]_t, \tag{4.1.9}$$

we can write (4.1.5) in the form

$$X_t = R_t + \int_0^t F(s, X_{s-})\,dW_s + \int_0^t G(X_{s-})\,d\bar{Y}_s + \int_0^t \frac{\partial G_k}{\partial x_i} G_{ij}(X_{s-})\,d\Gamma_s^{jk} + \int_0^t \frac{\partial G_k}{\partial x_i} F_{ij}(s, X_{s-})\,d\Lambda_s^{jk} \tag{4.1.10}$$

where G_k is the kth column of G. We have transformed equation (4.1.1) into (4.1.10); we could not apply directly the theorems of Sect. 2 on the former equation, but we are going to put assumptions which will imply that the study of the latter one becomes a regular perturbation problem.

4.2 A Contiguity Theorem

Let us first enumerate the set of assumptions about the model which was introduced in Sect. 4.1. Note that unlike W and w in Sect. 2, Y and $\bar{Y}$ must be semimartingales with respect to the same family of filtrations; otherwise, one cannot write (4.1.4).

Assumption 4.2.1. We are given two families of filtrations $\mathscr{F}_t$ and $\mathscr{H}_t$, families $W_t \in \mathbb{R}^{d_2}$, $Y_t \in \mathbb{R}^{d_3}$ and $\bar{Y}_t \in \mathbb{R}^{d_3}$ of $\mathscr{F}_t$ semimartingales, a family of $\mathscr{F}_0$ measurable variables X_0, a family of $\mathscr{P}(\mathscr{F}) \otimes \mathscr{B}(\mathbb{R}^{d_1})$ measurable functions $F(t, \omega, x)$ and a family of deterministic C^2 functions $G(x)$. We also consider the matrix-valued processes Γ_t and Λ_t defined by (4.1.8) and (4.1.9). We suppose that F satisfies

$$|F(t, \omega, x)| \leqq \rho_t(\omega), \qquad |F(t, \omega, x) - F(t, \omega, y)| \leqq \rho_t(\omega)\,|x-y| \tag{4.2.1}$$

for $\mathscr{F}_t$ predictable processes ρ_t which are bounded in probability and that $G(x)$ and its first and second derivatives are bounded uniformly in (x, ε). We suppose that X_t and $\bar{X}_t$ are solutions of (4.1.1) and

$$\begin{aligned}\bar{X}_t = X_0 &+ \int_0^t F(s, \bar{X}_{s-})\, dW_s + \int_0^t G(\bar{X}_{s-})\, d\bar{Y}_s \\ &+ \int_0^t \frac{\partial G_k}{\partial x_i} G_{ij}(\bar{X}_{s-})\, d\Gamma_s^{jk} + \int_0^t \frac{\partial G_k}{\partial x_i} F_{ij}(s, \bar{X}_{s-})\, d\Lambda_s^{jk}.\end{aligned} \tag{4.2.2}$$

Theorem 4.2.2. *Under Assumption 4.2.1, suppose that W_t, $\bar{Y}_t$, Γ_t, Λ_t and $\langle\langle Y, Y\rangle\rangle_t$ are $\mathscr{H}_t$ dominated, that Y_t, $\bar{Y}_0$ and X_0 are bounded in probability, that Y and $\bar{Y}$ are contiguous in $L^0(\mathscr{H})$ and that $\sup_{t \leqq T} |\Delta \bar{Y}_t - \Delta Y_t|$ converges in probability to 0 for any T. Then X and $\bar{X}$ are contiguous in $L^0(\mathscr{H})$.*

Proof. In order to apply Theorem 2.4.3, it is sufficient to prove that the process R_t defined by (4.1.7) is bounded in probability and contiguous in $L^0(\mathscr{H})$ to the constant process X_0. We study separately the terms of the right side of (4.1.7) and only the three last ones are not trivially estimated. The functions H_i are bounded and the semimartingales S^i are $\mathscr{H}_t$ dominated, so from Proposition 2.2.4(c), the 'sup$_{t \leqq T}$' of the first stochastic integral is bounded in probability and converges to 0. One has

$$\Delta Z_t \leqq C\,|\Delta X_t|^2 \leqq C \rho_t^2\, \Delta [\![W, W]\!]_t + C\, \Delta [\![Y, Y]\!]_t \tag{4.2.3}$$

and the processes $\langle\langle W, W\rangle\rangle_t$ and $\langle\langle Y, Y\rangle\rangle_t$ are $\mathscr{H}_t$ dominated, so Z_t is prelocally $\mathscr{H}_t$ dominated (see the discussion following Definition 2.2.2) and therefore the integral with respect to Z_t is bounded and converges to 0 in L^0. Finally, the jumping term is dominated by

$$\Big|\sum_{s \leqq t} (\Delta \bar{Y}_s - \Delta Y_s)\, \Delta Z_s\Big| \leqq \sup_{s \leqq t} |\Delta \bar{Y}_s - \Delta Y_s|\, Z_t \tag{4.2.4}$$

which also converges to 0. □

Remark 1. The assumption $|F(t, x)| \leqq \rho_t$ can be removed in some cases. We have supposed it because if $F(t, x)$ has linear growth as $|x| \to \infty$, some of the functions H_i may have quadratic growth; however, if $\langle W, W\rangle_t^c = 0$, the semimartingales S^i corresponding to the coefficients H_i which have quadratic growth are zero, so we are reduced to the case of functions H_i with linear growth which can be dealt with. Moreover, the condition (4.2.1) and the boundedness of G and its derivatives can also be localized as in Theorem 2.4.5.

Remark 2. From Proposition 2.7.2, we can also prove the contiguity in $L^q(\mathscr{H})$ with integrability assumptions on the semimartingales.

Remark 3. As a second step, one can apply the techniques of Sect. 2 to (4.2.2); if the processes W_t, Y_t, Γ_t and Λ_t converge in $L^0(\mathscr{H})$ to semimartingales w_t, y_t, γ_t, λ_t, if X_0 converges to x_0, if the functions F, G and the first derivatives of G converge to f, g and the first derivatives of g, we can get the convergence of X_t to

$$x_t = x_0 + \int_0^t f(s, x_{s-})\, dw_s + \int_0^t g(x_{s-})\, dy_s$$

$$+ \int_0^t \frac{\partial g_k}{\partial x_i} g_{ij}(x_{s-})\, d\gamma_s^{jk} + \int_0^t \frac{\partial g_k}{\partial x_i} f_{ij}(s, x_{s-})\, d\lambda_s^{jk}. \tag{4.2.5}$$

With this remark in mind, our result can be viewed as a generalization of [22]. In particular, one can recover several previously known results ([23, 28, 11]).

We now give some properties of the processes Γ_t and Λ_t involved in Theorem 4.2.2.

Proposition 4.2.3. *Denote by '$\simeq$' the contiguity in $L^0(\mathscr{H})$. Assuming that W and $\bar{Y}$ are $\mathscr{H}_t$ dominated, that W, Y and $\bar{Y}$ are bounded in probability, that $Y_t \simeq \bar{Y}_t$, we have*

$$\Gamma_t + \Gamma_t^* \simeq [\bar{Y}, \bar{Y}]_t - [Y, Y]_t \tag{4.2.6}$$

and

$$\Gamma_t \simeq \int_0^t Y_{s-}\, dY_s^* - \int_0^t \bar{Y}_{s-}\, d\bar{Y}_s^*, \qquad \Lambda_t \simeq \int_0^t W_{s-}\, dY_s^* - \int_0^t W_{s-}\, d\bar{Y}_s^*. \tag{4.2.7}$$

Proof. Write Itô's formula

$$(\bar{Y}_t - Y_t)(\bar{Y}_t - Y_t)^* = \int_0^t (\bar{Y}_{s-} - Y_{s-})\, d\bar{Y}_s^*$$

$$+ \int_0^t d\bar{Y}_s (\bar{Y}_{s-} - Y_{s-})^* - \Gamma_t - \Gamma_t^* + [\bar{Y}, \bar{Y}]_t - [Y, Y]_t. \tag{4.2.8}$$

By applying Proposition 2.2.4 to the two stochastic integrals, one deduces (4.2.6). The estimates (4.2.7) are checked with a similar method by applying Itô's formula to $Y_t(\bar{Y}_t - Y_t)^*$ and $W_t(\bar{Y}_t - Y_t)^*$. □

Note that the behaviour of Γ_t and Λ_t is closely related to the behaviour of the double integrals of W, Y and $\bar{Y}$. The second part of (4.2.7) means that

$$\int_0^t W_{s-}\, dY_s^* + [W, Y]_t \simeq \int_0^t W_{s-}\, d\bar{Y}_s^* + [W, \bar{Y}]_t. \tag{4.2.9}$$

This relation may surprise because it is generally admitted that the Stratonovich integrals are more stable than Itô's ones and here, since there is no coefficient 1/2 multiplying the brackets, we obtain the stability of another type of integral – a sort of backward integral; however, this phenomenon will be soon explained. Let us consider the process Γ_t; the contiguity relation $\bar{Y}_t \simeq Y_t$ will be said to be symmetric if Γ_t is asymptotically symmetric; from (4.2.6) this means that

$$\Gamma_t \simeq ([\bar{Y}, \bar{Y}]_t - [Y, Y]_t)/2 \tag{4.2.10}$$

and in this case

$$\int_0^t Y_{s-}\, dY_s^* + \tfrac{1}{2}[Y, Y]_t \simeq \int_0^t \bar{Y}_{s-}\, d\bar{Y}_s^* + \tfrac{1}{2}[\bar{Y}, \bar{Y}]_t. \tag{4.2.11}$$

Then, with the framework of Sect. 2, if $\bar{Y}_t$ converges to some $\mathscr{G}_t$ semimartingale y_t, (4.2.11) converges to $\int_0^t y_{s-}\, dy_s^* + \frac{1}{2}[y, y]_t$ and the processes Y_t will be said to be symmetric approximations of y_t; in this case we get the stability of integrals which coincide with the Stratonovich integrals in the continuous case and which will be called symmetric integrals (see [22]); they are defined by

$$\int_0^t Z_{s-}^1 \diamond dZ_s^2 = \int_0^t Z_{s-}^1\, dZ_s^2 + \tfrac{1}{2}[Z^1, Z^2]_t = \int_0^t Z_{s-}^1 \circ dZ_s^2 + \tfrac{1}{2} \sum_{s \leq t} \Delta Z_s^1\, \Delta Z_s^2. \tag{4.2.12}$$

In the general case (non symmetric contiguity), the difference between the symmetric double integrals of Y and $\bar{Y}$ is asymptotically skewsymmetric. Now, if $Y_t \simeq \bar{Y}_t$ is symmetric and if one considers the contiguity relation $(W_t, Y_t) \simeq (W_t, \bar{Y}_t)$, it appears that it is symmetric if and only if Λ_t converges to 0; in such a case, one can put any multiplying coefficient in front of the brackets of (4.2.9), so that both Itô's and symmetric integrals are stable; however Λ_t does not converge necessarily to 0 and this explains why we do not get in general the stability of symmetric integrals in (4.2.9).

Note that if Γ_t is asymptotically symmetric, then the solution of

$$X_t = X_0 + \int_0^t F(s, X_{s-})\, dW_s + \int_0^t G(X_{s-})\, dY_s$$
$$+ \frac{1}{2} \int_0^t \frac{\partial G_k}{\partial x_i} G_{ij}(X_{s-})\, d[Y^j, Y^k]_s + \int_0^t \frac{\partial G_k}{\partial x_i} F_{ij}(s, X_{s-})\, d[W^j, Y^k]_s \tag{4.2.13}$$

and the solution $\bar{X}_t$ of the same equation with Y replaced by $\bar{Y}$ are contiguous: (4.2.13) is stable for symmetric perturbations on Y. The same result holds for non symmetric approximations when the vector fields associated to the columns of G commute.

To conclude this subsection, let us explain briefly how some more singular perturbations can be studied. We have proved in Sect. 3 that if an equation is regularly perturbed, the limit of the approximating semimartingales directly enters the limit equation; in this subsection, we have proved that for some other types of perturbations, it is not sufficient to know the limit of the approximations Y_t but one also has to look for the limit of iterated integrals of order 2. Now if Γ_t is no more dominated but converges to a dominated semimartingale γ_t, we can repeat the technique used in Sect. 4.1 and apply the integration by parts formula to express the integral with respect to Γ_t in (4.1.10); in some cases this leads to a limit theorem involving iterated integrals of Y of order 3. This procedure can be repeated, so that we can say that the order of singularity of the approximation is characterized by the maximal order of iterated integrals which have to be considered; however, in this work, we will not go beyond the order 2.

