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Probability Theory and Related Fields

Derivation of the Hydrodynamical Equation for One-dimensional Ginzburg-Landau Model

Dedicated to Professor Takeyuki Hida on his 60th birthday

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Summary. The hydrodynamical behavior of one-dimensional scalar Ginzburg-Landau model with conservation law is investigated. The dynamics of the system is given by solving a stochastic partial differential equation. Under appropriate space-time scaling, a deterministic limit is obtained and the limit is described by a certain nonlinear diffusion equation.

Contents

1. Introduction

We shall study a model of the spin configuration $S: \mathbb{R} \to \mathbb{R}$ over the real line changing randomly with time. The evolution law is given by a stochastic partial differential equation (SPDE)

$$dS_t(x) = -\Delta^2 S_t(x)\,dt + \Delta\{U'(S_t(x))\}\,dt + \sqrt{2}\,\nabla dw_t(x),$$
$$t>0,\ x\in\mathbb{R};\ \Delta = d^2/dx^2,\ \nabla = d/dx, \tag{1.1}$$

where w_t is a cylindrical Brownian motion on the space $L^2(\mathbb{R}, dx)$, that is, an $\mathscr{S}'(\mathbb{R})$-valued continuous process such as $\langle w_t, \varphi\rangle$ is a standard Brownian motion for every $\varphi \in \mathscr{S}(\mathbb{R})$ satisfying $\|\varphi\|_{L^2}=1$. Throughout the paper we assume the following condition on the self-potential $U: \mathbb{R} \to \mathbb{R}$.

$$U(s)=\frac{\gamma}{2}s^2+V(s); \qquad \gamma>0, \quad V\in C_b^3(\mathbb{R}). \tag{1.2}$$

The SPDE (1.1) is called the *time-dependent Ginzburg-Landau equation* (TDGL Eq.).

The purpose of the present paper is to know the macroscopic behavior of this model. We introduce the hydrodynamical space-time scaling: $t\to t/\varepsilon^2$, $x\to x/\varepsilon$, $\varepsilon>0$, for the TDGL Eq. and investigate the asymptotic behavior of the scaled process $S_t^\varepsilon(x)=S_{t/\varepsilon^2}(x/\varepsilon)$ as $\varepsilon\downarrow 0$. Note that $S_t^\varepsilon(x)$ satisfies the following SPDE, correctly speaking, in the sense of law:

$$dS_t^\varepsilon(x)=-\varepsilon^2\Delta^2 S_t^\varepsilon(x)\,dt+\Delta\{U'(S_t^\varepsilon(x))\}\,dt+\sqrt{2\varepsilon}\,\nabla dw_t(x), \qquad t>0,\ x\in\mathbb{R}. \tag{1.3}$$

We shall prove that S_t^ε converges to a non-random function $\rho_t=\rho_t(x)$ which is a solution of the following type of nonlinear diffusion equation

$$\frac{\partial\rho_t}{\partial t}=\frac{\partial}{\partial x}\left\{d(\rho_t)\frac{\partial\rho_t}{\partial x}\right\}. \tag{1.4}$$

We introduce some more notations to explain the coefficient $d(\rho)$. Let $H_\lambda=-\frac{1}{2}d^2/ds^2+\{U(s)-\lambda s\}$, $\lambda\in\mathbb{R}$, be a self-adjoint operator on the space $L^2(\mathbb{R}, ds)$ and let Ω_λ be a positive and normalized eigenfunction of H_λ corresponding to its minimal eigenvalue $\kappa(\lambda)$. Define the *mean spin function* $\bar\rho$, which is real analytic and strictly increasing (see Sect. 2), by

$$\bar\rho(\lambda)=\int s\Omega_\lambda^2(s)\,ds, \qquad \lambda\in\mathbb{R}. \tag{1.5}$$

Then the diffusion coefficient $d(\rho)$ is the derivative

$$d(\rho)=\bar\lambda'(\rho), \tag{1.6}$$

of an inverse function $\bar\lambda=\bar\lambda(\rho)$ of $\bar\rho=\bar\rho(\lambda)$.

Let $C_b^{2+\beta}(\mathbb{R})$, $0<\beta<1$, be the class of all $S\in C_b^2(\mathbb{R})$ satisfying $\sup\{|S''(x)-S''(y)|/|x-y|^\beta;\ x, y\in\mathbb{R}, |x-y|<1\}<\infty$. We can now state our main result.

Theorem 1.1. *Let S_t^ε and ρ_t be the solutions of the scaled* TDGL *Eq.* (1.3) *and the nonlinear diffusion Eq.* (1.4), *respectively, with same initial value $S\in C_b^{2+\beta}(\mathbb{R})$. We assume $\gamma_0\equiv\|V''\|_\infty<\min(\gamma,\gamma_1)$, where γ_1 is an absolute constant appearing in Sect.* 10. *Then S_t^ε converges to ρ_t as ε tends to* 0 *in the following sense:* $\lim_{\varepsilon\downarrow 0} P(|\int_{\mathbb{R}}\{S_t^\varepsilon(x)-\rho_t(x)\}\,\varphi(x)\,dx|>\delta)=0$ *for every* $\delta>0$, $t>0$ *and* $\varphi\in C_0^\infty(\mathbb{R})$.

The existence and uniqueness theorems for the Eqs. (1.3) and (1.4) will be discussed in Sects. 5 and 11, respectively.

The SPDE (1.1) has a one-parameter family of invariant measures $\{\mu_\lambda\}_{\lambda\in\mathbb{R}}$ where μ_λ is a certain probability measure on the configuration space $\mathscr{C}=C(\mathbb{R})$, which is called (U,λ)-Gibbs distribution having constant profile λ (see Sect. 3).

The parameter λ represents the strength of the external field. The limiting PDE (1.4) can be derived quickly if we assume the so-called principle of hydrodynamics (see [5]): There exists a function $\lambda(t, x)$ such that for each $(t, x) \in (0, \infty) \times \mathbb{R}$ the distribution of $S_t^\varepsilon(x)$ converges weakly as $\varepsilon \downarrow 0$ to a probability measure $\nu_{\lambda(t,x)}$ on $\mathbb{R}$. Here ν_λ is the 1-dimensional marginal distribution of μ_λ given by $\nu_\lambda(ds) = \Omega_\lambda^2(s)\, ds$. This principle seems plausible, since the hydrodynamical scaling makes the system evolve rapidly and it is in result expected that $S_t^\varepsilon(x)$ converges to one of the equilibrium states. However establishing it is not easy. We shall follow and extend the method due to Fritz [9, 10] in which the discretized version of the Ginzburg-Landau model was discussed.

The proof of Theorem 1.1 consists of three main parts. In the first part we shall introduce a significant class of probability measures $\{\mu_{\lambda(\cdot)}\}$ on $\mathscr{C}$ called spatially inhomogeneous Gibbs distributions and investigate their asymptotic behavior under the spatial scaling limit (Sects. 3 and 4). The SPDE (1.3) determines a semigroup T_t^ε with a formal infinitesimal generator $\mathscr{G}^\varepsilon$ which is a functional differential operator of second order. The second task after some preparations (Sects. 6 and 7) is proving a formula of integration by parts on the space $\mathscr{C}$ based on a probability measure $\mu_{\lambda(\cdot),\varepsilon}$, which is obtained by acting the spatial scaling transformation on $\mu_{\lambda(\cdot)}$ (Sect. 8). This is a key formula which expresses the time derivative $\frac{\partial}{\partial t} \int_{\mathscr{C}} T_t^\varepsilon \Psi \, d\mu_{\lambda(\cdot),\varepsilon}$ in terms of the functional derivative $DT_t^\varepsilon \Psi$ for a certain class of functions Ψ on $\mathscr{C}$. Finally we shall prove the compactness of semigroups $\{T_t^\varepsilon \Psi\}_{0<\varepsilon<1}$ and their functional derivatives $\{DT_t^\varepsilon \Psi\}_{0<\varepsilon<1}$ in a proper sense (Sects. 9 and 10). Taking the limit $\varepsilon \downarrow 0$ in the key formula leads us to the conclusion of Theorem 1.1 (Sect. 11).

The results were already announced in [13]. This article explains briefly how we can arrive at the PDE (1.4) starting from the principle of hydrodynamics and also exposes the outline of the proof of Theorem 1.1 in slightly more detail than stated above. The present paper is a result of shortening some tedious parts in the proof of my preprint [Fu] (IMA preprint series no. 328, University of Minnesota, 1987). These two papers [13] and [Fu], however, must help understanding the present paper.

2. Notations and Preliminary Facts

In this section we first introduce some notations which will be used throughout the paper and then summarize known properties and their simple consequences on the so-called Schrödinger operators on $\mathbb{R}$.

2.1 Notations

(i) Generally for a topological space X, $C_b(X)$ and $\mathscr{P}(X)$ stand for the space of all bounded continuous functions on X and the space of all Borel probability measures on X, respectively. We denote $\langle \mu, \Phi \rangle = \int_X \Phi \, d\mu$ for $\Phi \in C_b(X)$ and $\mu \in \mathscr{P}(X)$.

(ii) Configuration spaces and their dual spaces. We fix a positive even function $\chi \in C^\infty(\mathbb{R})$ satisfying $\chi(x)=|x|$ for $x; |x| \geqq 1$ and set $\theta(x, r)=e^{-r\chi(x)}$, $r \in \mathbb{R}$. Introduce a family of Hilbert spaces $\mathbf{H}_r = L^2(\mathbb{R}, \theta(x, r)\,dx)$, $r \in \mathbb{R}$, having norms defined by $|S|_r = \{\int_{\mathbb{R}} S(x)^2 \theta(x, r)\,dx\}^{1/2}$, $S \in \mathbf{H}_r$. The space $\mathbf{H}_{-r}$ can be identified with the dual space $\mathbf{H}_r^*$ of $\mathbf{H}_r$. Let $\mathbf{H}_e = \bigcap_{r>0} \mathbf{H}_r$ and $\mathbf{H}_e^* = \bigcup_{r>0} \mathbf{H}_r^*$ be a countably Hilbertian space and its dual, respectively. We shall also consider a weak topology $\sigma(\mathbf{H}_e, \mathbf{H}_e^*)$ on the space $\mathbf{H}_e$. With this topology it will be written by $\mathbf{H}_{e,w}$. We denote by $\mathscr{C}$ the space $C(\mathbb{R})$ with the usual topology of uniform-convergence on each bounded set. Let $\mathscr{B}_I$ be a σ-field of the space $\mathscr{C}$ generated by $\{S(x); x \in I\}$, $S \in \mathscr{C}$, for every subset I of $\mathbb{R}$. We simply write $\mathscr{B}$ for $\mathscr{B}_{\mathbb{R}}$ and sometimes use the same notation $\mathscr{B}_I$ to denote the Borel field of the space $C(I)$ for each interval I of $\mathbb{R}$. Let $\mathscr{C}_r$, $r \in \mathbb{R}$, be the space of all $S \in \mathscr{C}$ satisfying $|||S|||_r = \sup_{x \in \mathbb{R}} |S(x)|\,\theta(x, r) < \infty$ and set $\mathscr{C}_e = \bigcap_{r>0} \mathscr{C}_r$ the countably normed space. We also consider the space $\hat{\mathscr{C}}_{-r}$, $r>0$, of all $\varphi \in \mathscr{C}_{-r}$ satisfying

$$\lim_{|x| \to \infty} |\varphi(x)|\,\theta(x, -r)=0. \tag{2.1}$$

The space $\hat{\mathscr{C}}_{-r}$ is a Banach space with norm $|||\cdot|||_{-r}$.