4.3 Example

We want to apply previous results and prove that the polygonal interpolation used with stochastic subdivisions and for equations driven by a continuous semimartingale y_t is a symmetric approximation; if y_t is a one-dimensional Brownian motion, this result follows directly from [33]; when y_t is a more general multidimensional continuous semimartingale, it is proved in [28] in the case of deterministic subdivisions. So let $\mathscr{G}_t$ be some filtration, y_t a $\mathscr{G}_t$ continuous semimartingale and g a C_b^2 function. Let (t_k) be a family of subdivisions satisfying the conditions of Sect. 2.5, consider the polygonal interpolation

$$Y_t = y_{t_k} + \frac{t-t_k}{t_{k-1}-t_k}(y_{t_{k+1}} - y_{t_k}) \quad \text{for } t_k \leqq t < t_{k+1} \tag{4.3.1}$$

and the solution of

$$X_t = x_0 + \int_0^t g(X_s)\, dY_s \tag{4.3.2}$$

for some fixed $\mathscr{G}_0$ measurable variable x_0.

Theorem 4.3.1. *Let x_t be the solution of*

$$x_t = x_0 + \int_0^t g(x_s) \circ dy_s. \tag{4.3.3}$$

Then $\sup_{t \leqq T} |X_t - x_t|$ converges in probability to 0 as $\varepsilon \to \varepsilon_0$.

Proof. Consider the filtrations $\mathscr{F}_t$ defined in (2.5.4); then Y_t is an $\mathscr{F}_t$ adapted absolutely continuous process. Let $\bar{Y}_t$ be the right discretization of y_t: it is equal to $y_{t_{k+1}}$ on $[t_k, t_{k+1})$; $\bar{Y}_t$ is a family of $\mathscr{F}_t$ dominated semimartingales and it is contiguous to Y_t in $L^0(\mathscr{F})$ (from the continuity of the paths of y_t). In order to apply Theorem 4.2.2, we still have to study the process Γ_t defined by (4.1.8). On $\{t_n \leqq t < t_{n+1}\}$, we have

$$\bar{Y}_t - Y_t = \frac{t_{n+1}-t}{t_{n+1}-t_n}(y_{t_{n+1}} - y_{t_n}), \qquad \dot{Y}_t = \frac{1}{t_{n+1}-t_n}(y_{t_{n+1}} - y_{t_n}) \tag{4.3.4}$$

so

$$\Gamma_t = \frac{1}{2}\sum_{k=0}^{n-1}(y_{t_{k+1}} - y_{t_k})(y_{t_{k+1}} - y_{t_k})^* + \frac{1}{2}\left(1 - \frac{(t_{n+1}-t)^2}{(t_{n+1}-t_n)^2}\right)(y_{t_{n+1}} - y_{t_n})(y_{t_{n+1}} - y_{t_n})^*. \tag{4.3.5}$$

Let $\bar{\Gamma}_t$ be the right discretization of Γ_t; it is equal to $[\bar{Y}, \bar{Y}]_t/2$; write it in the form

$$\bar{\Gamma}_t = \tfrac{1}{2}\bar{Y}_t \bar{Y}_t^* - \tfrac{1}{2}\int_0^t \bar{Y}_{s-}\, d\bar{Y}_s^* - \tfrac{1}{2}\int_0^t d\bar{Y}_s\, \bar{Y}_{s-}^*. \tag{4.3.6}$$

Since $\bar{Y}_t$ is a regular perturbation of y_t, we can deduce from Theorem 2.4.4 that $\bar{\Gamma}_t$ converges in $L^0(\mathscr{F})$ to the process

$$\gamma_t = \tfrac{1}{2} y_t y_t^* - \tfrac{1}{2}\int_0^t y_s\, dy_s^* - \tfrac{1}{2}\int_0^t dy_s\, y_s^* = \tfrac{1}{2}\langle y, y\rangle_t, \tag{4.3.7}$$

and therefore Γ_t also converges to this process. By proceeding similarly, we can prove that $\int_0^t |d\Gamma_s|$ is $\mathscr{F}_t$ dominated, so Γ_t is a $\mathscr{F}_t$ dominated semimartingale. Thus, by applying Theorem 4.2.2, X_t is contiguous in $L^0(\mathscr{F})$ to the solution of

$$\bar{X}_t = x_0 + \int_0^t g(\bar{X}_{s-})\, d\bar{Y}_s + \int_0^t \frac{\partial g_k}{\partial x_i} g_{ij}(\bar{X}_{s-})\, d\Gamma_s^{jk} \tag{4.3.8}$$

and from Theorem 2.4.4, $\bar{X}_t$ converges to x_t in $L^0(\mathscr{F})$. $\square$

Remark 1. By localizing as in Theorem 2.4.5, we can replace the assumption $g \in C_b^2$ by $g \in C^2$, provided that we assume that the solution of (4.3.3) does not explode.

Remark 2. The polygonal interpolation becomes a regular perturbation when it is restricted to continuous processes with finite variation. Thus, if some of the components of y_t have finite variation, the corresponding columns of g can be assumed to be only Lipschitz.

Remark 3. If y_t is not continuous, the preceding method cannot be applied; actually, even for simple processes such as $y_t = 1_{\{t \geqq 1\}}$, the convergence of X_t does not hold uniformly in the neighbourhood of the jump time, and the limit process x_t is solution of an equation which is quite different from (4.3.3). This type of approximation is considered in [24] and will not be studied here. With reference to the end of Sect. 4.2, we can say that this perturbation has an infinite order of singularity.

Remark 4. As in Sect. 2.6, one can consider the case of a process y_t depending on ε, and therefore obtain a convergence which is uniform over a family of equations.

4.4 The Case of a Time-Dependent Coefficient

In Sect. 4.2, we have considered equations in which the coefficient of the singularly perturbed semimartingale depends only on x; what happens when this coefficient has the form $G(t, \omega, x)$? If it has the form $G(U_t, x)$ for some family of $\mathscr{F}_t$ semimartingales U_t and if G is $C^{2,2}$, one can apply previous results by increasing the state dimension and adding the components of U_t to those of X_t; in this case indeed, we can write the Itô formula for $G(U_t, X_t)$; if U_t has bounded variation (for instance $U_t = t$) one only has to suppose that G is $C^{1,2}$. However, we are going to show in this subsection that one can also manage with some less regular functions; this will be applied in the next subsection to some approximations of the Brownian motion. Another point of view is given in [9].

So let us consider a family of $\mathscr{P}(\mathscr{F}) \otimes \mathscr{B}(\mathbb{R}^{d_1})$ measurable functions $G(t, \omega, x)$ and the equations

$$X_t = X_0 + \int_0^t F(s, X_{s-})\, dW_s + \int_0^t G(s, X_{s-})\, dY_s \tag{4.4.1}$$

where F, X_0, W and Y satisfy the set of Assumptions 4.2.1. First suppose that

$$G(t, \omega, x) = \sum_{l=1}^{N} \tilde{G}(l, x)\, \beta_t(l, \omega) \tag{4.4.2}$$

where $\beta_t(l)$ are $\mathscr{F}_t$ predictable processes and $\tilde{G}(l)$ are C^2 functions; by replacing processes Y_t by a vector consisting of the semimartingales $\int_0^t \beta_s(l)\, dY_s$, $1 \leqq l \leqq N$, we can write (4.4.1) in the form (4.1.1); consequently, with some contiguity assumptions about these integrals, we can sometimes extend the results of Sect. 4.2 and get the asymptotic behaviour of X_t. Generally, the contiguity of Y_t and $\bar{Y}_t$ does not imply the contiguity of $\int_0^t \beta_s(l)\, dY_s$ and $\int_0^t \beta_s(l)\, d\bar{Y}_s$, so that we have to take into account a new corrective term. In this subsection, we are going to study the case (4.4.2) with $N = \infty$. The vector consisting of the integrals $\int_0^t \beta_s(l)\, dY_s$ is in this case infinite-dimensional, so that we will rather use the theory of random fields. In order to simplify the results, we will limit ourselves to continuous processes and assume that W_t has finite variation (so that $A_t=0$). At the end of this subsection, we will give some hints for verifying the conditions which are given below; examples concerning the Brownian case will be dealt with in Sect. 4.5.

Assumption 4.4.1. We are given two families of filtrations $\mathscr{F}_t$ and $\mathscr{H}_t$, families $W_t\in\mathbb{R}^{d_2}$, $Y_t\in\mathbb{R}^{d_3}$ and $\bar{Y}_t\in\mathbb{R}^{d_3}$ of $\mathscr{F}_t$ continuous semimartingales, a family of $\mathscr{F}_0$ measurable variables X_0, three families of $\mathscr{P}(\mathscr{F})\otimes\mathscr{B}(\mathbb{R}^{d_1})$ measurable functions $F(t,\omega,x)$, $G(t,\omega,x)$ and $\bar{G}(t,\omega,x)$. We suppose that the $\mathscr{H}_t$ optional projection of a $\mathscr{F}_t$ optional process is $\mathscr{F}_t$ optional, that W_t has finite variation, that F satisfies (4.2.1) and that G and $\bar{G}$ are C^2 functions defined by

$$G(t,\omega,x)=\sum_{l=1}^{\infty}\tilde{G}(l,x)\,\beta_t(l,\omega),\qquad \bar{G}(t,\omega,x)=\sum_{l=1}^{\infty}\tilde{G}(l,x)\,\bar{\beta}_t(l,\omega),\tag{4.4.3}$$

where the series converge in $L^0(\mathscr{F})$, $\beta_t(l)$, $\bar{\beta}_t(l)$ are $\mathbb{R}^{d_4}\otimes\mathbb{R}^{d_3}$ valued $\mathscr{F}_t$ predictable processes and $\tilde{G}(l)$ are $\mathbb{R}^{d_1}\otimes\mathbb{R}^{d_4}$ valued C^2 functions (d_4 is a fixed integer); we suppose that

$$\sup_{\varepsilon}\sum_{l}\sup_{x}|\Psi(l,x)|<\infty\tag{4.4.4}$$

is satisfied for $\Psi(l)=\tilde{G}(l)$, $\partial\tilde{G}(l)/\partial x^k$ or $\partial^2\tilde{G}(l)/\partial x^j\,\partial x^k$. We let Γ_t be defined by (4.1.8), and we also define the processes

$$\begin{aligned}\Gamma_t(l,l')=&\int_0^t\beta_s(l)\,dY_s\left(\int_0^s\bar{\beta}_u(l')\,d\bar{Y}_u-\int_0^s\beta_u(l')\,dY_u\right)^*\\&+\int_0^t\beta_s(l)\,d\langle Y,\bar{Y}\rangle_s\,\bar{\beta}_s^*(l')-\int_0^t\beta_s(l)\,d\langle Y,Y\rangle_s\,\beta_s^*(l')\end{aligned}\tag{4.4.5}$$

and

$$\bar{\Gamma}_t(l,l')=\int_0^t\beta_s(l)\,d\Gamma_s\,\beta_s^*(l').\tag{4.4.6}$$

We suppose that there exists a family of admissible localization times τ_k such that for every k,

$$\sup_{\varepsilon,l}\mathbb{E}\sup_{t\leq\tau_k}\left[|\beta_t(l)|+|\bar{\beta}_t(l)|+\left|\int_0^t\beta_s(l)\,dY_s\right|+\left|\int_0^t\bar{\beta}_s(l)\,d\bar{Y}_s\right|\right]<\infty\tag{4.4.7}$$

and

$$\sup_{\varepsilon,l,l'}\mathbb{E}\big(|\Gamma(l,l')|_{\tau_k}+\langle\!\langle\Gamma(l,l'),\Gamma(l,l')\rangle\!\rangle_{\tau_k}^{1/2}\big)<\infty.\tag{4.4.8}$$

Finally let X_t and $\bar{X}_t$ be solutions of (4.4.1) and

$$\bar{X}_t=X_0+\int_0^t F(s,\bar{X}_s)\,dW_s+\int_0^t\bar{G}(s,\bar{X}_s)\,d\bar{Y}_s+\int_0^t\frac{\partial G_k}{\partial x_i}G_{ij}(s,\bar{X}_s)\,d\Gamma_s^{jk}.\tag{4.4.9}$$

Theorem 4.4.2. *Under Assumption* 4.4.1, *suppose that* W_t, $\bar{Y}_t$, Γ_t, $\Gamma_t(l, l')$, $\bar{\Gamma}_t(l, l')$ *and* $\langle\langle Y, Y\rangle\rangle_t$ *are* $\mathscr{H}_t$ *dominated and that* X_0 *is bounded in probability. Assume that* $\int_0^t \beta_s(l)\,dY_s$ *and* $\int_0^t \bar{\beta}_s(l)\,d\bar{Y}_s$, $\Gamma_t(l, l')$ *and* $\bar{\Gamma}_t(l, l')$ *are contiguous in* $L^0(\mathscr{H})$. *Then* X *and* $\bar{X}$ *are contiguous in* $L^0(\mathscr{H})$.