(iii) The class of tame functions on the configuration space. Let $\mathscr{D}$ be the class of all functions Ψ on the space $\mathscr{C}$ having the form:

$$\Psi(S)=\psi(\langle S, \varphi_1\rangle, \ldots, \langle S, \varphi_k\rangle), \qquad S \in \mathscr{C}, \tag{2.2}$$

with $k=1, 2, \ldots$, $\psi=\psi(\alpha_1, \ldots, \alpha_k) \in C_0^\infty(\mathbb{R}^k)$ and $\varphi_1, \ldots, \varphi_k \in C_0^\infty(\mathbb{R})$, where $\langle S, \varphi\rangle = \int_{\mathbb{R}} S(x)\,\varphi(x)\,dx$. We also denote by $\mathscr{D}$ without distinction the class of such functions on $\mathscr{C}_e$, $\mathbf{H}_e$ or $\mathbf{H}_r$ etc. instead of $\mathscr{C}$ as introduced similarly as above. For $l>0$, $\mathscr{D}_l$ is the class of all $\Psi \in \mathscr{D}$ of the form (2.2) with $\varphi_1, \ldots, \varphi_k$ having supports in an open interval $(-l, l)$. We shall sometimes denote by $\mathscr{D}_l$ again the class of functions Ψ on the space $C([-l, l])$ or $L^2([-l, l])$, which are defined in a similar manner.

2.2 Schrödinger operator

Recall the definition of the operator H_λ together with its minimal eigenvalue $\kappa(\lambda)$ and eigenfunction Ω_λ. The condition (1.2) on U implies that H_λ has purely discrete spectrum and the function Ω_λ decays exponentially fast, i.e., $\Omega_\lambda(s) \leqq \text{const.}\, e^{-r|s|}$, $s \in \mathbb{R}$, for every $r>0$ (see Reed and Simon [20]).

We can show that $\kappa(\lambda)$ is real-analytic in $\lambda \in \mathbb{R}$ and Ω_λ is strongly differentiable as an $L^2(\mathbb{R}, ds)$-valued function of $\lambda \in \mathbb{R}$. These facts follow from Kato-Rellich theorem by proving that the family of operators $\{H_\lambda; \lambda \in \mathscr{O}\}$ on a complex Hilbert space $L^2(\mathbb{R} \to \mathbb{C}, ds)$ is an analytic family in the sense of Kato for some neighborhood $\mathcal{O}$ of $\mathbb{R}$ in the complex plane $\mathbb{C}$, see Reed and Simon [20, vol. IV, pp. 14–17]. Another consequence of this theorem is that $\inf_{\lambda\in I}\delta(\lambda)>0$ for every bounded interval I of $\mathbb{R}$, where $\delta(\lambda)$ is the gap between the second least eigenvalue of H_λ and $\kappa(\lambda)$.

Final remark is on the positivity of the diffusion coefficient $d(\rho)$ defined by (1.6). Indeed differentiating both sides of an equality $H_\lambda \Omega_\lambda=\kappa(\lambda)\Omega_\lambda$ in λ we obtain $(H_\lambda-\kappa(\lambda))\frac{\partial\Omega_\lambda}{\partial\lambda}=(s+\kappa'(\lambda))\Omega_\lambda$. This implies $\kappa'(\lambda)=-\bar{\rho}(\lambda)$ and therefore $\frac{\partial\Omega_\lambda}{\partial\lambda}=(H_\lambda-\kappa(\lambda))^{-1}\eta_\lambda$, since $\frac{\partial\Omega_\lambda}{\partial\lambda}\in L_0^2$; the space of all $\eta\in L^2(\mathbb{R},ds)$ such that $(\eta,\Omega_\lambda)\equiv(\eta,\Omega_\lambda)_{L^2}=0$. Here $\eta_\lambda=(s-\bar{\rho}(\lambda))\Omega_\lambda\in L_0^2$ and we consider $(H_\lambda-\kappa(\lambda))^{-1}$ as a positive operator of $L_0^2\to L_0^2$. Hence we see $d(\rho)>0$ from

$$\bar{\rho}'(\lambda)=2\left(s\Omega_\lambda,\frac{\partial\Omega_\lambda}{\partial\lambda}\right)=2(\eta_\lambda,(H_\lambda-\kappa(\lambda))^{-1}\eta_\lambda)>0.$$

3. Spatially Inhomogeneous Gibbs Distributions

3.1. Definition and Construction

Let $\mu_{x,s_1;y,s_2}$, $x<y$, $s_1,s_2\in\mathbb{R}$, be a probability distribution on the space $(C([x,y]),\mathscr{B}_{[x,y]})$ of the pinned Brownian motion $S=\{S(z);z\in[x,y]\}$ with time parameter z satisfying $S(x)=s_1$ and $S(y)=s_2$. To specify a family of profile functions describing the strength of the external field we consider a class Λ of all functions $\lambda\in C^2(\mathbb{R})$ satisfying $\lambda'\in C_0(\mathbb{R})$. For every $\lambda=\lambda(\cdot)\in\Lambda$ the *local specification* is a probability measure on $C([x,y])$ defined by

$$\mu_{\lambda(\cdot)}^{x,y}(dS;s_1,s_2)=\Xi^{-1}\exp\left\{-\int_x^y U(z,S(z);\lambda(\cdot))\,dz\right\}\mu_{x,s_1;y,s_2}(dS),$$

for each $x<y$ and $s_1,s_2\in\mathbb{R}$, where $U(z,s;\lambda(\cdot))=U(s)-\lambda(z)s$ and $\Xi=\Xi_{\lambda(\cdot)}^{x,y}(s_1,s_2)$ is a normalizing constant. We sometimes regard $\mu_{\lambda(\cdot)}^{x,y}(\cdot\,;s_1,s_2)$ as a probability measure on the space $(\mathscr{C},\mathscr{B})$ by considering $S(z)=s_1$ for $z\leq x$ and $S(z)=s_2$ for $z\geq y$ under this probability distribution. A probability measure μ on $(\mathscr{C},\mathscr{B})$ will be called a *$(U,\lambda(\cdot))$-Gibbs distribution* if and only if it satisfies the so-called DLR equation:

$$\mu(A\,|\,\mathscr{B}_{(x,y)^c})(S)=\mu_{\lambda(\cdot)}^{x,y}(A;S(x),S(y)),\qquad \mu\text{-a.e. }S,$$

for every $x<y$ and $A\in\mathscr{B}_{[x,y]}$.

The Gibbs distribution can be constructed in the following manner. For $\lambda(\cdot)\in\Lambda$ we can find $x_-<x_+$ such that $\lambda(\cdot)=$constant on two intervals $(-\infty,x_-]$

and $[x_+, \infty)$. Let two functions $\{\Omega^{(\pm)}(x, s)\}$ be the solutions of two diffusion equations:

$$\frac{\partial}{\partial x}\Omega^{(\pm)}(x, s) = \pm \bar{H}_x \Omega^{(\pm)}(x, s), \qquad x \in \mathbb{R}, \tag{3.1}$$

with initial (or terminal) conditions $\Omega^{(\pm)}(x_\pm, s) = \Omega(x_\pm, s)$, where $\bar{H}_x = H_{\lambda(x)} - \kappa(\lambda(x))$ and $\Omega(x, s) = \Omega_{\lambda(x)}(s)$. The double signs $\pm$ are taken in same order. The Eqs. (3.1) are used to see

$$Z(x) = \int_{\mathbb{R}} \Omega^{(+)}(x, s)\,\Omega^{(-)}(x, s)\,ds \tag{3.2}$$

is independent of x, so that it will be denoted simply by Z. We define $\mu_{\lambda(\cdot)}(A)$ for $A \in \mathscr{B}_{[x,y]}$, $x < y$, by

$$\mu_{\lambda(\cdot)}(A) = Z^{-1} \int_{\mathbb{R}^2} ds_1\, ds_2\, \Omega^{(-)}(x, s_1)\,\Omega^{(+)}(y, s_2)\, p(y - x, s_1, s_2)$$

$$\cdot E^{\mu_{x,s_1;y,s_2}}\left[\exp\left(-\int_x^y \{U(z, S(z); \lambda(\cdot)) + \kappa(\lambda(z))\}\, dz\right); A\right],$$

where $p(z, s_1, s_2)$ is the transition probability of standard Brownian motion. Then the Feynman-Kac formula can be applied to prove that $\mu_{\lambda(\cdot)}$ is well-defined as probability measure on $(\mathscr{C}, \mathscr{B})$, namely $\mu_{\lambda(\cdot)}(A)$ is determined independently of the choice of x and y. Moreover, it is not difficult to see $\mu_{\lambda(\cdot)}$ constructed as above is a $(U, \lambda(\cdot))$-Gibbs distribution; cf. Iwata [15] discussed the case of $\lambda(\cdot)$ being constant. This is a spatially-inhomogeneous extension of the $P(\phi)_1$-measure, see Simon [23]. Under the distribution $\mu_{\lambda(\cdot)}$, $\{S(x); x \in \mathbb{R}\}$ can be regarded as a temporally-inhomogeneous diffusion process with 1-dimensional marginal distribution $Z^{-1}\Omega^{(+)}(x, s)\,\Omega^{(-)}(x, s)\,ds$ and infinitesimal generator $\frac{1}{2} d^2/ds^2 + \left\{\frac{\partial}{\partial s} \log \Omega^{(+)}(x, s)\right\} d/ds$, $x \in \mathbb{R}$. Although generally there exist another μ's satisfying the DLR equation, in the following we mean by the $(U, \lambda(\cdot))$-Gibbs distribution the probability measure $\mu_{\lambda(\cdot)}$ which has been constructed in this manner. We introduce spatial scalings σ_ε and τ_ε by $(\sigma_\varepsilon S)(x) = S(x/\varepsilon)$ and $(\tau_\varepsilon \varphi)(x) = \Phi(\varepsilon x)$, respectively. Define a *scaled* $(U, \lambda(\cdot))$*-Gibbs distribution by* $\mu_{\lambda(\cdot),\varepsilon} = \mu_{\tau_\varepsilon \lambda(\cdot)} \circ \sigma_\varepsilon^{-1}$, $0 < \varepsilon < 1$.

3.2 FKG inequality

Here we prove the monotonicity property of $\mu_{\lambda(\cdot)}$ with respect to $\lambda(\cdot)$ and derive some uniform moment-estimates on $\{\mu_{\lambda(\cdot),\varepsilon}\}$ as its consequence.

On the space $C(I)$ with interval I of $\mathbb{R}$ a usual partial order $S \leqq \bar{S}$ is defined by the relation: $S(x) \leqq \bar{S}(x)$ for every $x \in I$. Generally for a Polish space X equipped with a partial order, let $\mathscr{M}(X)$ be the class of all monotone-increasing $\Phi \in C_b(X)$. We say $\mu_1 \leqq \mu_2$ for $\mu_1, \mu_2 \in \mathscr{P}(X)$ if $\langle \mu_1, \Phi \rangle \leqq \langle \mu_2, \Phi \rangle$ holds for every

$\Phi \in \mathcal{M}(X)$. We also say that $\mu \in \mathcal{P}(X)$ satisfies an FKG inequality on the space X if $\langle \mu, \Phi\Psi \rangle \geqq \langle \mu, \Phi \rangle \langle \mu, \Psi \rangle$ holds for every $\Phi, \Psi \in \mathcal{M}(X)$.

For $v_i \in \mathcal{P}(\mathbb{R})$, $i=1, 2$, satisfying $\langle v_i, \theta(\cdot, -r) \rangle < \infty$ for every $r>0$ we define $\mu_{\lambda(\cdot)}^{x,y}(\cdot\,; v_1, v_2) \in \mathcal{P}(C([x, y]))$ by

$$\mu_{\lambda(\cdot)}^{x,y}(\cdot\,; v_1, v_2) = \int_{\mathbb{R}^2} \mu_{\lambda(\cdot)}^{x,y}(\cdot\,; s_1, s_2)\, v_1(ds_1)\, v_2(ds_2).$$

Lemma 3.1. *The probability measures* $\mu_{\lambda(\cdot)}^{x,y}(\cdot\,; s_1, s_2)$, $\mu_{\lambda(\cdot)}^{x,y}(\cdot\,; v_1, v_2)$ *and* $\mu_{\lambda(\cdot)}$ *each satisfy the FKG inequality on the spaces on which these measures are defined.*

Proof. The conclusion can be shown by usual method. We first prove the FKG condition for the measure on a finite-dimensional space obtained by discretizing the interval $[x, y]$ and then take the limit. We omit the detail (see Simon [22, 23], Iwata [15]). □

We recall the definition of $v_\lambda \in \mathcal{P}(\mathbb{R})$: $v_\lambda(ds) = \Omega_\lambda^2(s)\, ds$, $\lambda \in \mathbb{R}$.

Proposition 3.1. (i) For local specifications, we have $\mu_{\lambda_1(\cdot)}^{x,y}(\cdot\,; s_1, s_2) \geqq \mu_{\lambda_2(\cdot)}^{x,y}(\cdot\,; \bar{s}_1, \bar{s}_2)$ if $s_1 \geqq \bar{s}_1$, $s_2 \geqq \bar{s}_2$ and if $\lambda_1(\cdot), \lambda_2(\cdot) \in \Lambda$ satisfy $\lambda_1(\cdot) \geqq \lambda_2(\cdot)$ in the space $\mathcal{C}$.

(ii) For Gibbs distributions, we have $\mu_{\lambda_1(\cdot)} \geqq \mu_{\lambda_2(\cdot)}$ if $\lambda_1(\cdot)$, $\lambda_2(\cdot) \in \Lambda$ satisfy $\lambda_1(\cdot) \geqq \lambda_2(\cdot)$ in the space $\mathcal{C}$. Especially $\lambda_1 \geqq \lambda_2$ implies an inequality $v_{\lambda_1} \geqq v_{\lambda_2}$ in the space $\mathcal{P}(\mathbb{R})$.

(iii) $\sup_{x \in \mathbb{R},\, 0<\varepsilon<1} \langle \mu_{\lambda(\cdot),\varepsilon}, |S(x)|^p \rangle < \infty, \quad p \geqq 1, \quad \lambda(\cdot) \in \Lambda.$

Proof. The first assertion of (ii) is an easy consequence of the FKG inequality for $\mu_{\lambda_2(\cdot)}$ with the help of the fact:

$$\langle \mu_{\lambda_1(\cdot)}, \Phi \rangle = \lim_{l \to \infty} \langle \mu_{\lambda_2(\cdot)}, \Psi^l \rangle^{-1} \langle \mu_{\lambda_2(\cdot)}, \Phi \Psi^l \rangle$$

which holds for every $\mathcal{B}_{(x,y)}$-measurable $\Phi \in C_b(\mathcal{C})$, $x<y$, where $\Psi^l(S) = \exp\left\{ \int_{-l}^{l} (\lambda_1(z) - \lambda_2(z))\, S(z)\, dz \right\}$ is an increasing function on $C([x, y])$ (see Theorem 6.9 in Simon [23]). Similarly the monotonicity of $\mu_{\lambda(\cdot)}^{x,y}(\cdot\,; s_1, s_2)$ in $\lambda(\cdot)$ can be proved. The second assertion of (ii) is shown immediately since v_λ is the 1-dimensional marginal distribution of $\mu_{\lambda(\cdot)}$ with $\lambda(\cdot) \equiv \lambda$. The assertion (iii) follows from the decay properties of $\Omega_{\lambda_\pm}$ by noting that (ii) shows $\mu_{\lambda_-} \leqq \mu_{\lambda(\cdot)} \leqq \mu_{\lambda_+}$, where $\lambda_- = \inf_x \lambda(x)$ and $\lambda_+ = \sup_x \lambda(x)$. Finally we prove the monotonicity of $\mu_{\lambda(\cdot)}^{x,y}(\cdot\,; s_1, s_2)$ in (s_1, s_2). Let $v_{s,v}$ be the Gaussian distribution on $\mathbb{R}$ with mean s and variance $v>0$. Then, since the function $\Psi(S) = \prod_{i=1}^{2} (dv_{s_i,v}/dv_{\bar{s}_i,v})(S(x_i))$, $x_1 = x$, $x_2 = y$, is increasing on the space $C([x, y])$, the FKG inequality proves $\langle \mu_{\lambda(\cdot)}^{x,y}(\cdot\,; v_{s_1,v}, v_{s_2,v}), \Phi \rangle \geqq \langle \mu_{\lambda(\cdot)}^{x,y}(\cdot\,; v_{\bar{s}_1,v}, v_{\bar{s}_2,v}), \Phi \rangle$, $\Phi \in \mathcal{M}(C([x, y]))$ if $s_1 \geqq \bar{s}_1$ and $s_2 \geqq \bar{s}_2$. Therefore the conclusion follows by taking the limit $v \downarrow 0$. □

Remark 3.1. (i) Iwata [15] proved that $\mu_{\lambda(\cdot)}(\mathscr{C}_e)=1$ when $\lambda(\cdot)=$constant. Therefore Proposition 3.1 (ii) shows that $\mu_{\lambda(\cdot)}(\mathscr{C}_e)=1$ for general $\lambda(\cdot)\in\Lambda$.

(ii) Since the inclusion map of $\mathscr{C}_e$ into $\mathbf{H}_e$ or $\mathbf{H}_{e,w}$ is continuous, we can regard $\mu_{\lambda(\cdot)}\in\mathscr{P}(\mathbf{H}_e)$ or $\in\mathscr{P}(\mathbf{H}_{e,w})$ by identifying it with its image measure under the inclusion map.

3.3 *Thermodynamical limit*

The $(U,\lambda(\cdot))$-Gibbs distribution $\mu_{\lambda(\cdot)}$ can be also obtained by taking the thermodynamical limit.

Proposition 3.2. *The probability measure* $\mu_{\lambda(\cdot)}^{-l,l}(\cdot\,;0,0)$ *converges weakly to* $\mu_{\lambda(\cdot)}$ *as* $l\to\infty$ *on the space* $\mathscr{C}_r$ *for every* $r>0$.

Proof. Assume $0<l'<l$ and $-l'<x_-<x_+<l'$. Then we have a representation for $\mu^l\equiv\mu_{\lambda(\cdot)}^{-l,l}(\cdot\,;0,0)$:

$$\begin{aligned}&\mu^l(S(-l')\in ds_1,\,S(l')\in ds_2)\\&\quad=Z_l^{-1}f_{l'}(s_1,s_2)\,e^{-(l-l')\bar{H}_-}(0,s_1)\,e^{-(l-l')\bar{H}_+}(0,s_2)\,ds_1\,ds_2,\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}f_{l'}(s_1,s_2)&=e^{l'\{\kappa(\lambda_-)+\kappa(\lambda_+)\}}\,\Xi_{\lambda(\cdot)}^{-l',l'}(s_1,s_2)\,p(2l',s_1,s_2)\\Z_l&=f_l(0,0),\ \lambda_\pm=\lambda(x_\pm),\ \bar{H}_\pm=\bar{H}_{x_\pm},\end{aligned}$$

and $e^{-l\bar{H}_\pm}(s,s')$ are the integral kernel functions for the semigroup operators $e^{-l\bar{H}_\pm}$ on $L^2(\mathbb{R},ds)$. Since it is easy to see that $e^{-l\bar{H}_\pm}(0,\cdot)$ converge to $c_\pm\,\Omega(x_\pm,\cdot)$ in the space $L^2(\mathbb{R},ds)$ as $l\to\infty$ with some positive constants $c_\pm$, we can prove using (3.3) and the Markov property of the local specifications that $\inf_{l\geqq1}Z_l>0$ and $\lim_{l\to\infty}\mu^l(A)=\mu_{\lambda(\cdot)}(A)$ for every $A\in\mathscr{B}_{(-l',l')}$, $l'>0$. To conclude the proof it is now enough to show the tightness of $\{\mu^l\}_{l\geqq1}$ on the space $\mathscr{C}_r$ for every $r>0$. To this end, we show (i) the tightness of $\{\mu^l\}_{l\geqq1}$ on the space $\mathscr{C}$ and (ii) an estimate: $\sup_{l\geqq1}\langle\mu^l,|||S|||_r\rangle<\infty$ for every $r>0$. Indeed (ii) is proved by noting that Proposition 3.1 (i) implies

$$\langle\mu^l,\Phi\rangle\cdot\mu_{\lambda_-}(S(-l)>0,\,S(l)>0)\leqq\langle\mu_{\lambda_+},\Phi\rangle$$

for every non-negative and $\mathscr{B}_{(-l,l)}$-measurable function $\Phi\in\mathscr{M}(\mathscr{C})$ and similarly

$$\langle\mu^l,\Phi'\rangle\cdot\mu_{\lambda_+}(S(-l)<0,\,S(l)<0)\leqq\langle\mu_{\lambda_-},\Phi'\rangle$$

for every non-negative and $\mathscr{B}_{(-l,l)}$-measurable Φ' such that $-\Phi'\in\mathscr{M}(\mathscr{C})$. We note $\inf_{l\geqq1}\mu_{\lambda_\pm}(\pm S(-l)<0,\,\pm S(l)<0)>0$ (see Iwata [16]) and $\langle\mu_{\lambda_\pm},|||S|||_r\rangle<\infty$,

$r>0$, which follows easily from the stationarity of $\mu_{\lambda_\pm}$. On the other hand, the assertion (i) follows if we can prove for every sufficiently large l'

$$\langle \mu^l, |S(x)-S(y)|^4\rangle \leqq \text{const.}\,|x-y|^2, \; -l'<x<y<l', \tag{3.4}$$

with some positive const. independent of l; $l>l'$. However simple calculation shows

$$\begin{aligned}&\langle \Xi_{\lambda(\cdot)}^{-l',l'}(s_1,s_2)\,\mu_{\lambda(\cdot)}^{-l',l'}(\cdot\,;s_1,s_2), |S(x)-S(y)|^4\rangle\\ &\quad\leqq \text{const.}\,\{1+|s_1-s_2|^4\}\,|x-y|^2, \qquad -l'<x<y<l'.\end{aligned}$$

Therefore we obtain (3.4) by using the representation (3.3) with the help of Markov property of μ^l and then noting $\inf_l Z_l>0$ and the asymptotic behavior as $l\to\infty$ of the kernel functions $e^{-l\bar{H}_\pm}(0,\cdot)$ mentioned above. □

4. Law of Large Numbers for Gibbs Distributions

The purpose of this section is to investigate the asymptotic behavior as $\varepsilon\downarrow 0$ of the scaled $(U, \lambda(\cdot))$-Gibbs distributions $\mu_{\lambda(\cdot),\varepsilon}$ constructed in Sect. 3. We define a function $\rho\in H_e$ by $\rho(x)=\bar{\rho}(\lambda(x))$, $x\in\mathbb{R}$; see (1.5) for the definition of the mean spin function $\bar{\rho}$. We shall prove the following theorem.

Theorem 4.1. *The probability measure $\mu_{\lambda(\cdot),\varepsilon}$ converges weakly to the δ-distribution δ_ρ on the space $\mathbf{H}_{e,w}$ as $\varepsilon\downarrow 0$, namely $\lim_{\varepsilon\downarrow 0}\langle\mu_{\lambda(\cdot),\varepsilon}, \Psi\rangle=\Psi(\rho)$ for every $\Psi\in C_b(\mathbf{H}_{e,w})$.*

The proof of the theorem will be divided into three parts.