We have put in the following lemma some results which will be used in order to prove Theorem 4.4.2.

Lemma 4.4.3. *Under the assumptions of Theorem* 4.4.2, *the processes consisting of the suprema over* x *of the norms of* $G(t, x)$, $\bar{G}(t, x)$ *and their first and second derivatives are bounded in probability; if* $G^L(t, x)$ *is the truncated sum* $\sum_{l=1}^{L} \tilde{G}(l)\,\beta(l)$, *the supremum over* x *of* $|G(t, x) - G^L(t, x)|$ *converges to* 0 *in* $L^0(\mathscr{H})$ *as* $(\varepsilon, L) \to (\varepsilon_0, \infty)$ *and a similar property holds for the derivatives of* G *and* G^L. *Moreover, there exist* $\mathscr{P}(\mathscr{F})\otimes\mathscr{B}(\mathbb{R}^{d_1})$ *measurable functions* $D(t, \omega, x)$ *and* $\bar{D}(t, \omega, x)$ *which are* C^2 *with respect to* x, *such that for every fixed* x,

$$D(t, x) = \int_0^t G(s, x)\,dY_s, \qquad \bar{D}(t, x) = \int_0^t \bar{G}(s, x)\,d\bar{Y}_s. \tag{4.4.10}$$

The suprema over x *of* $D(t, x)$, $\bar{D}(t, x)$ *and their first and second derivatives are bounded in probability, the processes* $D(t, X_t)$ *and* $\bar{D}(t, X_t)$ *are contiguous in* $L^0(\mathscr{H})$, *and a similar property holds for the derivatives of* D *and* $\bar{D}$. *Finally, if one defines the process*

$$\begin{aligned} Z_t = &\int_0^t \left(\frac{\partial \bar{D}}{\partial x^i}(s, X_s) - \frac{\partial D}{\partial x^i}(s, X_s)\right) G_{ij}(s, X_s)\,dY_s^j \\ &+ \int_0^t \frac{\partial \bar{G}_k}{\partial x^i} G_{ij}(s, X_s)\,d\langle Y^j, \bar{Y}^k\rangle_s - \int_0^t \frac{\partial G_k}{\partial x^i} G_{ij}(s, X_s)\,d\langle Y^j, Y^k\rangle_s, \end{aligned} \tag{4.4.11}$$

$$\bar{Z}_t = \int_0^t \frac{\partial G_k}{\partial x^i} G_{ij}(s, \bar{X}_s)\,d\Gamma_s^{jk}, \tag{4.4.12}$$

$$Z_t^L = \sum_{l,l'=1}^{L} \int_0^t \frac{\partial \tilde{G}_k}{\partial x^i}(l', X_s)\,\tilde{G}_{ij}(l, X_s)\,d\Gamma_s^{jk}(l, l'), \tag{4.4.13}$$

$$\bar{Z}_t^L = \int_0^t \frac{\partial G_k^L}{\partial x^i} G_{ij}^L(s, \bar{X}_s)\,d\Gamma_s^{jk}, \tag{4.4.14}$$

then these processes are $\mathscr{H}_t$ *Riemann for each* L *fixed; moreover,* Z_t *and* Z_t^L *on one hand,* $\bar{Z}_t$ *and* $\bar{Z}_t^L$ *on the other hand are bounded in probability and contiguous in* $L^0(\mathscr{F})$ *as* $(\varepsilon, L) \to (\varepsilon_0, \infty)$.

Proof. If $\Psi(l)$ is $\tilde{G}(l)$ or one of its derivatives, it follows from (4.4.4) and (4.4.7) that

$$\sup_x \left| \sum_{l=L}^{\infty} \Psi(l,x)\,\beta_{t\wedge\tau_k}(l) \right|$$

converges to 0 in $L^1(\mathscr{F})$ as $L\to\infty$, uniformly in ε. Since τ_k are admissible localization times, we easily deduce that the series $\sum \Psi(l,x)\,\beta(l)$ defines a function, the supremum of which over x is bounded in probability; we also check from classical theorems about limits of functions that for the different choices of $\Psi(l)$, we obtain $G(t,x)$ and its derivatives; the contiguity of G and G^L is also easily verified. By considering similarly

$$\sum_l \Psi(l,x)\,\bar{\beta}_t(l), \qquad \sum_l \Psi(l,x)\int_0^t \beta_s(l)\,dY_s, \qquad \sum_l \Psi(l,x)\int_0^t \bar{\beta}_s(l)\,d\bar{Y}_s$$

we check the same properties for $\bar{G}(t,x)$, $D(t,x)$ and $\bar{D}(t,x)$. Moreover

$$\bar{D}(t,X_t)-D(t,X_t)=\sum_{l=1}^{\infty}\tilde{G}(l,X_t)\left(\int_0^t \bar{\beta}_s(l)\,d\bar{Y}_s-\int_0^t \beta_s(l)\,dY_s\right) \tag{4.4.15}$$

where the convergence of the series holds in $L^1(\mathscr{F})$ on $\{t\leqq\tau_k\}$, uniformly in ε; since each term converges to 0 in $L^0(\mathscr{H})$ as $\varepsilon\to\varepsilon_0$, we deduce the contiguity of $D(t,X_t)$ and $\bar{D}(t,X_t)$; we prove in a similar way the contiguity of the derivatives of D and $\bar{D}$ taken at (t,X_t). The processes Z_t^L, $\bar{Z}_t^L$ and $\bar{Z}_t$ are $\mathscr{H}_t$ Riemann from Proposition 2.3.2. By developing the functions D, $\bar{D}$, G and $\bar{G}$ in the Definition (4.4.11) of Z_t, a calculation shows that Z_t is the limit of Z_t^L as $L\to\infty$ for each fixed ε; on the other hand, we can deduce from (4.4.8) that this convergence is uniform in ε and that $|Z|_t$ and $\langle\!\langle Z,Z\rangle\!\rangle_t$ are bounded in probability, so in particular Z_t is $\mathscr{H}_t$ Riemann from Proposition 2.3.2. Finally, since $G^L(t,\bar{X}_t)$ and $G(t,\bar{X}_t)$ as well as their first derivatives are bounded in probability and contiguous as $(\varepsilon,L)\to(\varepsilon_0,\infty)$ and since Γ_t is $\mathscr{H}_t$ dominated, we deduce the boundedness in probability and the contiguity of $\bar{Z}_t^L$ and $\bar{Z}_t$. $\square$

Proof of Theorem 4.2.2. The analogue of (4.1.4) consists of developing $\bar{D}(t,X_t)-D(t,X_t)$; we deduce from Lemma 4.4.3 that we can apply an Itô's formula for random fields (see for instance Theorem I.8.1 of [17]); we obtain

$$\begin{aligned} D(t,X_t)=D(0,X_0)&+\int_0^t G(s,X_s)\,dY_s+\int_0^t \frac{\partial D}{\partial x^i}(s,X_s)\,dX_s^i\\ &+\frac{1}{2}\int_0^t \frac{\partial^2 D}{\partial x^i\,\partial x^j}(s,X_s)\,d\langle X^i,X^j\rangle_s+\int_0^t \frac{\partial G_j}{\partial x^i}(s,X_s)\,d\langle X^i,Y^j\rangle_s \end{aligned} \tag{4.4.16}$$

and a similar formula for $\bar{D}(t, X_t)$. Then, if one defines

$$R_t = D(t, X_t) - \bar{D}(t, X_t) + \bar{D}(0, X_0) - D(0, X_0) + \int_0^t \left(\frac{\partial \bar{D}}{\partial x^i} - \frac{\partial D}{\partial x^i}\right) F_{ij}(s, X_s)\, dW_s^j$$

$$+ \frac{1}{2} \int_0^t \left(\frac{\partial^2 \bar{D}}{\partial x^i \partial x^j} - \frac{\partial^2 D}{\partial x^i \partial x^j}\right)(s, X_s)\, d\langle X^i, X^j \rangle_s, \tag{4.4.17}$$

we deduce from (4.4.1), (4.4.11), (4.4.16), the similar equation for $\bar{D}(t, X_t)$ and (4.4.17) that

$$X_t = X_0 + R_t + Z_t + \int_0^t F(s, X_s)\, dW_s + \int_0^t \bar{G}(s, X_s)\, d\bar{Y}_s. \tag{4.4.18}$$

Now consider some integer L, put $R_t^L = Z_t - Z_t^L$ so that

$$X_t = X_0 + R_t + R_t^L + \int_0^t F(s, X_s)\, dW_s + \int_0^t \bar{G}(s, X_s)\, d\bar{Y}_s$$

$$+ \sum_{l, l' = 1}^{L} \int_0^t \frac{\partial \tilde{G}_k}{\partial x^i}(l', X_s)\, \tilde{G}_{ij}(l, X_s)\, d\Gamma_s^{jk}(l, l'), \tag{4.4.19}$$

and let $\bar{X}_t^L$ be the solution of

$$\bar{X}_t^L = X_0 + R_t^L + \int_0^t F(s, \bar{X}_s^L)\, dW_s + \int_0^t \bar{G}(s, \bar{X}_s^L)\, d\bar{Y}_s$$

$$+ \sum_{l, l' = 1}^{L} \int_0^t \frac{\partial \tilde{G}_k}{\partial x^i}(l', \bar{X}_s^L)\, \tilde{G}_{ij}(l, \bar{X}_s^L)\, d\bar{\Gamma}_s^{jk}(l, l'). \tag{4.4.20}$$

From Lemma 4.4.3, the process R_t is bounded in probability and converges to 0 in $L^0(\mathscr{H})$, the process R_t^L is bounded in probability and $\mathscr{H}_t$ Riemann for L fixed and the function $\bar{G}$ satisfies $\mathbb{H}_0'$; thus we deduce from Theorem 2.4.3 that for L fixed, X_t and $\bar{X}_t^L$ are contiguous as $\varepsilon \to \varepsilon_0$. On the other hand, note that the last term of (4.4.20) is $\bar{Z}_t^L$; if one defines $\bar{R}_t^L = \bar{Z}_t - \bar{Z}_t^L$, it appears that the Eqs. (4.4.9) and (4.4.20) can be written in the form

$$\bar{X}_t = X_0 + \bar{R}_t^L + \int_0^t F(s, \bar{X}_s)\, dW_s + \int_0^t \bar{G}(s, \bar{X}_s)\, d\bar{Y}_s + \int_0^t \frac{\partial G_k^L}{\partial x^i} G_{ij}^L(s, \bar{X}_s)\, d\Gamma_s^{jk},$$

$$\tag{4.4.21}$$

$$\bar{X}_t^L = X_0 + R_t^L + \int_0^t F(s, \bar{X}_s^L)\, dW_s + \int_0^t \bar{G}(s, \bar{X}_s^L)\, d\bar{Y}_s + \int_0^t \frac{\partial G_k^L}{\partial x^i} G_{ij}^L(s, \bar{X}_s^L)\, d\Gamma_s^{jk}.$$

From Lemma 4.4.3, R_t^L and $\bar{R}_t^L$ are bounded in probability and contiguous in $L^0(\mathscr{H})$ as $(\varepsilon, L) \to (\varepsilon_0, \infty)$, so from Theorem 2.4.3, $\bar{X}_t$ and $\bar{X}_t^L$ are also contiguous. Thus X_t and $\bar{X}_t$ are contiguous as $\varepsilon \to \varepsilon_0$. □

Remark 1. When $\mathscr{H}_t = \mathscr{F}_t$, condition (4.4.4) can be localized as in Theorem 2.4.5.