4.1 Convergence of 1-dimensional Distribution

Let $\Omega_\varepsilon^{(\pm)}(x)$ be two solutions of the diffusion Eq. (3.1) with $\bar{H}_x$ replaced by $\frac{1}{\varepsilon}\bar{H}_x$ having the same initial (or terminal) conditions. These functions play same roles for $\mu_{\lambda(\cdot),\varepsilon}$ as $\Omega^{(\pm)}(x)$ do for $\mu_{\lambda(\cdot)}$. For example, the distribution of $S(x)$ under $\mu_{\lambda(\cdot),\varepsilon}$ is given by $Z_\varepsilon^{-1}\,\Omega_\varepsilon^{(+)}(x,s)\,\Omega_\varepsilon^{(-)}(x,s)\,ds$, where Z_ε is a constant defined by the right hand side (RHS) of (3.2) with $\Omega^{(\pm)}$ replaced by $\Omega_\varepsilon^{(\pm)}$.

First we analyze the asymptotic behavior of the solution $\eta(y,s)=\eta_\varepsilon(y,s)$, $0<\varepsilon<1$, of the diffusion equation:

$$\begin{aligned}&\frac{\partial\eta}{\partial y}(y,s)=-\frac{1}{\varepsilon}\bar{H}_y\,\eta(y,s), \qquad y>x,\\ &\eta(x,s)=\bar{\eta}(s),\end{aligned} \tag{4.1}$$

with a given initial value $\bar{\eta}\in L^2(\mathbb{R}, ds)$. It is easy to know the existence and uniqueness of solutions of (4.1) in the space $L^2(\mathbb{R}, ds)$. Put $c_\varepsilon(y)\equiv c_\varepsilon(y; x, \bar{\eta}) =(\eta_\varepsilon(y), \Omega(y))$, where $\Omega(y)=\Omega_{\lambda(y)}$ (see Sect. 3). We shall denote the norm and inner product of the space $L^2(\mathbb{R}, ds)$ simply by $\|\cdot\|$ and $(\,,\,)$, respectively, in this section.

Lemma 4.1. (i) *We have two estimates:*

$$|c_\varepsilon(y)| \leqq \|\bar{\eta}\|, \qquad \left|\frac{\partial}{\partial y} c_\varepsilon(y)\right| \leqq \|\bar{\eta}\| \left\|\frac{\partial \Omega}{\partial y}(y)\right\|, \qquad y \geqq x, \varepsilon > 0.$$

(ii) $\lim_{\varepsilon \downarrow 0} \|\eta_\varepsilon(y) - c_\varepsilon(y)\,\Omega(y)\| = 0$, $y > x$.

Proof. The first estimate in (i) follows from the bound $\|\eta_\varepsilon(y)\| \leqq \|\bar{\eta}\|$, $y \geqq x$, which is a consequence of non-negativity of the operator $\bar{H}_y$. The second estimate can be shown since $\bar{H}_y\,\Omega(y) = 0$ implies

$$\left|\frac{\partial}{\partial y} c_\varepsilon(y)\right| = \left|\left(\eta_\varepsilon(y), \frac{\partial \Omega}{\partial y}(y)\right)\right| \leqq \|\bar{\eta}\| \left\|\frac{\partial \Omega}{\partial y}(y)\right\|.$$

Note the differentiability of $\Omega(y)$ in y; see Sect. 2. To prove the assertion (ii), we derive an equality

$$\frac{\partial}{\partial y} \|\zeta_\varepsilon(y)\|^2 = -\frac{2}{\varepsilon}(\bar{H}_y \zeta_\varepsilon, \zeta_\varepsilon) - 2\left(\frac{\partial}{\partial y}\{c_\varepsilon \Omega\}, \zeta_\varepsilon\right), \tag{4.2}$$

for $\zeta_\varepsilon(y) = \eta_\varepsilon(y) - c_\varepsilon(y)\,\Omega(y)$. Since $(\zeta_\varepsilon(y), \Omega(y)) = 0$, the first term in the RHS of (4.2) can be bounded from above by $-2\delta\|\zeta_\varepsilon\|^2/\varepsilon$, where $\delta = \inf_{x \in [x_-, x_+]} \delta_x > 0$ and δ_x is the second least eigenvalue of $\bar{H}_x$; see Sect. 2. On the other hand, since (i) implies $K = \sup_{y \geqq x, \varepsilon > 0} \left\|\frac{\partial}{\partial y}\{c_\varepsilon \Omega\}\right\| < \infty$, the second term is bounded from above by $2K\|\zeta_\varepsilon\|$ and therefore by $K\{1 + \|\zeta_\varepsilon\|^2\}$. These estimates are now summed up into

$$\frac{\partial}{\partial y}\|\zeta_\varepsilon(y)\|^2 \leqq -\left(\frac{2\delta}{\varepsilon} - K\right)\|\zeta_\varepsilon(y)\|^2 + K,$$

from which one can complete the proof of (ii). □

Put $\bar{c}_\varepsilon^{(\pm)}(x) = (\Omega_\varepsilon^{(\pm)}(x), \Omega(x))$. Then the following is an immediate consequence of Lemma 4.1. Consider in the reverse direction for $\Omega_\varepsilon^{(+)}$.

Corollary 4.1. (i) *We have for every* $x \in \mathbb{R}$ *and* $\varepsilon > 0$:

$$|\bar{c}_\varepsilon^{(\pm)}(x)| \leqq 1, \qquad \left|\frac{\partial}{\partial x}\bar{c}_\varepsilon^{(\pm)}(x)\right| \leqq \left\|\frac{\partial \Omega}{\partial x}(x)\right\|.$$

(ii) $\lim_{\varepsilon \downarrow 0} \|\Omega_\varepsilon^{(\pm)}(x) - \bar{c}_\varepsilon^{(\pm)}(x)\,\Omega(x)\| = 0$, $x \in \mathbb{R}$.

Now we introduce the following additional condition on the profile $\lambda(\cdot)\in\Lambda$:

$$\beta=(x_+-x_-)\sup_{x\in\mathbb{R}}\left\|\frac{\partial\Omega}{\partial x}(x)\right\|<1. \tag{4.3}$$

Corollary 4.2. *Assume the condition* (4.3). *Then we have*

$$\inf_{0<\varepsilon<1} Z_\varepsilon>0 \tag{4.4}$$

and

$$\lim_{\varepsilon\downarrow 0}\|Z_\varepsilon^{-1}\Omega_\varepsilon^{(+)}(x)\Omega_\varepsilon^{(-)}(x)-\Omega^2(x)\|_{L^1(\mathbb{R},ds)}=0, \qquad x\in\mathbb{R}. \tag{4.5}$$

Proof. Since $\bar{c}_\varepsilon^{(\pm)}(x_\pm)=1$, Corollary 4.1 (i) combined with the condition (4.3) proves that $\bar{c}_\varepsilon^{(\pm)}(x)\geqq 1-\beta$, $\varepsilon>0$, $x\in[x_-,x_+]$. We therefore see (4.4) with the help of Corollary 4.1 (ii). Now the assertion (4.5) follows also from Corollary 4.1 (ii) noting (4.4). □

4.2 Asymptotic Independence

Recall the definition of $\nu_\lambda\in\mathscr{P}(\mathbb{R})$.

Lemma 4.2. *Under the condition* (4.3) *we have*

$$\lim_{\varepsilon\downarrow 0}|\langle\mu_{\lambda(\cdot),\varepsilon},\xi(S(x))\Phi\rangle-\langle\nu_{\lambda(x)},\xi\rangle\langle\mu_{\lambda(\cdot),\varepsilon},\Phi\rangle|=0,$$

for every $\xi\in C_b(\mathbb{R})$ and $\mathscr{B}_{(y,z)}$-measurable bounded function Φ on the space $\mathscr{C}$, but we assume (i) *$x<y<z$ and $y\leqq x_+$ or* (ii) *$y<z<x$ and $x_-\leqq z$.*

Proof. We may only discuss the case (i) because of the symmetry. Let $\tilde{\eta}_\varepsilon=\tilde{\eta}_\varepsilon(y,s;x,\xi)$ be a solution of the Eq. (4.1) with initial value $\bar{\eta}(s)=\xi(s)\Omega_\varepsilon^{(-)}(x,s)$ for given $\xi\in C_b(\mathbb{R})$. Then we have the following representation for the conditional distribution:

$$E^{\mu_{\lambda(\cdot),\varepsilon}}[\xi(S(x))\,|\,\mathscr{B}_{\{y\}}](S(y)=s)=\Omega_\varepsilon^{(-)}(y,s)^{-1}\tilde{\eta}_\varepsilon(y,s;x,\xi),\ x<y,$$

and therefore

$$\begin{aligned}&\langle\mu_{\lambda(\cdot),\varepsilon},\xi(S(x))\Phi\rangle\\ &\quad=Z_\varepsilon^{-1}\int\tilde{\eta}_\varepsilon(y,s)\Omega_\varepsilon^{(+)}(y,s)E^{\mu_{\lambda(\cdot),\varepsilon}}[\Phi\,|\,\mathscr{B}_{\{y\}}](S(y)=s)\,ds.\end{aligned}$$

Hence, putting $\tilde{c}_\varepsilon(y)\equiv\tilde{c}_\varepsilon(y;x,\xi)=(\tilde{\eta}_\varepsilon(y),\Omega(y))$, we obtain

$$\begin{aligned}I_\varepsilon&\equiv|\langle\mu_{\lambda(\cdot),\varepsilon},\xi(S(x))\Phi\rangle-\{\bar{c}_\varepsilon^{(-)}(y)\}^{-1}\tilde{c}_\varepsilon(y)\langle\mu_{\lambda(\cdot),\varepsilon},\Phi\rangle|\\ &\leqq Z_\varepsilon^{-1}\|\Omega_\varepsilon^{(+)}(y)\|\,\|\Phi\|_\infty I'_\varepsilon.\end{aligned}$$

The term I'_ε is defined and estimated as follows:

$$\begin{aligned}I'_\varepsilon &\equiv \|\tilde{\eta}_\varepsilon(y)-\{\bar{c}^{(-)}_\varepsilon(y)\}^{-1}\tilde{c}_\varepsilon(y)\,\Omega^{(-)}_\varepsilon(y)\| \\ &\leqq \|\tilde{\eta}_\varepsilon(y)-\tilde{c}_\varepsilon(y)\,\Omega(y)\| + \{\bar{c}^{(-)}_\varepsilon(y)\}^{-1}\tilde{c}_\varepsilon(y)\,\|\Omega^{(-)}_\varepsilon(y)-\bar{c}^{(-)}_\varepsilon(y)\,\Omega(y)\|.\end{aligned}$$

Here Lemma 4.1 (ii) shows that the first term in the RHS converges to 0 as $\varepsilon\downarrow 0$. On the other hand, since $\bar{c}^{(-)}_\varepsilon(y)\geqq 1-\beta$, $y\leqq x_+$, and $|\tilde{c}_\varepsilon(y)|\leqq\|\xi\|_\infty$, Corollary 4.1 (ii) proves that the second term also converges to 0. Therefore using (4.4) we get $\lim_{\varepsilon\downarrow 0} I_\varepsilon=0$. Especially this is true for $\Phi\equiv 1$. Now these calculations combined with (4.5) lead us to the conclusion. □

Use an inductive method to prove the following.