Remark 2. Suppose that $\beta_t(l) = \alpha_{t-}(l)$ for càdlàg $\mathscr{H}_t$ dominated semimartingales $\alpha_t(l)$ such that $\alpha_0(l) = 0$. From the integration by parts formula

$$\alpha_t(l)(\bar{Y}_t - Y_t) = \int_0^t \beta_s(l)\, d\bar{Y}_s - \int_0^t \beta_s(l)\, dY_s + \int_0^t d\alpha_s(l)(\bar{Y}_s - Y_s) + \langle \alpha(l), \bar{Y} - Y \rangle_t, \tag{4.4.22}$$

we deduce that if $\bar{Y}_t$ and Y_t are contiguous and if one can find processes $\bar{\beta}_t(l)$ such that

$$\int_0^t \bar{\beta}_s(l)\, d\bar{Y}_s = \int_0^t \beta_s(l)\, d\bar{Y}_s + \langle \alpha(l), \bar{Y} - Y \rangle_t, \tag{4.4.23}$$

then $\int_0^t \beta_s(l)\, dY_s$ and $\int_0^t \bar{\beta}_s(l)\, d\bar{Y}_s$ will be contiguous as requested in Theorem 4.4.2. The representation (4.4.23) is often made possible by increasing the dimension d_3 of Y: if the semimartingales $\alpha_t(l)$ can be written as $\int_0^t \phi_s(l)\, dN_s$ for some semimartingale N_t, then one can add to the components of Y_t and $\bar{Y}_t$ the components of $\langle \bar{Y} - Y, N \rangle_t$; the process $\beta_t(l)$ is then defined to be equal to $\beta_t(l)$ on the old components and to 0 on the new ones, and $\bar{\beta}_t(l)$ is defined to be equal to $\beta_t(l)$ on the old components and to $\phi_t(l)$ on the new ones. Admitting (4.4.23), we also deduce from (4.4.22) that

$$\begin{aligned} \Gamma_t(l, l') = {} & \int_0^t \beta_s(l)\, dY_s \left(\alpha_s(l')(\bar{Y}_s - Y_s) - \int_0^s d\alpha_u(l')(\bar{Y}_u - Y_u) \right)^* \\ & + \int_0^t \beta_s(l)\, d\langle Y, \bar{Y} - Y \rangle_s \beta_s^*(l') \end{aligned} \tag{4.4.24}$$

so that

$$\bar{\Gamma}_t(l, l') - \Gamma_t(l, l') = \int_0^t \beta_s(l)\, dY_s \int_0^{s-} (\bar{Y}_u - Y_u)^*\, d\alpha_u^*(l'). \tag{4.4.25}$$

By using an integration by parts formula for this last expression, since $\int_0^t \beta_s(l)\, dY_s$ is bounded in probability, we also get the contiguity of $\Gamma_t(l, l')$ and $\bar{\Gamma}_t(l, l')$.

Remark 3. In examples, one is given the function $G(t, x)$ and one has to find a representation formula (4.4.3) so that assumptions of the theorem are satisfied.

Taking $d_4 = d_3$, one can choose for $\tilde{G}(l)$ a complete orthonormal system of $L^2(\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_3})$ multiplied by some coefficient c_l such that (4.4.4) holds; then

$$\beta_t(l) = \frac{I}{c_l^2} (G(t,\cdot), \tilde{G}(l,\cdot))_{L^2(\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_3})} \tag{4.4.26}$$

where I is the identity matrix of $\mathbb{R}^{d_3}$. If G is regular enough with respect to x, the processes $\beta_t(l)$ will not explode as $l \to \infty$ and we will be able to verify the assumptions of the theorem; in particular, if $G(t, x)$ is, for each x, a semimartingale, then $\beta_t(l)$ is (with some boundedness assumptions) also a semimartingale, so that we can apply Remark 2 and recover some results from [9]. However, we can sometimes also study some functions $G(t, x)$ which are not semimartingales; this problem is dealt with in next subsection for some approximations of the Brownian motion.

4.5 *Absolutely Continuous Approximations of a Brownian Motion*

We now consider the case where Y_t are absolutely continuous $\mathscr{F}_t$ adapted processes which are contiguous as $0 < \varepsilon \to \varepsilon_0 = 0$ to some $\mathscr{F}_t$ Brownian motions $\bar{Y}_t$ (which may depend, as well as the filtration, on ε); more precisely, we will study a class of approximations Y_t which are Gaussian processes included in the Gaussian space generated by $\bar{Y}$; we want to make easier the application of the theorems of Sects. 4.2 and 4.4 – for Sect. 4.4, we will limit ourselves to deterministic functions $G(t, x)$. We will use some assumptions which are not all necessary for the contiguity but which will be useful for the study of the rate of convergence in Sect. 5. Let us fix a constant terminal time T. An example of such an approximation is the delayed polygonal interpolation

$$\begin{aligned} &Y_t = \bar{Y}_{(k-1)\varepsilon} + \frac{t-k\varepsilon}{\varepsilon}(\bar{Y}_{k\varepsilon} - \bar{Y}_{(k-1)\varepsilon}) \quad \text{for } k\varepsilon \leqq t \leqq (k+1)\varepsilon, \quad k \geqq 1 \\ &Y_t = 0 \quad \text{for } t \leqq \varepsilon \end{aligned} \tag{4.5.1}$$

or more generally, if A is a $d_3 \times d_3$ matrix, the delayed quadratic interpolation

$$\begin{aligned} Y_t = \sum_{k \geqq 1} \Bigg(&\bar{Y}_{(k-1)\varepsilon} + \frac{t-k\varepsilon}{\varepsilon}(\bar{Y}_{k\varepsilon} - \bar{Y}_{(k-1)\varepsilon}) \\ &+ A\frac{(t-k\varepsilon)((k+1)\varepsilon - t)}{\varepsilon^2}(\bar{Y}_{k\varepsilon} - \bar{Y}_{(k-1)\varepsilon})\Bigg) 1_{\{k\varepsilon \leqq t < (k+1)\varepsilon\}}(t). \end{aligned} \tag{4.5.2}$$

One can also consider the solution of the equation

$$\dot{Y}_t = \frac{A}{\varepsilon}(Y_t - \bar{Y}_t), \qquad Y_0 = 0, \tag{4.5.3}$$

where A is a stable $d_3 \times d_3$ matrix. Note that all these examples can be included in the framework

$$Y_t = \int_0^t \dot{Y}_u\, du = \int_0^t du \int_0^u L(u, s)\, d\bar{Y}_s \tag{4.5.4}$$

where $L(u, s)$ is a family of $\mathbb{R}^{d_3} \otimes \mathbb{R}^{d_3}$ valued measurable maps which are square-integrable on $[0, T] \times [0, T]$ and satisfy $L(u, s) = 0$ if $u < s$; for (4.5.2), one has

$$L(u, s) = \frac{1}{\varepsilon} \sum_{k=1}^{\infty} \left(I + A \frac{(2k+1)\varepsilon - 2u}{\varepsilon} \right) 1_{\{k\varepsilon \leqq u < (k+1)\varepsilon\}}\, 1_{\{(k-1)\varepsilon \leqq s < k\varepsilon\}} \tag{4.5.5}$$

and for (4.5.3)

$$L(u, s) = -\frac{A}{\varepsilon} \exp \frac{A}{\varepsilon}(u - s) \qquad \text{if } s < u. \tag{4.5.6}$$

Before the study of these approximations, let us explain briefly how one can deal with cases which do not satisfy $L(u, s) = 0$ for $u < s$ i.e. which are not $\mathscr{F}_t$ adapted. The first method is similar to the one used in Sect. 4.3 for classical polygonal interpolation (1.3); if indeed Y_t is $\mathscr{F}_{t+\varepsilon}$ measurable, one can firstly compare the equations driven by Y_t and $\bar{Y}_{t+\varepsilon}$, and secondly notice that $\bar{Y}_{t+\varepsilon}$ is a regular perturbation of $\bar{Y}_t$ (see Theorem 4.5.6 below). If this trick cannot be applied, another method consists of using the Skorohod stochastic integral and the stochastic calculus which was recently developed for it: this technique is developed in [29].

In all this subsection, $\mathscr{H}_t$ will be the filtration of deterministic events; first, we want to find a sufficient condition on L for the contiguity of Y and $\bar{Y}$; since these processes are Gaussian, note that the contiguity in $L^0(\mathscr{H})$ implies the contiguity in $L^q(\mathscr{H})$ for every q. It is easily seen that

$$\mathbb{E}\, |Y_t - \bar{Y}_t|^2 = \int_0^t \left| \int_0^t L(u, s)\, du - I \right|^2 ds \tag{4.5.7}$$

so the contiguity in $L^0(\mathscr{H})$ of Y_t and $\bar{Y}_t$ is equivalent to the convergence to 0 of the above integral uniformly in $t \leqq T$. Henceforth, we will assume

$$\sup_{0 \leqq t \leqq T} \mathbb{E}\, |Y_t - \bar{Y}_t|^2 = O(\varepsilon). \tag{4.5.8}$$

This is easily verified in the case of approximations (4.5.2) and (4.5.3). Another assumption which will be used in the following one; we will suppose that

$$|L(t, u)| \leqq \frac{1}{\varepsilon}\, \Psi\!\left(\frac{t-u}{\varepsilon} \right), \tag{4.5.9}$$

for some positive bounded function Ψ defined on $\mathbb{R}$ such that $\Psi(z)=0$ for $z<0$ and

$$\int_0^\infty \Psi(z)\sqrt{z}\,dz<\infty. \tag{4.5.10}$$

This condition means that $\dot{Y}_t$ is nearly independent from old values of $\bar{Y}$ and it is satisfied for the examples (4.5.2) and (4.5.3): one can take for Ψ on $[0,\infty)$ an exponential function $\Psi(z)=C\exp-\mu z$ with some $\mu>0$.

Definition 4.5.1. A family of processes Y_t of type (4.5.4) will be called an admissible perturbation of $\bar{Y}_t$ if it satisfies (4.5.8) and (4.5.9).

Let $C^{1/2}$ be the space of real functions defined on $[0,T]$ which are Hölder continuous with coefficient 1/2 and consider on it the norm

$$|\phi|_{1/2}=\sup_{0\leqq t\leqq T}|\phi_t|+\sup_{0\leqq s<t\leqq T}\frac{|\phi_t-\phi_s|}{\sqrt{t-s}}. \tag{4.5.11}$$

If ϕ is a family of deterministic elements of $C^{1/2}$, we now study the contiguity of

$$X_t=\int_0^t \phi_s\,dY_s, \qquad \bar{X}_t=\int_0^t \phi_s\,d\bar{Y}_s. \tag{4.5.12}$$

Lemma 4.5.2. *If Y_t is an admissible perturbation of $\bar{Y}_t$ and if ϕ_t is a family of functions which are bounded in $C^{1/2}$, the processes $\int_0^t \phi_s\,dY_s$ and $\int_0^t \phi_s\,d\bar{Y}_s$ are contiguous in $L^2(\mathscr{H},T)$ and their difference is of order $\sqrt{\varepsilon}$.*

Proof. We estimate the difference of X_t and $\bar{X}_t$ in $L^2(\mathscr{H})$. We have

$$\begin{aligned}
\mathbb{E}|X_t-\bar{X}_t|^2&=\mathbb{E}\left|\int_0^t\left(\int_u^t L(s,u)\,\phi_s\,ds-\phi_u\right)d\bar{Y}_u\right|^2\\
&=\int_0^t\left|\int_u^t L(s,u)\,\phi_s\,ds-\phi_u\right|^2 du\\
&\leqq 2\int_0^t\left|\int_u^t L(s,u)\,ds-I\right|^2|\phi_u|^2\,du+2\int_0^t\left|\int_u^t L(s,u)(\phi_s-\phi_u)\,ds\right|^2 du\\
&\leqq 2|\phi|_{1/2}^2\left(\mathbb{E}|Y_t-\bar{Y}_t|^2+\int_0^t\left|\int_u^t|L(s,u)|\sqrt{s-u}\,ds\right|^2 du\right).
\end{aligned} \tag{4.5.13}$$

From (4.5.9), we deduce

$$\int_u^t|L(s,u)|\sqrt{s-u}\,ds\leqq\frac{1}{\varepsilon}\int_u^\infty\Psi\left(\frac{s-u}{\varepsilon}\right)\sqrt{s-u}\,ds\leqq\sqrt{\varepsilon}\int\Psi(z)\sqrt{z}\,dz \tag{4.5.14}$$

so the last integral of (4.5.13) is of order ε and therefore

$$\mathbb{E}|X_t - \bar{X}_t|^2 \leqq C\varepsilon |\phi|_{1/2}^2. \quad \square \tag{4.5.15}$$

Of course, this implies the contiguity in $L^q(\mathscr{H})$ for every q; it turns out that the contiguity also holds in $L^q(\mathscr{F})$. This is the aim of the following lemma which is also useful for the estimation (4.4.7).

Lemma 4.5.3. *If Y_t is an admissible perturbation of $\bar{Y}_t$ and if ϕ_t is a family of uniformly bounded functions, the variables $\sup_{t \leqq T} \left| \int_0^t \phi_s \, dY_s \right|$ are bounded in L^q for every q; moreover if ϕ_t is uniformly bounded in $C^{1/2}$,*

$$\sup_{t \leqq T} \left| \int_0^t \phi_s \, dY_s - \int_0^t \phi_s \, d\bar{Y}_s \right|$$

converges to 0 in L^q.

Proof. We have for $t_1 \leqq t_2$,

$$\mathbb{E}\left| \int_{t_1}^{t_2} \phi_s \, dY_s \right|^2 = \int_0^T \left| \int_0^T L(s,u)\, \phi_s \, 1_{\{t_1 \leqq s \leqq t_2\}} \, ds \right|^2 du$$

$$\leqq C \int_0^T \iint_{[t_1, t_2]^2} |L(s_1, u)|\,|L(s_2, u)| \, ds_1 \, ds_2 \, du. \tag{4.5.16}$$

Note that (4.5.9) implies

$$\int_0^{s_1 \wedge s_2} |L(s_1, u)|\,|L(s_2, u)| \, du \leqq \frac{1}{\varepsilon} \tilde{\Psi}\left(\frac{|s_1 - s_2|}{\varepsilon} \right) \tag{4.5.17}$$

with

$$\tilde{\Psi}(z) = \int_0^\infty \Psi(z + z')\, \Psi(z') \, dz'. \tag{4.5.18}$$

From Fubini's theorem, one can prove that $\int \tilde{\Psi}(z) \, dz < \infty$ and from (4.5.16),

$$\mathbb{E}\left| \int_{t_1}^{t_2} \phi_s \, dY_s \right|^2 \leqq \frac{C}{\varepsilon} \iint_{[t_1, t_2]^2} \tilde{\Psi}\left(\frac{|s_1 - s_2|}{\varepsilon} \right) ds_1 \, ds_2 \leqq C(t_2 - t_1) \int_0^\infty \tilde{\Psi}(z) \, dz. \tag{4.5.19}$$

Since the processes are Gaussian, this implies that for every $q \geqq 1$,

$$\mathbb{E}\left| \int_{t_1}^{t_2} \phi_s \, dY_s \right|^q \leqq C(t_2 - t_1)^{q/2} \tag{4.5.20}$$

so we can deduce from Lemma 2.1.3 (b) the boundedness in $L^{q'}$ for $q'<q$. Since $\int_0^t \phi_s\,d\bar{Y}_s$ satisfies an estimate similar to (4.5.20), we also deduce the contiguity from Lemma 2.1.3 (c). □

We now want to study the process

$$\Gamma_t=\int_0^t \dot{Y}_s(\bar{Y}_s-Y_s)^*\,ds \tag{4.5.21}$$

for Y_t an admissible perturbation of $\bar{Y}_t$.

Lemma 4.5.4. *If Y_t is an admissible perturbation of $\bar{Y}_t$, then Γ_t and $\mathbb{E}\Gamma_t$ are contiguous in $L^q(\mathscr{H}, T)$ and their difference is of order $\sqrt{\varepsilon}$.*

Proof. We have

$$\Gamma_t=\int_0^t ds\int_0^s L(s,u)\,d\bar{Y}_u\int_0^s d\bar{Y}_u^*\left(I-\int_u^s L(v,u)\,dv\right)^*. \tag{4.5.22}$$

We first consider the case $q=2$; we have to estimate the variance of Γ_t; by an elementary calculation, we get

$$\begin{aligned}
\operatorname{var}\Gamma_t &\leqq C\iint_{[0,t]^2} du_1\,du_2\left|\int_{u_1\vee u_2}^t |L(s,u_1)|\left|\int_{u_2}^s L(v,u_2)\,dv-I\right|ds\right|^2\\
&\leqq C\iiiint_{[0,t]^4} du_1\,du_2\,ds_1\,ds_2\,|L(s_1,u_1)|\,|L(s_2,u_1)|\,1_{\{u_2\leqq s_1\wedge s_2\}}\\
&\quad\cdot\left|\int_{u_2}^{s_1} L(v,u_2)\,dv-I\right|\left|\int_{u_2}^{s_2} L(v,u_2)\,dv-I\right|\\
&\leqq C\iiint_{[0,t]^3} du\,ds_1\,ds_2\,|L(s_1,u)|\,|L(s_2,u)|\,\|Y_{s_1}-\bar{Y}_{s_1}\|_2\,\|Y_{s_2}-\bar{Y}_{s_2}\|_2\\
&\leqq\frac{C}{\varepsilon}\iint_{[0,t]^2} ds_1\,ds_2\,\tilde{\Psi}\left(\frac{|s_1-s_2|}{\varepsilon}\right)|Y-\bar{Y}|^2_{2,\mathscr{H},T}\\
&\leqq C\varepsilon
\end{aligned} \tag{4.5.23}$$

with the Definition (4.5.18) for $\tilde{\Psi}$. In order to prove the case $q>2$, we are going to see that since $\Gamma_t-\mathbb{E}\Gamma_t$ is in the second Wiener chaos of the Brownian motion $\bar{Y}_t$, its L^q norm is dominated by its L^2 norm; studying separately each component, it indeed appears that if one notes

$$\Gamma_t^{ij}-\mathbb{E}\Gamma_t^{ij}=\int_0^t d\bar{Y}_s^*\int_0^s h(s,u)\,d\bar{Y}_u, \tag{4.5.24}$$

using the inequality

$$\left(\int_0^t \alpha_s\, ds\right)^{q/2} \leqq \int_0^t \alpha_s^{q/2}\, \beta_s^{1-q/2}\, ds \left(\int_0^t \beta_s\, ds\right)^{-1+q/2} \tag{4.5.25}$$

valid for positive functions α_s and β_s, one deduces

$$\begin{aligned}
&\mathbb{E}\,|\Gamma_t^{ij} - \mathbb{E}\Gamma_t^{ij}|^q \\
&\leqq C\,\mathbb{E}\left(\int_0^t \left|\int_0^s h(s,u)\, d\bar{Y}_u\right|^2 ds\right)^{q/2} \\
&\leqq C\left(\int_0^t \int_0^s |h(s,u)|^2\, du\, ds\right)^{-1+q/2} \mathbb{E}\int_0^t \left|\int_0^s h(s,u)\, d\bar{Y}_u\right|^q \left(\int_0^s |h(s,u)|^2\, du\right)^{1-q/2} ds \\
&\leqq C\left(\int_0^t \int_0^s |h(s,u)|^2\, du\, ds\right)^{q/2}
\end{aligned} \tag{4.5.26}$$

and the term of the last line is just $\|\Gamma_t^{ij} - \mathbb{E}\Gamma_t^{ij}\|_2^q$. □

After Lemma 4.5.4, we still have to study asymptotically the mean of Γ_t, or equivalently from Proposition 4.2.3, study the mean of $\int_0^t Y_s \dot{Y}_s^*\, ds$; the perturbation will be symmetric if it converges to $I t/2$, and if it converges to another function γ_t, the difference $\gamma_t - I t/2$ will be skew-symmetric. Let us consider our two examples. For the delayed quadratic interpolation, one has

$$\begin{aligned}
\mathbb{E}\,\dot{Y}_t(\bar{Y}_t - Y_t)^* = &\left(I + \frac{(2k+1)\varepsilon - 2t}{\varepsilon} A\right) \\
&\cdot\left(\frac{(k+1)\varepsilon - t}{\varepsilon} - \frac{((k+1)\varepsilon - t)(t - k\varepsilon)}{\varepsilon^2} A\right)^*
\end{aligned} \tag{4.5.27}$$

for $\varepsilon \leqq k\varepsilon \leqq t \leqq (k+1)\varepsilon$, so $\mathbb{E}\Gamma_t$ converges to

$$\gamma_t = \frac{I t}{2} + \frac{t}{6}(A - A^*). \tag{4.5.28}$$

Thus the approximation is symmetric if and only if A is symmetric – for instance for the non perturbed case $A = 0$ which was considered in Sect. 4.3. For (4.5.3), one can prove that $\mathbb{E}\Gamma_t$ converges to $\gamma_t = -AKt$ where K is symmetric and solution of $AK + KA^* + I = 0$ (see [29]). The approximation is symmetric if and only if A is symmetric.

We will apply Sect. 4.4 with $\beta_t(l) = \bar{\beta}_t(l) = \phi_t(l) I$ for real-valued functions $\phi_t(l)$, so in order to study $\Gamma_t(l, l')$, we still have to consider double integrals of the form

$$\zeta_t = \int_0^t \phi_s\, dY_s \int_0^s \phi'_u\, d(\bar{Y}_u - Y_u)^* \tag{4.5.29}$$

where ϕ_t and ϕ'_t are real-valued functions. We want to prove that their variation is bounded in L^1 – in order to obtain (4.4.8) – and that they are contiguous to $\int_0^t \phi_s \phi'_s \, d\Gamma_s$. In the following lemma, we also prove that the contiguity of Γ_t and its mean holds in $L^q(\mathscr{F})$.

Lemma 4.5.5. *Let Y_t be an admissible perturbation of $\bar{Y}_t$; then for any families of functions ϕ_t and ϕ'_t which are bounded in $C^{1/2}$, for every q, the process ζ_t defined by (4.5.29) is contiguous in $L^q(\mathscr{F})$ to $\int_0^t \phi_s \phi'_s \, d\Gamma_s$ and $\int_0^t \phi_s \phi'_s \mathbb{E}\dot{\Gamma}_s \, ds$, and $\dot{\zeta}_t$ is bounded in L^q.*

Proof. One easily deduce from (4.5.9) that $\|\dot{Y}_t\|_q$ is of order $1/\sqrt{\varepsilon}$, so from Lemma 4.5.2,

$$\|\dot{\zeta}_t\|_q \leqq |\phi_t| \, \|\dot{Y}_t\|_{2q} \left\| \int_0^t \phi'_s \, d(\bar{Y}_s - Y_s) \right\|_{2q} \leqq C |\phi_t| \, |\phi'|_{1/2}. \tag{4.5.30}$$