Corollary 4.3. *Assume the condition* (4.3). *Then we have*

$$\lim_{\varepsilon\downarrow 0}\left\langle \mu_{\lambda(\cdot),\varepsilon}, \prod_{i=1}^{n} \xi_i(S(x_i))\right\rangle = \prod_{i=1}^{n} \langle \nu_{\lambda(x_i)}, \xi_i\rangle \tag{4.6}$$

for every $x_1<x_2<\ldots<x_n$ *and* $\xi_i\in C_b(\mathbb{R})$, $1\leqq i\leqq n$, $n=1,2,\ldots$.

Let $\mu^{x,y}_{\lambda(\cdot),\varepsilon}(\cdot\,;s_1,s_2)\equiv\mu^{x/\varepsilon,y/\varepsilon}_{\tau_\varepsilon\lambda(\cdot)}(\cdot\,;s_1,s_2)\circ\sigma_\varepsilon^{-1}$ be the local specifications corresponding to the scaled Gibbs distribution $\mu_{\lambda(\cdot),\varepsilon}$.

Lemma 4.3. *Let* $\lambda(\cdot)\in\Lambda$ *be given and assume that we can find another profile* $\tilde{\lambda}(\cdot)\in\Lambda$ *which satisfies* (4.3) *and coincides with* $\lambda(\cdot)$ *on an interval* $[x,y]$. *Then the convergence* (4.6) *holds with* $\mu^{x,y}_{\tilde{\lambda}(\cdot),\varepsilon}(\cdot\,;s_1,s_2)$ *instead of* $\mu_{\lambda(\cdot),\varepsilon}$ *for every* $s_1,s_2\in\mathbb{R}$, $x<x_1<\ldots<x_n<y$ *and* $\xi_i\in C_b(\mathbb{R})$. *This convergence is uniform in* (s_1,s_2) *on each bounded subset of* $\mathbb{R}^2$.

Proof. To complete the proof we may assume that the functions $\xi_i\in C_b(\mathbb{R})$, $1\leqq i\leqq n$, are non-negative and monotone increasing. Then Proposition 3.1 (i) proves that

$$f_\varepsilon(s_1,s_2):=\langle\mu^{x,y}_{\tilde{\lambda}(\cdot),\varepsilon}(\cdot\,;s_1,s_2), \prod_{i=1}^{n}\xi_i(S(x_i))\rangle$$

is monotone in $(s_1,s_2)\in\mathbb{R}^2$. Put $c=\prod_{i=1}^{n}\langle\nu_{\lambda(x_i)},\xi_i\rangle$ and denote the 2-dimensional marginal distribution of $(S(x),S(y))$ under $\mu_{\tilde{\lambda}(\cdot),\varepsilon}$ by $\nu^{(2)}_{\tilde{\lambda}(\cdot),\varepsilon}\in\mathscr{P}(\mathbb{R}^2)$. Then Corollary 4.3 shows that the measures $f_\varepsilon(s_1,s_2)\,\nu^{(2)}_{\tilde{\lambda}(\cdot),\varepsilon}(ds_1\,ds_2)$ and $\nu^{(2)}_{\tilde{\lambda}(\cdot),\varepsilon}(ds_1\,ds_2)$ on $\mathbb{R}^2$ converge weakly to $c\,\nu_{\lambda(x)}\otimes\nu_{\lambda(y)}$ and $\nu_{\lambda(x)}\otimes\nu_{\lambda(y)}$, respectively, as $\varepsilon\downarrow 0$. Since the limiting measure $\nu_{\lambda(x)}\otimes\nu_{\lambda(y)}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$, every interval I in $\mathbb{R}^2$ is its continuity set (see, e.g., Billingsley [3]). Therefore $\int_I f_\varepsilon\,d\nu^{(2)}_{\tilde{\lambda}(\cdot),\varepsilon}$ and $\nu^{(2)}_{\tilde{\lambda}(\cdot),\varepsilon}(I)$ converge to $c\,\nu_{\lambda(x)}\otimes\nu_{\lambda(y)}(I)$

and $\nu_{\lambda(x)} \otimes \nu_{\lambda(y)}(I)$, respectively, as $\varepsilon \downarrow 0$. However, noting the monotonicity of the function f_ε, this proves $\lim_{\varepsilon \downarrow 0} f_\varepsilon(s_1, s_2) = c$ for every $(s_1, s_2) \in \mathbb{R}^2$. The uniformity of the convergence follows from Dini's theorem. □

Lemma 4.4. *The conclusion of Corollary* 4.3 *still holds without assuming the condition* (4.3).

Proof. For given $x_1 < \ldots < x_n$ we can find a sequence $y_0 < y_1 < \ldots < y_m$ in such a manner that $y_0 < x_1 < x_n < y_m$, $x_i \neq y_k$ for every i, k and there exists a profile $\tilde{\lambda}_k(\cdot) \in \Lambda$ for each k; $1 \leqq k \leqq m$, which satisfies (4.3) and coincides with $\lambda(\cdot)$ on the interval $[y_{k-1}, y_k]$. Then Lemma 4.3 proves that

$$f_\varepsilon(s_0, \ldots, s_m) := \prod_{k=1}^{m} \langle \mu_{\tilde{\lambda}(\cdot), \varepsilon}^{y_{k-1}, y_k}(\cdot\,; s_{k-1}, s_k), \prod_{i;\, x_i \in [y_{k-1}, y_k]} \xi_i(S(x_i)) \rangle$$

converges to the RHS of (4.6) as $\varepsilon \downarrow 0$ uniformly on each compact set of $\mathbb{R}^{m+1}$. Since Proposition 3.1 (iii) proves that the family of marginal distributions

$$\{\nu_{\lambda(\cdot), \varepsilon}^{(m)}(ds_0 \ldots ds_m) = \mu_{\lambda(\cdot), \varepsilon}(S(y_0) \in ds_0, \ldots, S(y_m) \in ds_m); 0 < \varepsilon < 1\}$$

is tight in $\mathscr{P}(\mathbb{R}^{m+1})$, the equality $\langle \mu_{\lambda(\cdot), \varepsilon}, \prod_{i=1}^{n} \xi_i(S(x_i)) \rangle = \langle \nu_{\lambda(\cdot), \varepsilon}^{(m)}, f_\varepsilon \rangle$ completes the proof of the lemma. □

4.3 The proof of Theorem 4.1

Lemma 4.5. *For every* $\Psi \in \mathscr{D}$, $\langle \mu_{\lambda(\cdot), \varepsilon}, \Psi \rangle$ *converges to* $\Psi(\rho)$ *as* $\varepsilon \downarrow 0$.

Proof. First we consider the case where $\Psi \in \mathscr{D}$ has the form $\Psi(S) = \psi(\langle S, \varphi_1 \rangle, \ldots, \langle S, \varphi_n \rangle)$ with $\varphi_i \in C_0^\infty(\mathbb{R})$ and $\psi(\alpha_1, \ldots, \alpha_n) = \prod_{i=1}^{n} \alpha_i^{m_i}$, $m_i \in \mathbb{N}$. We may assume $m_i = 1$, $1 \leqq i \leqq n$, by making n large if necessary. For Ψ of this form, we have

$$\langle \mu_{\lambda(\cdot), \varepsilon}, \Psi \rangle = \int_{\mathbb{R}^n} \prod_{i=1}^{n} \varphi_i(x_i) \left\langle \mu_{\lambda(\cdot), \varepsilon}, \prod_{i=1}^{n} S(x_i) \right\rangle \prod_{i=1}^{n} dx_i. \tag{4.7}$$

Here we notice that Proposition 3.1 (iii) guarantees taking $\xi_i(s) = s$, $1 \leqq i \leqq n$, in (4.6). The RHS of (4.6) becomes $\prod_{i=1}^{n} \rho(x_i)$ in this case. Therefore, Lebesgue's dominated convergence theorem proves that $\langle \mu_{\lambda(\cdot), \varepsilon}, \Psi \rangle$ tends to $\Psi(\rho)$ as $\varepsilon \downarrow 0$, since Proposition 3.1 (iii) again implies that the integrand of the RHS of (4.7) is uniformly bounded. It is now standard to show this convergence for every $\Psi \in \mathscr{D}$. □

Now we are ready to give the proof of Theorem 4.1. First we note that the locally convex space $\mathbf{H}_{e,w}$ is completely regular and a Radon space (Schwartz

[21]). Its balls $B(\{b_r\})=\{S\in\mathbf{H}_e;\ |S|_r\leq b_r$ for every $r>0\}$ are compact in this space for all sequences $\{b_r>0\}_{r>0}$ (Dunford and Schwartz [6, p. 423]). Conversely, every compact subset B of $\mathbf{H}_{e,w}$ is a closed set which is contained in some ball. Moreover, each ball is metrizable. From these observations, we see that Prokhorov's theorem still holds on the space $\mathbf{H}_{e,w}$ (Smolyanov and Fomin [24]). Proposition 3.1(iii) proves the tightness of the family $\{\mu_{\lambda(\cdot),\varepsilon}\}_{0<\varepsilon<1}$ in $\mathscr{P}(\mathbf{H}_{e,w})$; for every $\delta>0$, there exists a ball $B=B(\{b_r\})$ such that $\inf_{0<\varepsilon<1}\mu_{\lambda(\cdot),\varepsilon}(B)>1-\delta$. Let μ be an arbitrary weak limit of $\{\mu_{\lambda(\cdot),\varepsilon}\}$ as $\varepsilon\downarrow 0$. Then Lemma 4.5 shows that $\langle\mu,\Psi\rangle=\langle\delta_\rho,\Psi\rangle$ for every $\Psi\in\mathscr{D}$ and this proves $\mu=\delta_\rho$ since $\mathscr{D}$ is a determining class for the space $\mathscr{P}(\mathbf{H}_{e,w})$. The proof of Theorem 4.1 is completed.

5. Existence and Uniqueness Theorem for the TDGL Equation

From the assumption (1.2) on U the TDGL eq. described by the SPDE (1.1) can be rewritten at least formally into

$$dS_t(x)=-AS_t(x)\,dt+\Delta\{V'(S_t(x))\}\,dt+\sqrt{2}\,\nabla dw_t(x),\qquad t>0,\,x\in\mathbb{R},\tag{5.1}$$

where $A=\Delta^2-\gamma\Delta$. The cylindrical Brownian motion $w_t(x)$ on a Hilbert space $L^2(\mathbb{R},dx)$ is defined on a probability space $(\Omega,\mathscr{F},P)$ with reference family $\{\mathscr{F}_t\}$ and we assume that w_t is $\{\mathscr{F}_t\}$-adapted and its increment w_t-w_u is independent of $\mathscr{F}_u$ for every $0\leqq u\leqq t$.

Until now the general theory of SPDE's is developed pretty well and basically two approaches are known; the semigroup method (see Dawson [4], Marcus [19] and others) and the variational one (see Krylov and Rozovskii [17] and references of this paper). Here in this paper we shall adopt the former approach and the construction of solutions is accomplished by a usual contraction mapping method. Some parts of the proof are omitted when the argument is quite standard. See [Fu] for detail if necessary. Similar calculations were developed in [11].