The case $\phi \equiv \phi' \equiv 1$ yields the same estimation for $\dot{\Gamma}_t$. In particular

$$|\zeta|_{\mathscr{H},2,T} \leqq C |\phi|_{1/2} |\phi'|_{1/2} \tag{4.5.31}$$

so the bilinear map $(\phi, \phi') \mapsto \zeta$ is continuous from $C^{1/2} \times C^{1/2}$ into $L^2(\mathscr{H})$ uniformly in ε; the maps

$$(\phi, \phi') \mapsto \int_0^t \phi_s \phi'_s \, d\Gamma_s \quad \text{and} \quad (\phi, \phi') \mapsto \int_0^t \phi_s \phi'_s \mathbb{E}\dot{\Gamma}_s \, ds$$

satisfy the same continuity property. On the other hand, consider families of C^1 functions (ϕ, ϕ') which are uniformly bounded as well as their derivatives; applying two integrations by parts, we get

$$\begin{aligned} \zeta_t = {} & \int_0^t \phi_s \phi'_s \, d\Gamma_s - \phi'_0 \int_0^t \phi_s \, dY_s (\bar{Y}_0 - Y_0)^* \\ & - \int_0^t \phi_s \, dY_s \int_0^t (\bar{Y}_s - Y_s)^* \, d\phi'_s + \int_0^t \int_0^s \phi_u \, dY_u (\bar{Y}_s - Y_s)^* \, d\phi'_s \end{aligned} \tag{4.5.32}$$

so ζ_t and $\int_0^t \phi_s \phi'_s \, d\Gamma_s$ are contiguous in $L^2(\mathscr{H})$ $\left(\text{use Lemma 4.5.3 to estimate } \int_0^t \phi_s \, dY_s\right)$; from Lemma 4.5.4, these processes are also contiguous with $\int_0^t \phi_s \phi'_s \mathbb{E}\dot{\Gamma}_s \, ds$. Since C^1 is dense in $C^{1/2}$, by approximating (ϕ, ϕ') by such families, we get the contiguity in $L^2(\mathscr{H})$ for all (ϕ, ϕ') in $C^{1/2}$. Finally, from Lemma 2.1.3 (c) and (4.5.30), we check the contiguity in $L^q(\mathscr{F})$. □

By putting together previous results, we obtain the

Theorem 4.5.6. *Let y_t be a $\mathscr{F}_t$ Brownian motion, put $\bar{Y}_t = y_t$ or $y_{t+\varepsilon} - y_\varepsilon$, suppose that Y_t is an admissible perturbation of the Brownian motion $\bar{Y}_t$, and that X_t is the solution of*

$$\dot{X}_t = F(t, X_t) + G(t, X_t)\,\dot{Y}_t, \qquad X_0 = \Xi \tag{4.5.33}$$

where Ξ is a family of $\mathscr{F}_0$ measurable variables with bounded moments, $F(t,x)$ is a family of $\mathscr{F}_t$ predictable functions which are uniformly bounded and Lipschitz with respect to x, and

$$G(t,x) = \sum_{l=1}^{\infty} \beta_t(l)\,\tilde{G}(l,x). \tag{4.5.34}$$

We assume that $\tilde{G}(l)$ and its first and second derivatives satisfy (4.4.4) *and that $\beta_t(l)$ is a family of deterministic real-valued functions which are uniformly bounded in $C^{1/2}$; suppose also that the mean of the variable Γ_t converges to some function γ_t, that Ξ converges in the spaces L^q, $q \geqq 1$, to a variable ξ and that x_t is the solution of*

$$d x_t = F(t, x_t)\,dt + \frac{\partial G_k}{\partial x_i} G_{ij}(t, x_t)\,\dot{\gamma}_t^{jk}\,dt + G(t, x_t)\,d y_t, \qquad x_0 = \xi. \tag{4.5.35}$$

Then for every $q \geqq 1$, the variable $\sup_{t \leqq T} |X_t - x_t|$ converges to 0 in L^q.

Sketch of the Proof. Let $\bar{\mathscr{F}}_t$ be $\mathscr{F}_t$ or $\mathscr{F}_{t+\varepsilon}$, so that $\bar{Y}_t$ is a $\bar{\mathscr{F}}_t$ Brownian motion. From Lemmas 4.5.3 and 4.5.5 one can verify that one can apply Theorem 4.4.2 with $\bar{\beta}_t(l) = \beta_t(l)$ (the filtrations $\mathscr{F}_t$ and $\mathscr{H}_t$ of this theorem are taken to be $\bar{\mathscr{F}}_t$); thus one gets the contiguity in $L^0(\bar{\mathscr{F}})$ of X_t and the solution of

$$d\bar{X}_t = F(t, \bar{X}_t)\,dt + \frac{\partial G_k}{\partial x_i} G_{ij}(t, \bar{X}_t)\,d\Gamma_t + G(t, \bar{X}_t)\,d\bar{Y}_t, \qquad \bar{X}_0 = \Xi. \tag{4.5.36}$$

Then one uses Theorem 2.4.3 in order to get the contiguity of $\bar{X}_t$ and x_t in $L^0(\bar{\mathscr{F}})$. Moreover, the variable $\sup_t |x_t|$ is easily shown to be bounded in the spaces L^q, and to prove the theorem, it is sufficient to prove the same property for $\sup_t |X_t|$. One writes the equation for X_t in the form (4.4.18); the two integrals with respect to $W_t = t$ and $\bar{Y}_t$ are easily estimated; from the development of $D(t,x)$ and Lemma 4.5.3, one deduces that the suprema over x of the norms of $D(t,x)$ and its derivatives are bounded in L^q, so $\sup_t |R_t|$ (given by (4.4.17)) is also bounded in L^q; finally, by writing Z_t as the uniform limit as $L \to \infty$ of Z_t^L (given by (4.4.13)) and using Lemma 4.5.5, one also proves that $\sup_t |Z_t|$ is bounded in L^q. □

In particular, if the perturbation is symmetric, we obtain the stability of the Stratonovich equation. As it was explained previously, the case $\bar{Y}_t = y_{t+\varepsilon} - y_\varepsilon$ leads to anticipating approximations of y_t (such as those of [11]); for instance, in example (4.5.2), we obtain the quadratic interpolation without delay of y_t.

5. Rate of Convergence

When a Brownian motion is perturbed, the rate of convergence of the approximations can often be estimated; for instance, for the approximations of Sect. 4.5, this rate is of order $\sqrt{\varepsilon}$. The problem which is dealt with in this section is to study how this rate is transmitted to the solutions of the SDEs driven by the Brownian motion and its approximations. We will consider this question for equations driven by general semimartingales as studied in Sects. 2 and 4, and will estimate the rate of convergence in $L^q(\mathcal{H})$ for $q \geqq 1$. Of course, our regularity assumptions will be more stringent than in the previous sections. The cases of regular and singular perturbations will be respectively dealt with in Sects. 5.1 and 5.2; applications to the Euler discretization scheme and the approximations of Sect. 4.5 will be given.

5.1 Regular Perturbations

We are going to use the framework of Sect. 2; however, we will assume that $\mathcal{F}_t = \mathcal{G}_t$, as well as some regularity conditions; for instance, we will not study the general case of monotone coefficients but will ask them to be at least Lipschitz; actually, the monotone case can also be dealt with, but the resulting rate of convergence is generally worse (compare (3.3.4) and (3.3.23), or look at the rate of convergence in [2]); in this section, we have chosen to study only smooth situations in order to get sharper estimates. We first consider the basic case where only r_t is perturbed (situation of Lemma 3.4.3).

Theorem 5.1.1. *Suppose that we are given two families of filtrations $\mathcal{F}_t$ and $\mathcal{H}_t$, a family of $\mathcal{F}_t$ semimartingales W_t, two families of $\mathcal{F}_t$ adapted càdlàg processes R_t and r_t and a family of $\mathcal{F}_t$ predictable functions $F(t, \omega, x)$ such that*

$$|F(t, \omega, x) - F(t, \omega, y)| \leqq C_0 |x - y|. \tag{5.1.1}$$

Suppose that (X, x) is solution of

$$\begin{aligned} X_t &= R_t + \int_0^t F(s, X_{s-})\, dW_s, \\ x_t &= r_t + \int_0^t F(s, x_{s-})\, dW_s. \end{aligned} \tag{5.1.2}$$

Fix $T > 0$, $q \geqq 2$, $1 \leqq q' < q$, $p > 1$ and define q'' by $1 = q/q'' + 1/p$; suppose that

$$A_t = |W|_t + \langle\!\langle W, W \rangle\!\rangle_t + \langle\!\langle W \rangle\!\rangle_t^{(q)} \tag{5.1.3}$$

satisfies (3.2.1) *for some admissible $\mathcal{H}_t$ dominating processes L_t. Let η_ε be a family of positive numbers converging to 0 as $\varepsilon \to \varepsilon_0$. Then there exists a $K > 0$ depending only on C_0, q, q' such that if the exponential moment of order K of A_T is bounded,*

$$|R - r|_{\mathcal{H}, q'', T} = O(\eta_\varepsilon) \Rightarrow |X - x|_{\mathcal{H}, q', T} = O(\eta_\varepsilon). \tag{5.1.4}$$

If dA_t/dL_t is uniformly bounded by some constant number (case $p=\infty$ and $\mathscr{H}_t \subset \mathscr{F}_t$), then one can take $q'=q=q''$; moreover, if A_t is uniformly bounded (case $p=\infty$ and $\mathscr{H}_t=\mathscr{F}_t$),

$$\|\sup_{t\leqq T} |R_t - r_t|\,\|_q = O(\eta_\varepsilon) \Rightarrow \|\sup_{t\leqq T} |X_t - x_t|\,\|_q = O(\eta_\varepsilon). \tag{5.1.5}$$

When one writes equation

$$X_t - x_t - R_t + r_t = \int_0^t (F(s, X_{s-}) - F(s, x_{s-}))\, dW_s, \tag{5.1.6}$$

this theorem is an easy corollary of Lemma 3.3.1. As an example, we now apply this result to the Euler scheme of Sect. 2.5 when the predictable characteristics of the semimartingale w_t are strongly dominated by a deterministic function.

Proposition 5.1.2. *Consider the equation*

$$x_t = r_t + \int_0^t f(x_{s-})\, dw_s \tag{5.1.7}$$

for a Lipschitz bounded function f, a semimartingale w_t and a càdlàg process r_t such that the moments of $\sup_{t\leqq T} |r_t|$ are finite. Fix $q \geqq 2$ and suppose that

$$a_t = |w|_t + \langle\!\langle w, w \rangle\!\rangle_t + \langle\!\langle w \rangle\!\rangle_t^{(q)} \tag{5.1.8}$$

satisfies $a_t - a_s \leqq \ell_t - \ell_s$ for $s \leqq t$ for some deterministic function ℓ_t. Consider as in Sect. 2.5 a family of discretizations (t_k), define $X(k)$ by (2.5.3) and let N be an integer-valued random variable such that t_N is a $\mathscr{G}_t$ stopping time which is less than T. Let $\tau(t)$ be the greatest time t_k such that $t_k \leqq t$. Then

$$\|\sup_{k\leqq N} |X(t) - x_{t_k}|\,\|_q \leqq C \sup_{t\leqq T} \|\,|x_t - x_{\tau(t)}|\, 1_{\{t \leqq t_N\}}\|_q. \tag{5.1.9}$$

Remark. For classical SDEs driven by Brownian motion and dt, we recover the classical bound $\sqrt{\varepsilon}$ for the rate of convergence of the Euler scheme with discretization step ε; more generally, we can consider equations driven by semimartingales with independent increments.

Proof. By stopping equation (5.1.7) after time t_N, we can suppose that $t_N = T$ and that w_t and r_t are constant after T; as in Sect. 2.5, we can also suppose $w_t = w_0$ for $t \leqq t_1$. We will use the notations of Sect. 2.5 and will verify that we can apply Theorem 5.1.1 to the system (2.5.9) with $\mathscr{H}_t = \mathscr{F}_t$, $L_t = \arctan A_t$ and $p = \infty$. Formulas (2.5.11) and (2.5.13) provide us with $|W|_t$ and $\langle\!\langle W, W \rangle\!\rangle_t$, and one checks similarly

$$\langle\!\langle W \rangle\!\rangle_T^{(q)} = \sum_{k=1}^{N-1} \mathbb{E}[|w_{t_{k+1}} - w_{t_k}|^q | \mathscr{G}_{t_k}]. \tag{5.1.10}$$

By developing the qth power with Itô's formula and using some inequalities as in the proof of Lemma 3.3.1, we deduce

$$\langle\!\langle W\rangle\!\rangle_T^{(q)} \leqq C\sum_{k=1}^{N-1}\mathbb{E}\Bigg[\int_{t_k}^{t_{k+1}}|w_{s-}-w_{t_k}|^{q-2}\,d\langle\!\langle w,w\rangle\!\rangle_s$$

$$+\int_{t_k}^{t_{k+1}}|w_{s-}-w_{t_k}|^{q-1}\,d|w|_s+\langle\!\langle w\rangle\!\rangle_{t_{k+1}}^{(q)}-\langle w\rangle\!\rangle_{t_k}^{(q)}\,|\mathscr{G}_{t_k}\Bigg]. \tag{5.1.11}$$

Thus if one defines A_t by (5.1.3) and if one uses the domination of a_t by ℓ_t, one gets

$$A_T \leqq C\ell_T + C\sum_{k=1}^{N-1}\int_{t_k}^{t_{k+1}}\mathbb{E}[|w_{s-}-w_{t_k}|+|w_{s-}-w_{t_k}|^{q-1}$$

$$+|w_{s-}-w_{t_k}|^{q-2}\,|\mathscr{G}_{t_k}]\,d\ell_s. \tag{5.1.12}$$

Since a_T is bounded by some constant number, we can also deduce that the variables

$$\mathbb{E}[\sup_{t_k<s<t_{k+1}}|w_s-w_{t_k}|^{q'}\,|\mathscr{G}_{t_k}]$$

are also uniformly bounded for $q'\leqq q$ (conditional form of Proposition 2.2.4(a)), so A_t is uniformly bounded. Thus we can apply Theorem 5.1.1 and (5.1.5) yields

$$\|\sup_{t\leqq T}|X_t-\bar{x}_t|\,\|_q \leqq C\left\|\sup_{k\leqq N}\left|\int_0^{t_k}(f(x_{s-})-f(x_{\tau(s-)}))\,dw_s\right|\right\|_q. \tag{5.1.13}$$

The left-hand side is equal to the left-hand side of (5.1.9) and to estimate the right-hand side, we can use Proposition 2.2.4(b) with $p=\infty$ and $\mathscr{H}_t$ the filtration of deterministic events. □

In Theorem 5.1.1, the assumption concerning exponential moments of A_t may seem too strong and is sometimes difficult to prove. We have seen in Sect. 2.7 that if this condition is weakened for the bounded part of F, the contiguity in $L^{q'}(\mathscr{H})$ is preserved. We now verify that the rate of convergence is also sometimes preserved.

Proposition 5.1.3. *Assume the conditions of Theorem* 5.1.1 *except the boundedness of the exponential moments of A_t, and let $p'<\infty$ such that $1/q'>1/q+1/p'$.*

(a) *Suppose that (R, F, W), (r, F, W) satisfy $\mathbb{H}_q$ (see Definition* 2.7.1) *and that for any $K>0$, there exists a family of $\mathscr{F}_t$ stopping times τ such that for any T,*

$$\mathbb{P}[\tau<T]=O(\eta_\varepsilon^{p'}) \tag{5.1.14}$$

and the exponential moments of order K of $A_{T\wedge\tau}$ are bounded. Then (5.1.4) *holds.*

(b) *If there exists a family of $\mathscr{F}_t$ predictable nondecreasing processes P_t such that $P_0>0$, the exponential moments of order K of P_t are bounded and*

$$\mathbb{P}[\exists t \leqq T, A_t \geqq P_t] = O(\eta_\varepsilon^{p'}), \tag{5.1.15}$$

then one can find a family τ satisfying the conditions of (a).

Proof. We apply our assumption with the order K obtained in Theorem 5.1.1. Define q_1 by $1/q' = 1/q_1 + 1/p'$. Then $q_1 < q$ and if τ' is a family of times of $\mathscr{T}(\mathscr{H}, T)$,

$$\begin{aligned}\|X_{\tau'} - x_{\tau'}\|_{q'} &\leqq \|x_{\tau' \wedge \tau} - x_{\tau' \wedge \tau}\|_{q'} + \|(X_{\tau'} - x_{\tau'}) 1_{\{\tau < T\}}\|_{q'} \\ &\leqq \|X_{\tau' \wedge \tau} - x_{\tau' \wedge \tau}\|_{q'} + (|X|_{\mathscr{H}, q_1, T} + |x|_{\mathscr{H}, q_1, T}) \mathbb{P}[\tau < T]^{1/p'}.\end{aligned} \tag{5.1.16}$$

Let $\bar{X}_t$ and $\bar{x}_t$ be the solutions of an equation similar to (5.1.2), but with W_t replaced by $W_{t \wedge \tau}$; then (X_t, x_t) and $(\bar{X}_t, \bar{x}_t)$ coincide on $[0, \tau]$, so

$$\begin{aligned}\|X_{\tau'} - x_{\tau'}\|_{q'} &\leqq \|\bar{X}_{\tau' \wedge \tau} - \bar{x}_{\tau' \wedge \tau}\|_{q'} + (|X|_{\mathscr{H}, q_1, T} + |x|_{\mathscr{H}, q_1, T}) \mathbb{P}[\tau < T]^{1/p'} \\ &\leqq \|\bar{X}_{\tau'} - \bar{x}_{\tau'}\|_{q'} + (|X|_{\mathscr{H}, q_1, T} + |x|_{\mathscr{H}, q_1, T} \\ &\quad + |\bar{X}|_{\mathscr{F}, q_1, T} + |\bar{x}|_{\mathscr{F}, q_1, T}) \mathbb{P}[\tau < T]^{1/p'}.\end{aligned} \tag{5.1.17}$$

The first term is of order η_ε from Theorem 5.1.1 and for the second one, one applies Proposition 2.7.2 and (5.1.14), so that one gets (a). To prove (b), define

$$\tau_1 = \inf\{t; A_t \geqq P_t\}. \tag{5.1.18}$$

The stopping time τ_1 satisfies (5.1.14) and is predictable (it is the beginning of a right continuous predictable set), so there exists another stopping time $\tau < \tau_1$ which also satisfies (5.1.14); since $A_{t \wedge \tau}$ is dominated by P_t, its exponential moment of order K is bounded. □

Let us now describe a SDE where the coefficients, the semimartingales and the initial condition are all perturbed. In order to avoid multiplicity of the exponents, we will not give the more general statement.

Theorem 5.1.4. *We are given two families of filtrations $\mathscr{F}_t$ and $\mathscr{H}_t$, two families of $\mathscr{F}_t$ semimartingales W_t and w_t, a family of $\mathscr{F}_t$ predictable functions $F(t, \omega, x)$ satisfying* (5.1.1), *a family of deterministic C_b^2 functions $f(x)$ (with bounds uniform in ε), a family of $\mathscr{F}_t$ adapted càdlàg processes R_t and a family of $\mathscr{F}_0$ measurable variables x_0. Let (X, x) be the solution of*

$$\begin{aligned}X_t &= R_t + \int_0^t F(s, X_{s-})\, dW_s, \\ x_t &= x_0 + \int_0^t f(x_{s-})\, dw_s.\end{aligned} \tag{5.1.19}$$

Let $q>2$ and $q'<q$. Defining A_t with (5.1.3), *assume that the exponential moments of A_t are bounded and that A_t satisfies* (3.2.1) *for every $p<\infty$; define also*

$$a_t = |w|_t + \langle\langle w, w\rangle\rangle_t + \langle\langle w\rangle\rangle_t^{(2q)} \tag{5.1.20}$$

and assume that it satisfies (3.2.1) *for every $p<\infty$. If the $L^q(\mathscr{H}, T)$ norm of $R.-x_0$, $\sup_x |F(\cdot, x)-f(x)|$, $W-w$ and of the variation of $[W-w, w]$ are of order η_ε, then $|X-x|_{\mathscr{H},q',T}$ is also of order η_ε.*

Remark. The condition concerning exponential moments of A_t can be modified as in Proposition 5.1.3.

Proof. We have

$$x_t = x_0 + \int_0^t f(x_{s-})\, d(w_s - W_s) + \int_0^t (f(x_{s-}) - F(s, x_{s-}))\, dW_s + \int_0^t F(s, x_{s-})\, dW_s. \tag{5.1.21}$$

Using an integration by parts formula in order to express the integral of $f(x_{s-})$ with respect to $w_s - W_s$, we deduce that if we put

$$\begin{aligned} r_t = x_0 + f(x_t)(w_t - W_t) &+ \int_0^t df(x_s)(W_{s-} - w_{s-}) \\ &+ [f(x), W-w]_t + \int_0^t (f(x_{s-}) - F(s, x_{s-}))\, dW_s, \end{aligned} \tag{5.1.22}$$

we are in the framework of Theorem 5.1.1; choosing $2 \vee q' < q_1 < q$, W_t satisfies the assumptions of this theorem for every $p<\infty$ and with q replaced by q_1; thus we only have to prove that $R-r$ is of order η_ε in $L^{q_2}(\mathscr{H}, T)$ for some $q_2 > q_1$; we will choose $q_1 < q_2 < q$. The last integral of (5.1.22) is estimated by means of Proposition 2.2.4 (b). From Itô's formula,

$$\begin{aligned} \left|\int_0^t df(x_s)(W_{s-} - w_{s-})\right| \leqq \left|\int_0^t \frac{\partial f}{\partial x^i} f_{ij}(x_{s-})(W_{s-} - w_{s-})\, dw_s^j\right| \\ + C \int_0^t |W_{s-} - w_{s-}|\, d[\![w, w]\!]_s. \end{aligned} \tag{5.1.23}$$

These two terms are again estimated with Proposition 2.2.4 (b): for the second one, note that the predictable compensator of $[\![w, w]\!]_t$ is $\langle\langle w, w\rangle\rangle_t$, that the compensator of its bracket is $\langle\langle w\rangle\rangle_t^{(4)}$ and that the compensator of its qth power jumps is $\langle\langle w\rangle\rangle_t^{(2q)}$. Finally, by studying separately its continuous and discontinuous parts, the bracket of $f(x)$ and $W-w$ is easily estimated by the variation of $[W-w, w]$. □

Nevertheless, one has to remark that the assumptions are here much more restrictive than in Sect. 2, because we want the processes W and w to be semimartingales with respect to the same family of filtrations $\mathscr{F}_t$. As it was noticed in Corollary 2.2.5, this implies a rather strong convergence of $W-w$ to 0; thus perturbations on absolutely continuous processes will be dealt with this method, but for perturbations on more general semimartingales, we will generally need singular perturbation methods.

5.2 Singular Perturbations

Here, we use the framework of Sects. 4.1 and 4.2 (we will limit ourselves to the case of time-independent coefficients G) and explain how the rate of convergence can be deduced from Theorem 5.1.1.

Theorem 5.2.1. *Under Assumption* 4.2.1, *suppose that the functions* $F(t,\omega,x)$ *are uniformly Lipschitz and bounded. Fix* $q>2$, $q'<q$ *and define*

$$\begin{aligned} A_t = {} & |W|_t + \langle\!\langle W,W\rangle\!\rangle_t + \langle\!\langle W\rangle\!\rangle_t^{(2q)} + |\bar{Y}|_t + \langle\!\langle \bar{Y},\bar{Y}\rangle\!\rangle_t + \langle\!\langle \bar{Y}\rangle\!\rangle_t^{(2q)} \\ & + \langle\!\langle Y,Y\rangle\!\rangle_t + \langle\!\langle Y\rangle\!\rangle_t^{(2q)} + \int_0^t |\bar{Y}_{s-} - Y_{s-}|\, d|Y|_s \\ & + \int_0^t |\bar{Y}_{s-} - Y_{s-}|^2\, d\langle\!\langle Y,Y\rangle\!\rangle_s + \int_0^t |\bar{Y}_{s-} - Y_{s-}|^q\, d\langle\!\langle Y\rangle\!\rangle_s^{(q)}. \end{aligned} \tag{5.2.1}$$

Suppose that the exponential moments of A_t *are bounded and that* (3.2.1) *is satisfied for every* $p<\infty$. *Let* η_ε *be a family of positive numbers converging to* 0; *assume that the* $L^q(\mathcal{H},T)$ *norm of* $\bar{Y}-Y$ *and that for any* p *the* L^p *norm of* $\sup_{s\leq T}|\Delta\bar{Y}_s-\Delta Y_s|$ *are of order* η_ε; *then the* $L^{q'}(\mathcal{H},T)$ *norm of* $\bar{X}-X$ *is also of order* η_ε. *If one removes the conditions concerning exponential moments of* A_t *but if for any* K, *there exist stopping times* τ *satisfying the conditions of Proposition* 5.1.3, *then the conclusion still holds.*