Let $q=q(t,x)$ be the fundamental solution of the parabolic operator $\frac{\partial}{\partial t}+A$. Then the following estimates are known:

$$\left|\frac{\partial^j}{\partial t^j}\frac{\partial^k}{\partial x^k}q(t,x)\right|\leqq K_1\,t^{-j-\frac{1+k}{4}}\exp\left\{-L_1\left(\frac{x^4}{t}\right)^{1/3}\right\},$$
$$0<t\leqq T,\ x\in\mathbb{R};\ j=0,1,\ k=0,1,2,3,\ T>0,\tag{5.2}$$

with positive constants K_1 and L_1, which depend only on T (see Eidel'man [7]). The mathematical meaning of the SPDE (1.1) will be given by rewriting (5.1) again formally into a stochastic integral equation:

$$S_t(x)=\int_{\mathbb{R}}q(t,x,y)\,S_0(y)\,dy-\sqrt{2}\int_0^t\int_R q_y(t-u,x,y)\,dw_u(y)\,dy$$
$$+\int_0^t\int_{\mathbb{R}}q_{yy}(t-u,x,y)\,V'(S_u(y))\,du\,dy,\qquad t\geqq 0,\ x\in\mathbb{R},\tag{5.3}$$

where $q(t, x, y)=q(t, x-y)$ and the subscripts to q mean derivatives with respect to those variables; e.g., $q_y=\partial q/\partial y$ etc. The initial data S_0 of the SPDE (1.1) accordingly of the integral Eq. (5.3) is always taken from the space $\mathbf{H}_e$.

Let $\mathscr{T}$ be the class of all stochastic processes $S_t=\{S_t(x;\omega); x\in\mathbb{R}\}$, $t\geqq 0$, defined on the probability space $(\Omega, \mathscr{F}, P; \{\mathscr{F}_t\})$, which are $\{\mathscr{F}_t\}$-adapted and jointly measurable in $(t, x, \omega)\in[0,\infty)\times\mathbb{R}\times\Omega$. We call S_t a *solution of the SPDE* (1.1) if S_t belongs to the class $\mathscr{T}$ and satisfies the integral Eq. (5.3) with probability one. We denote by $\mathscr{T}'$ the subclass of $\mathscr{T}$ consisting of all S_t such that $S_t\in C((0,\infty), \mathscr{C}_e)$ (a.s.) and $\sup_{0<t\leqq T} t^{1/8}|||S_t|||_r<\infty$ (a.s.) for every $T>0$ and $r>0$.

The existence and uniqueness result for the SPDE (1.1) is formulated as follows.

Theorem 5.1. (i) *There exists a solution S_t of the* SPDE (1.1). *Every solution belongs to the class $\mathscr{T}'$.*

(ii) *Let S_t and S_t' be two solutions of the* SPDE (1.1). *If $S_0=S_0'$, then we have $S_t=S_t'$, $t\geqq 0$* (a.s.).

(iii) *Suppose $S_0\in\mathscr{C}_e$, then we have $S_t\in C([0,\infty), \mathscr{C}_e)$* (a.s.).

For given $S_0\in\mathbf{H}_e$ *and* $S_\cdot\in\mathscr{T}$, we denote

$$S_{t,1}(x)\equiv S_{t,1}(x;S_0)=\int_{\mathbb{R}} q(t,x,y)\,S_0(y)\,dy,$$

$$S_{t,3}(x)\equiv S_{t,3}(x;S_\cdot)=\int_0^t\int_{\mathbb{R}} q_{yy}(t-u,x,y)\,V'(S_u(y))\,du\,dy,$$

and set

$$S_{t,2}(x)=\int_0^t\int_{\mathbb{R}} q_y(t-u,x,y)\,dw_u(y)\,dy,\qquad t\geqq 0,\quad x\in\mathbb{R}.$$

Then the following two lemmas can be shown by using the estimate (5.2). Remember that V' is bounded.

Lemma 5.1. *For every $T>0$ and $0<\alpha<1$, there exists a positive constant C such that*

$$E[|S_{t,2}(x)-S_{t',2}(x')|^2]\leqq C\{|t-t'|^{1/4}+|x-x'|^\alpha\},$$
$$0\leqq t,t'\leqq T,\quad x,x'\in\mathbb{R}.$$

Lemma 5.2. (i) $\sup\left\{\left|\frac{\partial S_{t,3}}{\partial x}(x;S_\cdot)\right|;\ 0\leqq t\leqq T,\ x\in\mathbb{R},\ S_\cdot\in\mathscr{T}\right\}<\infty,\ T>0.$

(ii) *For every $T>0$, there exists a positive constant C such that*

$$|S_{t,3}(x)-S_{t',3}(x)|\leqq C|t-t'|^{1/2},\qquad 0\leqq t,t'\leqq T,\quad x\in\mathbb{R}.$$

These lemmas imply the following consequence. We use Kolmogorov-Totoki's regularization theorem (see Walsh [26] for example) for $\{S_{t,2}\}$ noting that it is a Gaussian system.

Corollary 5.1. *The processes $S_{t,2}$ and $S_{t,3}\in C([0,\infty),\mathscr{C}_e)$* (a.s.)

We shall use frequently an estimate which follows from (5.2):

$$\int \left| \frac{\partial^j}{\partial t^j} \frac{\partial^k}{\partial x^k} q(t, x, y) \right| \theta(y, r)\, dy \leqq K t^{-j-k/4} \theta(x, r),$$
$$0 < t \leqq T,\ x \in \mathbb{R};\ j = 0, 1,\ k = 0, 1, 2, 3, \tag{5.4}$$

for every $r \in \mathbb{R}$ with a positive constant K depending only on r and T.

Lemma 5.3. (i) *If* $S_0 \in \mathscr{C}_e$, *then* $S_{t,1} = S_{t,1}(\cdot\,; S_0) \in C([0, \infty), \mathscr{C}_e)$.

(ii) *If* $S_0 \in \mathbf{H}_e$, *then* $S_{t,1} \in C((0, \infty), \mathscr{C}_e)$ *and* $\sup_{0<t\leqq T} t^{1/8} |||S_{t,1}|||_r < \infty$ *for every* $T > 0$ *and* $r > 0$.

Proof. For $S_0 \in \mathbf{H}_e$ the usage of (5.2), (5.4) and Schwarz's inequality proves

$$\begin{aligned} |S_{t,1}(x)| &\leqq |S_0|_{2r} \Big\{ \int_{\mathbb{R}} |q(t, x, y)|^2 \theta(y, -2r)\, dy \Big\}^{1/2} \\ &\leqq |S_0|_{2r} \{\text{const. } t^{-1/4} \theta(x, -2r)\}^{1/2}, \qquad 0 < t \leqq T, \end{aligned}$$

and this implies $\sup_{0<t\leqq T} t^{1/8} |||S_{t,1}|||_r < \infty$, $T > 0$, $r > 0$. Especially we see that $S_{t,1} \in \mathscr{C}_e$ for each $t > 0$ because $S_{t,1} \in \mathscr{C}$ is easy to be shown. Therefore, if (i) is proved, then the semigroup property of $S_{t,1}$ completes the proof of (ii). From now on we assume $S_0 \in \mathscr{C}_e$. Then it is not difficult to show from (5.4) with $j = 1$ and $k = 0$ that $S_{t,1} \in C((0, \infty), \mathscr{C}_e)$. We only need proving the continuity of $S_{t,1} \in \mathscr{C}_e$ at $t = 0$. To this end, take a function $\psi \in C_0^\infty(\mathbb{R})$, $0 \leqq \psi \leqq 1$, satisfying $\psi \equiv 1$ on the interval $[-1, 1]$ and put $\psi_\varepsilon(x) = \psi(\varepsilon x)$, $\varepsilon > 0$. Then $S_{t,1}(x)$ is decomposed into the sum of $S_{t,1}^{\varepsilon,1}(x) \equiv S_{t,1}(x; \psi_\varepsilon \cdot S_0)$ and $S_{t,1}^{\varepsilon,2}(x) \equiv S_{t,1}\{x; (1 - \psi_\varepsilon\} S_0)$. From the next estimation

$$\begin{aligned} |S_{t,1}^{\varepsilon,2}(x)| &\leqq |||S_0|||_{r/2}\, \theta(\varepsilon^{-1}, r/2) \int_{\mathbb{R}} |q(t, x, y)|\, \theta(y, -r)\, dy \\ &\leqq K |||S_0|||_{r/2}\, \theta(\varepsilon^{-1}, r/2)\, \theta(x, -r), \qquad r > 0, \end{aligned}$$

we get

$$|||S_{t,1}^{\varepsilon,2}|||_r \leqq K |||S_0|||_{r/2}\, \theta(\varepsilon^{-1}, r/2), \qquad r > 0,\ 0 < t \leqq T. \tag{5.5}$$

On the other hand, since (5.4) can be used to show

$$|||S_{t,1}|||_r \leqq K |||S_0|||_r, \qquad 0 \leqq t \leqq T,\ r \in \mathbb{R}, \tag{5.6}$$

we obtain for $r > 0$

$$\begin{aligned} &\sup_{|x|\geqq l} |S_{t,1}^{\varepsilon,1}(x) - (\psi_\varepsilon \cdot S_0)(x)|\, \theta(x, r) \\ &\quad \leqq \sup_{|x|\geqq l} \{\text{const. } |||S_0|||_{r/2}\, \theta(x, -r/2)\, \theta(x, r)\} \to 0 \quad \text{as} \quad l \to \infty. \end{aligned}$$

However, $\psi_\varepsilon \cdot S_0 \in C_b(\mathbb{R})$ implies $\lim_{t \downarrow 0} \sup_{|x| \leq l} |S^{\varepsilon,1}_{t,1}(x) - (\psi_\varepsilon \cdot S_0)(x)|\, \theta(x, r) = 0$ for every $l, \varepsilon > 0$ (see Arima [2]). We therefore get $\lim_{t \downarrow 0} |||S^{\varepsilon,1}_{t,1} - \psi_\varepsilon S_0|||_r = 0$ for every $\varepsilon > 0$. This combined with (5.5) now proves the continuity of $S_{t,1}$ in the space $\mathscr{C}_e$ at $t = 0$. □

Lemma 5.4. *For every $2 < p < 8$, $T > 0$ and $r > 0$, there exists a positive constant C such that*

$$|||S_{t,3}(\cdot\,; S) - S_{t,3}(\cdot\,; \bar{S})|||_r^p \leq C \int_0^t |||S_u - \bar{S}_u|||_r^p\, du, \qquad 0 \leq t \leq T,\ S, \bar{S} \in \mathscr{T}'.$$

Proof. We obtain the conclusion from the following calculation:

$$\begin{aligned} &|S_{t,3}(x; S) - S_{t,3}(x; \bar{S})| \\ &\leq \gamma_0 \int_0^t |||S_u - \bar{S}_u|||_r\, du \int_{\mathbb{R}} |q_{xx}(t-u, x, y)|\, \theta(y, -r)\, dy \\ &\leq \gamma_0 K \theta(x, -r) \int_0^t (t-u)^{-1/2} |||S_u - \bar{S}_u|||_r\, du \\ &\leq \gamma_0 K \theta(x, -r) \left\{ \int_0^t (t-u)^{-p'/2} du \right\}^{1/p'} \left\{ \int_0^t |||S_u - \bar{S}_u|||_r^p\, du \right\}^{1/p}, \\ &0 \leq t \leq T, \end{aligned}$$

where $1 < p' < 2 < p < \infty$ such that $1/p + 1/p' = 1$. Note that the RHS is finite if $p < 8$. □

Proof of Theorem 5.1. Corollary 5.1 and Lemma 5.3 prove that $S_t \in \mathscr{T}'$ for every solution S_t of the SPDE (1.1) if exists. Therefore the uniqueness of solutions may be discussed in the class $\mathscr{T}'$. However this follows from Lemma 5.4 immediately. Lemma 5.4 can be also applied to construct solutions of the SPDE (1.1) by using the usual method of successive approximation. The assertion (iii) follows from Lemma 5.3 (i). □

The theorem implies the existence and uniqueness result for the scaled TDGL Eq. (1.3).