Sketch of the Proof. The equation of $\bar{X}_t$ is (4.2.2) and we write the equation of X_t in the form (4.1.10). Consider q_1 such that $2\vee q'<q_1<q$; we first have to verify that the semimartingales W_t, $\bar{Y}_t$, Γ_t and Λ_t satisfy the assumptions of Theorems 5.1.1 with q replaced by q_1 and for every $p<\infty$; actually it is not difficult to prove that the increments of their predictable characteristics are dominated by the increments of the process A_t defined in (5.2.1). Then, as in the proof of Theorem 5.1.4, we check that for $q_1<q_2<q$, the $L^{q_2}(\mathcal{H},T)$ norm of $R-X_0$ is of order η_ε: in (4.1.7), the integrals with respect to S_t and Z_t are estimated by means of Proposition 2.2.4 (b) (one notes that the variation of Z_t is dominated by the sum of increments of $[\![W,W]\!]_t$ and $[\![Y,Y]\!]_t$); for the last term, we use the assumption on $\sup_s|\Delta\bar{Y}_s-\Delta Y_s|$. The generalization concerning exponential moments of A_t is obtained from Proposition 5.1.3 and by proving that the Eq. (4.1.10) of X_t satisfies $\mathbb{H}_{\bar{q}}$ for any $\bar{q}<q$. □

Then, if W_t, $\bar{Y}_t$, Γ_t and Λ_t are regular perturbations (in the sense of Sect. 5.1) of w_t, y_t, γ_t and λ_t, we can estimate the difference between $\bar{X}_t$ and the solution x_t of (4.2.5) by applying Theorem 5.1.4. All these results can be applied to the approximations of Brownian motions by absolutely continuous processes, in particular the examples of Sect. 4.5 and (4.5.33) when $F(t, x)$ and $G(t, x)$ depend only on x. In this case, one takes for $\mathscr{H}_t$ the filtration of deterministic events; actually, the rate of convergence in $L^q(\mathscr{F})$ can also be estimated but it is generally slower (it is related to the modulus of continuity of the Brownian motion); note that if Y_t is an admissible perturbation of $\bar{Y}_t$ (see Definition 4.5.1), the $L^q(\mathscr{H}, T)$ norm of $\bar{Y}-Y$ is of order $\sqrt{\varepsilon}$ from (4.5.8), so we want to deduce a similar rate of convergence for the process X_t of Theorem 4.5.6.

Theorem 5.2.2. *Suppose that $F(x)$ and $G(x)$ are families of uniformly C_b^2 and C_b^3 functions and that Ξ and ξ are $\mathscr{F}_0$ measurable variables such that $\Xi-\xi$ is of order $\sqrt{\varepsilon}$ in the spaces L^q, $q \geqq 1$. Let y_t be a Brownian motion, $\bar{Y}_t = y_t$ or $y_{t+\varepsilon}-y_t$, let Y_t be an admissible perturbation of the Brownian motion $\bar{Y}_t$ (see Definition* 4.5.1) *and suppose that besides* (4.5.9), *the estimate*

$$\left|\int_s^t L(u, s)\, du - I\right| \leqq \Psi\left(\frac{t-s}{\varepsilon}\right) \tag{5.2.2}$$

is satisfied for $s \leqq t$. Suppose that γ_t is a deterministic absolutely continuous function such that $\mathbb{E}\Gamma_t - \gamma_t$ is of order $\sqrt{\varepsilon}$. If X_t and x_t are solutions of

$$\dot{X}_t = F(X_t) + G(X_t)\dot{Y}_t, \qquad X_0 = \Xi, \tag{5.2.3}$$

$$d x_t = F(x_t)\, dt + \frac{\partial G_k}{\partial x_i} G_{ij}(x_t)\, \dot{\gamma}_t^{jk}\, dt + G(x_t)\, dy_t, \qquad x_0 = \xi, \tag{5.2.4}$$

then for any $q < \infty$, the L^q norm of $|X_t - x_t|$ is of order $\sqrt{\varepsilon}$, uniformly on bounded time intervals $[0, T]$.

Proof. Let $\bar{X}_t$ be the solution of

$$d\bar{X}_t = F(\bar{X}_t)\, dt + \frac{\partial G_k}{\partial x_i} G_{ij}(\bar{X}_t)\, \dot{\Gamma}_t^{jk}\, dt + G(\bar{X}_t)\, d\bar{Y}_t \tag{5.2.5}$$

with $\bar{X}_0 = \Xi$. We will first estimate $\bar{X}-X$ with Theorem 5.2.1; then we will estimate $\bar{X}-x$ with Theorem 5.1.4, We have $W_t = t$ and the process A_t of (5.2.1) reduces to

$$A_t = 2t + \int_0^t |\dot{\Gamma}_s|\, ds. \tag{5.2.6}$$

Since $\|\dot{\Gamma}_t\|_p$ is bounded, A_t satisfies (3.2.1) for $L_t = \arctan t$ and any $p < \infty$; let us now study its exponential integrability. For $K > 0$,

$$\begin{aligned} \mathbb{E}\exp\left(K \int_0^t |\dot{\Gamma}_s|\, ds\right) &\leqq \frac{1}{t} \int_0^t \mathbb{E}\exp(K t\, |\dot{\Gamma}_s|)\, ds \\ &\leqq \frac{1}{t} \int_0^t \left(\mathbb{E}\exp K\varepsilon t\, |\dot{Y}_s|^2\, \mathbb{E}\exp \frac{K t\, |Y_s - \bar{Y}_s|^2}{\varepsilon}\right)^{1/2} ds. \end{aligned} \tag{5.2.7}$$

On the other hand, $\sqrt{\varepsilon}\,\dot{Y}_s$ and $(\bar{Y}_s - Y_s)/\sqrt{\varepsilon}$ are Gaussian variables, the variances of which are bounded as $\varepsilon \to 0$; thus from standard estimates, for any K, the last line of (5.2.7) is shown to be uniformly bounded if t is at most equal to some number δ which does not depend on ε; however, the problem of the exponential integrability for larger t does not look easy so we are going to avoid it with Proposition 5.1.3; thus we now construct the process P_t. For any $s \leqq t$, let $\mathscr{Y}_t^s$ be the σ-algebra generated by $\bar{Y}_u - \bar{Y}_s$ for $s \leqq u \leqq t$; consider some $\delta > 0$ (which will be chosen later) and define the process

$$U_t = \mathbb{E}[\dot{Y}_t \mid \mathscr{Y}_t^{t-\delta}] = \int_{t-\delta}^{t} L(t,s)\, d\bar{Y}_s. \tag{5.2.8}$$

By comparing with (4.5.4) we deduce

$$\begin{aligned} \mathbb{E}\,|U_t - \dot{Y}_t|^2 &= \int_0^{t-\delta} |L(t,s)|^2\, ds \\ &\leqq \frac{1}{\varepsilon^2} \int_0^{t-\delta} \Psi^2\left(\frac{t-s}{\varepsilon}\right) ds \\ &\leqq \frac{1}{\varepsilon} \int_{\delta/\varepsilon}^{\infty} \Psi^2(z)\, dz. \end{aligned} \tag{5.2.9}$$

Since Ψ is bounded,

$$\int_u^{\infty} \Psi^2(z)\, dz \leqq C \int_u^{\infty} \Psi(z)\, dz \leqq \frac{C}{\sqrt{u}} \int_0^{\infty} \sqrt{z}\, \Psi(z)\, dz, \tag{5.2.10}$$

so from (4.5.10), the L^2 norm of $U_t - \dot{Y}_t$ is therefore of order $\varepsilon^{-1/4}$; since it is a Gaussian variable, its L^p norm has the same order for any $p < \infty$. Define

$$V_t = \mathbb{E}[\bar{Y}_t - Y_t \mid \mathscr{Y}_t^{t-\delta}] = \int_{t-\delta}^{t} \left(I - \int_s^t L(u,s)\, du\right) d\bar{Y}_s. \tag{5.2.11}$$

By means of (5.2.2), we deduce from a calculus similar to (5.2.9) that the L^p norm of $V_t - \bar{Y}_t + Y_t$ is of order $\varepsilon^{3/4}$; thus if we define

$$\tilde{\Gamma}_t = \int_0^t U_s V_s^*\, ds, \tag{5.2.12}$$

we deduce that the L^p norm of $\dot{\tilde{\Gamma}}_t - \dot{\Gamma}_t$ is of order $\varepsilon^{1/4}$. Define also

$$P_t = 2t + \int_0^t |\dot{\tilde{\Gamma}}_s|\, ds + 1. \tag{5.2.13}$$

Then the L^p norm of $\dot{P}_t - \dot{A}_t$ is of order $\varepsilon^{1/4}$, so

$$\begin{aligned}\mathbb{P}[\exists t \leqq T, A_t \geqq P_t] &\leqq \mathbb{P}\left[\int_0^T |\dot{A}_t - \dot{P}_t|\,dt \geqq 1\right] \\ &\leqq \left\| \int_0^T |\dot{A}_t - \dot{P}_t|\,dt \right\|_p^p \\ &= O(\varepsilon^{p/4}).\end{aligned} \tag{5.2.14}$$

Since this is true for any p, in order to use Proposition 5.1.3(b), we only have to verify the exponential integrability of P_t. Fix K; from the Cauchy-Schwarz inequality,

$$\begin{aligned}&\mathbb{E}\exp K \int_0^T |\dot{\tilde{\Gamma}}_s|\,ds \\ &\leqq \left(\mathbb{E}\exp 2K \sum_{k\,\text{even}} \int_{k\delta}^{(k+1)\delta} |\dot{\tilde{\Gamma}}_s|\,ds \cdot \mathbb{E}\exp 2K \sum_{k\,\text{odd}} \int_{k\delta}^{(k+1)\delta} |\dot{\tilde{\Gamma}}_s|\,ds\right)^{1/2}\end{aligned} \tag{5.2.15}$$

where the sums are limited to k such that $k\delta < T$. But for $k\delta \leqq s \leqq (k+1)\delta$, the variable $\dot{\tilde{\Gamma}}_s$ is $\mathscr{Y}_{(k+1)\delta}^{(k-1)\delta}$ measurable so the different terms in each sum are independent, so

$$\mathbb{E}\exp K \int_0^T |\dot{\tilde{\Gamma}}_s|\,ds \leqq \prod_{k<T/\delta} \left(\mathbb{E}\exp 2K \int_{k\delta}^{(k+1)\delta} |\dot{\tilde{\Gamma}}_s|\,ds\right)^{1/2}. \tag{5.2.16}$$

By means of an estimate of type (5.2.7), one can prove that one can choose $\delta > 0$ such that each term of the product is bounded ($\sqrt{\varepsilon}\,U_t$ and $V_t/\sqrt{\varepsilon}$ are indeed Gaussian variables the variances of which are bounded uniformly in ε and δ); with this choice of δ, the conditions of Proposition 5.1.3(b) are satisfied. Thus we obtain the result of Theorem 5.2.1, so the $L^q(\mathscr{H}, T)$ norm of $X - \bar{X}$ is of order $\sqrt{\varepsilon}$. In order to conclude, first suppose that we are in the case $\bar{Y}_t = y_t$. From Lemma 4.5.4 and the assumption about γ_t, $\Gamma_t - \gamma_t$ is of order $\sqrt{\varepsilon}$ and it is not difficult to use Theorem 5.1.4 and obtain that $\bar{X} - x$ is also of order $\sqrt{\varepsilon}$ (it is in the application of this theorem that we need third derivatives for G, so that the coefficient of γ_t is C_b^2). If $\bar{Y}_t = y_{t+\varepsilon} - y_\varepsilon$, we consider the solution of

$$d\bar{x}_t = F(\bar{x}_t)\,dt + \frac{\partial G_k}{\partial x_i}\, G_{ij}(\bar{x}_t)\,\dot{\gamma}_{t+\varepsilon}^{jk}\,dt + G(\bar{x}_t)\,d\bar{Y}_t, \qquad \bar{x}_0 = x_\varepsilon. \tag{5.2.17}$$

Then $\bar{X}_t - \bar{x}_t$ is of order $\sqrt{\varepsilon}$ from Theorem 5.1.4 and $x_t = \bar{x}_{t-\varepsilon}$ so $x_t - \bar{x}_t$ is also of order $\sqrt{\varepsilon}$. □

Some further questions concerning approximations of Brownian motions are dealt with in [29]; in particular, it is explained how the rate $\sqrt{\varepsilon}$ can be improved with some additional commutativity conditions; some tools of the stochastic calculus of variations are used with this aim.

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