Remark 5.1. (i) We did not discuss the equivalence between the integral Eq. (5.3) and the SPDE (1.1) or (5.1). See Iwata [14] for such problem. See also Funaki [12] for a non-scalar TDGL eq. (ii) The variational method is also available for the construction of solutions of the SPDE (1.1). In fact, let $\tilde{W}_2^{-m}$, $m = 0, 2, 4$, be the class of all generalized functions S on $\mathbb{R}$ satisfying that the products $\theta(\cdot, r) S$ belong to the Sobolev space $W_2^{-m}(\mathbb{R})$. Here we have fixed $r > 0$. The norm is naturally defined by $\|S\|_{\tilde{W}_2^{-m}} = \|\theta(\cdot, r) S\|_{W_2^{-m}}$. Then the application of the theory of Krylov and Rozovskii [17] based on a Gelfand triple $(\mathbf{V}, \mathbf{H}, \mathbf{V}^*) \equiv (\tilde{W}_2^0, \tilde{W}_2^{-2}, \tilde{W}_2^{-4})$ proves the existence and uniqueness of solutions of the SPDE (1.1) satisfying $S_t \in \mathbf{V}$ (a.e. $-(t, \omega)$) and $\in C([0, \infty), \mathbf{H})$ (a.e.-ω).

6. Approximation Theorems for the TDGL Equation

In order to develop an infinite-dimensional analysis on the stochastic process S_t, we need to approximate it by finite-dimensional processes. Here we shall discuss two types of approximation theorems, namely, finite volume approximation and its further approximation using the so-called Galerkin method. The result will be applied in Sect. 8.

6.1 Finite Volume Approximation

Consider the following SPDE on a finite interval $[-l, l]$, $l \in \mathbb{N}$:

$$dS_t^l(x) = -A^l S_t^l(x)\,dt + \Delta\{V'(S_t^l(x))\}\,dt + \sqrt{2}\,\nabla dw_t(x), \qquad t>0,\ x\in(-l,l), \tag{6.1}$$

with an initial condition: $S_0^l = S^l$ on $[-l, l]$. Here $S^l \in L_l^2 \equiv L^2([-l,l], dx)$ and $A^l = (-\Delta)^2 - \gamma\Delta$ should be understood as an operator defined as a function of the self-adjoint operator $-\Delta$ on the space L_l^2 having the Dirichlet 0-boundary condition at $\pm l$. The precise mathematical meaning of the Eq. (6.1) is given similarly to the Eq. (5.1) by rewriting it into an integral equation:

$$\begin{aligned} S_t^l(x) = & \int_{-l}^{l} q^l(t,x,y)\,S^l(y)\,dy - \sqrt{2}\int_0^t \int_{-l}^{l} q_y^l(t-u,x,y)\,dw_u(y)\,dy \\ & + \int_0^t \int_{-l}^{l} q_{yy}^l(t-u,x,y)\,V'(S_u^l(y))\,du\,dy, \qquad t\geqq 0,\ x\in[-l,l], \end{aligned} \tag{6.2}$$

where q^l is the fundamental solution of the operator $\frac{\partial}{\partial t} + A^l$. The second term in the RHS of (6.2) should be understood as

$$-\sqrt{2}\int_0^t \int_{\mathbb{R}} 1_{[-l,l]}(y)\,q_y^l(t-u,x,y)\,dw_u(y)\,dy$$

or being defined with the cylindrical Brownian motion on the Hilbert space L_l^2. We note the relation:

$$\begin{aligned} q^l(t,x,y) = \sum_{n=-\infty}^{\infty} \{ & q(t, x-y+4nl) - q(t, x+y+2(2n+1)l)\}, \\ & x,y\in[-l,l],\ t>0, \end{aligned} \tag{6.3}$$

where $q = q(t,x)$ is the function introduced in Sect. 5. In fact, this follows by seeing that both sides satisfy the same boundary conditions $q(\pm l) = q''(\pm l) = 0$ as functions of x. The existence and uniqueness theorem for the Eq. (6.1) can be formulated as follows.

Proposition 6.1. (i) *There exists a unique solution* S_t^l *of* (6.1) *satisfying* $S_t^l \in C([0,\infty), L_l^2)$ (a.s.).

(ii) *If the initial data* $S^l \in C([-l,l])$ *and* $S^l(\pm l) = 0$, *then the solution* S_t^l *satisfies* $S_t^l(\pm l) = 0$ *and* $S_t^l \in C([0,\infty), C([-l,l]))$ (a.s.).

The meaning of "uniqueness" in the statement of this proposition is the same as in Theorem 5.1 (ii). The first assertion is shown by using the Galerkin method (see Theorem 6.2 below) and the second one is proved by a similar method used in Sect. 5 noting the relation (6.3). We therefore omit the proof of the proposition.

Let S, $S^l \in \mathscr{C}_e$, $l \in \mathbb{N}$, be given and satisfy that $S^l(\pm l)=0$ and $S^l \to S$ as $l \to \infty$ in the space $\mathscr{C}_r$ with some $r>0$. The positive number r will be fixed throughout this paragraph. In the following we sometimes regard $S_t^l \in \mathscr{C}$ by setting $S_t^l(x)=0$ for $x \in \mathbb{R} \setminus [-l, l]$. Denote by P and P^l the distributions on the space $C([0, \infty), \mathscr{C})$ of the solution S_t of the SPDE (1.1) with initial data S respectively of the solution S_t^l of (6.1) with initial data S^l. The purpose of this paragraph is to prove the following theorem.

Theorem 6.1. *The probability distribution P^l converges weakly to P as $l \to \infty$ on the space $C((0, \infty), \mathscr{C})$.*

Similarly to the definitions of $S_{t,1}$, $S_{t,2}$ and $S_{t,3}$ given in Sect. 5, we denote three terms in the RHS of (6.2) in due order by $S_{t,1}^l$, $S_{t,2}^l$ and $S_{t,3}^l$, respectively (we neglect the factor $-\sqrt{2}$ for the definition of $S_{t,2}^l$). We regard $S_{t,1}^l$, $S_{t,2}^l$ and $S_{t,3}^l \in \mathscr{C}$ by setting $=0$ outside $[-l, l]$ similarly to S_t^l. The proof of Theorem 6.1 will be divided into four steps.

6.1.1 Convergence of $S_{t,1}^l$

We shall prove that $S_{t,1}^l$ converges to $S_{t,1}$ in the space $C((0, \infty), \mathscr{C})$ as $l \to \infty$. The following estimate is an easy consequence of (6.3):

$$\int_{-l}^{l} \left| \frac{\partial^j}{\partial t^j} \frac{\partial^k}{\partial x^k} q^l(t, x, y) \right| \theta(y, -r)\, dy \leqq e^r \int_{\mathbb{R}} \left| \frac{\partial^j}{\partial t^j} \frac{\partial^k}{\partial x^k} q(t, x, y) \right| \theta(y, -r)\, dy$$
$$t>0,\ x \in [-l, l];\ j=0, 1,\ k=0, 1, 2, 3, \tag{6.4}$$

for every $r \geqq 0$. Although this bound holds only for $r \geqq 0$, we can derive a supplementary estimate for $r>0$:

$$\int_{-l}^{l} \left| \frac{\partial^j}{\partial t^j} \frac{\partial^k}{\partial x^k} q^l(t, x, y) \right| \theta(y, r)\, dy \leqq K' t^{-j-k/4} \theta(x, r),$$
$$0<t \leqq T,\ x \in [-l, l],\ l \geqq 1;\ j=0, 1,\ k=0, 1, 2, 3, \tag{6.5}$$

with a positive constant K' depending only on r and T. Indeed, noting that χ is an even function, (6.5) follows from (6.3), (5.4) and

$$\sup_{l \geqq 1} \sup_{|x| \leqq l} \sum_{m=-\infty}^{\infty} \theta(x+2ml, r)/\theta(x, r) < \infty, \qquad r>0.$$

The estimate (5.4) combined with (6.4) gives a uniform bound:

$\sup_{l \in \mathbb{N}} \sup_{0 \leqq t \leqq T} t^{j+k/4} \left|\left|\left| \frac{\partial^j}{\partial t^j} \frac{\partial^k}{\partial x^k} S_{t,1}^l \right|\right|\right|_r < \infty, j, k=0, 1,$ and this proves the following.

Lemma 6.1. *The family of functions* $\{S^l_{t,1}\}_{l\in\mathbb{N}}$ *is relatively compact in the space* $C((0,\infty),\mathscr{C})=C((0,0,\infty)\times\mathbb{R})$.

Let $\mathscr{C}_{-r,l}$, $r>0$, be the space $C([-l,l])$ with norm $|||\varphi|||_{-r} = \sup_{x\in[-l,l]} |\varphi(x)|\,\theta(x,-r)$ and let $\hat{\mathscr{C}}_{-r,l}$ be its subspace consisting of all φ such that $\varphi(\pm l)=0$. We denote by e^{-tA} an integral operator with the kernel $q(t,x,y)$, which is defined on the space $\mathscr{C}_{-r}$; i.e., $e^{-tA}\varphi=S_{t,1}(\cdot\,;\varphi)$, $\varphi\in\mathscr{C}_{-r}$. Similarly e^{-tA^l} can be defined as an operator on the space $\mathscr{C}_{-r,l}$. Let $\hat{\mathscr{C}}_{-r}$ be the space introduced in Sect. 2.

Lemma 6.2. (i) $\{e^{-tA}\}_{t\geq 0}$ *and* $\{e^{-tA^l}\}_{t\geq 0}$ *are strongly continuous semigroups on the spaces* $\hat{\mathscr{C}}_{-r}$ *and* $\hat{\mathscr{C}}_{-r,l}$, *respectively.*

(ii) *With some constants* $M\geq 1$ *and* $\delta>0$ *which are independent of* l *and* t, *operator norms of* e^{-tA^l} *on* $\hat{\mathscr{C}}_{-r,l}$ *and of* e^{-tA} *on* $\hat{\mathscr{C}}_{-r}$ *can be estimated as follows:*

$$|||e^{-tA^l}|||\leq Me^{\delta t},\quad |||e^{-tA}|||\leq Me^{\delta t},\qquad l\in\mathbb{N},\ t>0.$$

Proof. (i) The estimate (5.6) shows that $\varphi\in\mathscr{C}_{-r}$ implies $e^{-tA}\varphi\in\mathscr{C}_{-r}$, $t\geq 0$. However, since the condition (2.1) on $\varphi\in\hat{\mathscr{C}}_{-r}$ proves with the help of (5.4)

$$\lim_{l\to\infty}\sup_{0\leq t\leq T}\sup_{|x|\geq l}|e^{-tA}\varphi(x)|\,\theta(x,-r)=0,\ T>0, \tag{6.6}$$

we see that $\varphi\in\hat{\mathscr{C}}_{-r}$ implies $e^{-tA}\varphi\in\hat{\mathscr{C}}_{-r}$ and therefore e^{-tA} is a semigroup on the space $\hat{\mathscr{C}}_{-r}$. The strong-continuity of this semigroup follows from (6.6) and the fact that $e^{-tA}\varphi$ converges to φ as $t\downarrow 0$ uniformly on each bounded interval of $\mathbb{R}$ if $\varphi\in C_b(\mathbb{R})$ (see Arima [2]). For the assertion on e^{-tA^l}, see Arima [2] noting that A^l is an operator with boundary conditions: $\varphi(\pm l)=\varphi''(\pm l)=0$. We notice that the uniform convergence of $e^{-tA^l}\varphi$ to φ on $[-l,l]$ as $t\downarrow 0$ can be shown by using (5.2), (6.3) and the property $\varphi(\pm l)=0$ of $\varphi\in\hat{\mathscr{C}}_{-r,l}$.

(ii) Since the estimate (6.5) proves $|||e^{-tA^l}\varphi|||_{-r}\leq K'|||\varphi|||_{-r}$, $0\leq t\leq T$, the assertion on $\{e^{-tA^l}\}$ follows from their semigroup property, see Tanabe [25, Theorem 3.1.1]. Remember (5.6) for e^{-tA}. □

Let Π_l, $l\in\mathbb{N}$, be a linear operator from $\hat{\mathscr{C}}_{-r}$ into $\hat{\mathscr{C}}_{-r,l}$ defined as follows:

$$\Pi_l\varphi(x)=\begin{cases}\varphi(x), & |x|\leq\frac{l}{2},\\ \varphi(l/2)\{2-2x/l\}\,\theta\left(\frac{l}{2},-r\right)\theta(x,r), & \frac{l}{2}\leq x\leq l,\\ \varphi(-l/2)\{2+2x/l\}\,\theta\left(\frac{l}{2},-r\right)\theta(x,r), & -l\leq x\leq -\frac{l}{2}.\end{cases}$$

Then, the operator norm $|||\Pi_l|||$ of Π_l is equal to 1. We also define a mapping Π_l^{-1} from $\hat{\mathscr{C}}_{-r,l}$ into $\hat{\mathscr{C}}_{-r}$ by setting $\Pi_l^{-1}\varphi(x)=\varphi(x)$, $x\in[-l,l]$, and $=0$, $x\in\mathbb{R}\setminus[-l,l]$, for $\varphi\in\hat{\mathscr{C}}_{-r,l}$. If $\varphi\in C_0(\mathbb{R})$ satisfies $\operatorname{supp}\varphi\subset[-l/2,l/2]$, then $\Pi_l^{-1}\Pi_l\varphi=\varphi$.

Lemma 6.3. *For every r, $T>0$ and $\varphi\in\mathscr{C}_{-r}$, we have*

$$\lim_{l\to\infty}\sup_{t\in[0,T]}|||\Pi_l^{-1}e^{-tA^l}\Pi_l\varphi-e^{-tA}\varphi|||_{-r}=0.$$

Proof. We use the results in the book of Ethier and Kurtz [8]. Let D be the space of all $\varphi\in C^\infty(\mathbb{R})$ such that its k-th derivative $\varphi^{(k)}$ belongs to $\mathscr{C}_{-r}$ for every $k=0, 1, 2, \ldots$. Then D is a core for the generator $-A$ of the semigroup e^{-tA} on the space $\mathscr{C}_{-r}$; use Proposition 3.3 of [8, p. 17] with the help of Lemma 6.2(i). Therefore, noting Lemma 6.2 again, we obtain

$$\lim_{l\to\infty}\sup_{t\in[0,T]}|||e^{-tA^l}\Pi_l\varphi-\Pi_l e^{-tA}\varphi|||_{-r}=0, \qquad T>0,\ \varphi\in\mathscr{C}_{-r},$$

by verifying the condition (*c*) of Theorem 6.1 of [8, p. 28] (see also Remark 1.3 of [8, p. 7]). This completes the proof of the lemma with the help of (6.6) and

$$\lim_{l\to\infty}\sup_{t\in[0,T]}\sup_{l/2\le|x|\le l}|e^{-tA^l}\Pi_l\varphi(x)|\,\theta(x,-r)=0$$

for all $\varphi\in\mathscr{C}_{-r}$ and $T>0$, which can be shown similarly to (6.6). □

Lemma 6.4. *For every $t>0$, $S_{t,1}^l$ converges to $S_{t,1}$ as $l\to\infty$ in the following sense: $\lim_{l\to\infty}\langle S_{t,1}^l,\varphi\rangle=\langle S_{t,1},\varphi\rangle$ for all $\varphi\in C_0(\mathbb{R})$.*

Proof. The symmetry of the fundamental solutions $q^l(t,x,y)$ and $q(t,x,y)$ in the variables (x,y) implies $\langle S_{t,1}^l,\varphi\rangle=\langle S^l,\Pi_l^{-1}e^{-tA^l}\Pi_l\varphi\rangle$ and $\langle S_{t,1},\varphi\rangle=\langle S,e^{-tA}\varphi\rangle$ for every $\varphi\in C_0(\mathbb{R})$ such that $\operatorname{supp}\varphi\subset[-l/2,l/2]$. We therefore obtain the conclusion by using Lemma 6.3. □

This lemma shows that an arbitrary limit as $l\to\infty$ of $\{S_{t,1}^l\}$ in the space $C((0,\infty),\mathscr{C})$, whose existence is guaranteed by Lemma 6.1, coincides with $S_{t,1}$. Therefore we have proved the following:

Proposition 6.2. *In the space $C((0,\infty),\mathscr{C})$, $S_{t,1}^l$ converges to $S_{t,1}$ as $l\to\infty$.*

6.1.2 Convergence of $S_{t,2}^l$

Let $\kappa_n\equiv\kappa_n^l=\left(\frac{n\pi}{2l}\right)^2$ and $e_n(x)\equiv e_n^l(x)=\sqrt{1/l}\sin\{\sqrt{\kappa_n}(x+l)\}$, $n\in\mathbb{N}$, be eigenvalues and their corresponding normalized eigenfunctions, respectively, of the operator $-\Delta$ with Dirichlet boundary conditions defined on the space L_l^2. We denote by $\langle\ ,\ \rangle_l$ the usual inner product of L_l^2. Let $\{w_t^n\}_{n\in\mathbb{N}}$ be a system of mutually independent 1-dimensional standard Brownian motions defined by w_t^n

$=\langle w_t(\cdot), \sqrt{1/l}\cos\{\sqrt{\kappa_n}(\cdot+l)\}\rangle_l$ (see [11] for the RHS of this expression). Then, using a representation

$$S_{t,2}^l(x)=\sum_{n=1}^{\infty} e_n^l(x)\int_0^t (\kappa_n^l)^{1/2}\exp[-(t-u)\{(\kappa_n^l)^2+\gamma\kappa_n^l\}]\,dw_u^n,$$

simple but somewhat lengthy calculations yield the following estimate. We omit the proof ([Fu] explains the detail).

Lemma 6.5. *For every $T>0$ and $0<\alpha<1$, there exists a positive constant C independent of l such that*

$$E[|S_{t,2}^l(x)-S_{t',2}^l(x')|^2]\leqq C\{|t-t'|^{1/4}+|x-x'|^\alpha\},$$
$$0\leqq t,t'\leqq T,\ x,x'\in[-l,l],\ l\in\mathbb{N}.$$

Now we can show the convergence of $S_{t,2}^l$.

Proposition 6.3. *The probability distribution of $S_{t,2}^l$ on the space $C([0,\infty),\mathscr{C})$ converges weakly to that of $S_{t,2}$ as $l\to\infty$.*

Proof. Since $\{S_{t,2}^l(x);\ t\geqq 0,\ x\in\mathbb{R}\}$ is a Gaussian system and $S_{t,2}^l(\pm l)=0$, Lemma 6.5 with the help of Kolmogorov-Totoki's theorem proves that $S_{\cdot,2}^l\in C([0,\infty),\mathscr{C})$ (a.s.) and the family of their distributions on the space $C([0,\infty),\mathscr{C})$ is tight. Therefore the conclusion follows from an observation that for every $t>0$ and $\varphi\in C_0^\infty(\mathbb{R})$ satisfying supp $\varphi\subset[-l/2,l/2]$ we have

$$E[\langle S_{t,2}^l-S_{t,2},\varphi\rangle^2]=\int_0^t \|\Pi_l^{-1}e^{-(t-u)A^l}\Pi_l\nabla\varphi-e^{-(t-u)A}\nabla\varphi\|_{L^2(\mathbb{R})}^2\,du$$

which converges to 0 as $l\to\infty$; see Lemma 6.3. □

6.1.3 Convergence of $S_{t,3}^l$

See Sect. 5 for the definition of $S_{t,3}(x)\equiv S_{t,3}(x,S.)$. For given $S.=\{S_t(x)\}\in C([0,\infty),\mathscr{C})$ we define similarly

$$S_{t,3}^l(x)\equiv S_{t,3}^l(x;S.)=\int_0^t\int_{-l}^l q_{yy}^l(t-u,x,y)\,V'(S_u(y))\,du\,dy.$$

Lemma 6.6. *For every $T>0$ there exists a positive constant C depending only on T such that*

$$|S_{t,3}^l(x)-S_{t',3}^l(x')|\leqq C\{|t-t'|^{1/2}+|x-x'|\},$$
$$0\leqq t,t'\leqq T,\ x,x'\in[-l,l],\ l\in\mathbb{N}.$$

Proof. The proof can be completed quite similarly to that of Lemma 5.2. We use the estimate (6.4) with $r=0$. □

Proposition 6.4. *Let functions $\{S^l_{\cdot}\}_{l\in\mathbb{N}}$ and $S_{\cdot}\in C([0,\infty),\mathscr{C})$ be given and assume that $S^l_{\cdot}$ converges to $S_{\cdot}$ as $l\to\infty$ in this space. Then $S^l_{t,3}(\cdot\,;S^l_{\cdot})$ converges to $S_t(\cdot\,;S_{\cdot})$ in the space $C([0,\infty),\mathscr{C})$.*

Proof. First we note that Lemma 6.6 shows the relative-compactness of the family of functions $\{S^l_{t,3}(\cdot\,;S^l_{\cdot})\}_{l\in\mathbb{N}}$ in the space $C([0,\infty),\mathscr{C})$. Therefore the conclusion follows by seeing that for $\varphi\in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\varphi\subset[-l/2,l/2]$,

$$\begin{aligned}
&|\langle S^l_{t,3}-S_{t,3},\varphi\rangle|\\
&\leqq\left|\int_0^t\langle V'(S^l_u),\Pi_l^{-1}e^{-(t-u)A^l}\Pi_l\Delta\varphi-e^{-(t-u)A}\Delta\varphi\rangle\,du\right|\\
&\quad+\left|\int_0^t\langle V'(S^l_u)-V'(S_u),e^{-(t-u)A}\Delta\varphi\rangle\,du\right|\\
&\leqq t\,\|V'\|_\infty\sup_{0\leqq u\leqq t}|||\Pi_l^{-1}e^{-uA^l}\Pi_l\Delta\varphi-e^{-uA}\Delta\varphi|||_{-1}\int_{\mathbb{R}}\theta(x,1)\,dx\\
&\quad+\sup_{0\leqq u\leqq t}|||e^{-uA}\Delta\varphi|||_{-1}\int_0^t du\int_{\mathbb{R}}|V'(S^l_u(x))-V'(S_u(x))|\,\theta(x,1)\,dx,
\end{aligned}$$

which converges to 0 as $l\to\infty$ for every $t>0$ and $\varphi\in C_0^\infty(\mathbb{R})$; see Lemma 6.3. □

6.1.4 Proof of Theorem 6.1

Now we give the proof of Theorem 6.1. Propositions 6.2, 6.3 and 6.4 are combined to show that the family of joint probability distributions $\{\tilde{P}^l\}_{l\in\mathbb{N}}$ of $\{(S^l_t,S^l_{t,1},S^l_{t,2})\}_{l\in\mathbb{N}}$ on the space $[C((0,\infty),\mathscr{C})]^3$ is tight. Take its arbitrary weak limit $\tilde{P}\in\mathscr{P}([C((0,\infty),\mathscr{C})]^3)$ and a subsequence $\{l'\}$ such that $\tilde{P}^{l'}\Rightarrow\tilde{P}$. We then apply Skorohod's representation theorem to construct $[C((0,\infty),\mathscr{C})]^3$-valued random variables $\bar{\mathbf{S}}^{l'}_t=(\bar{S}^{l'}_t,\bar{S}^{l'}_{t,1},\bar{S}^{l'}_{t,2})$ and $\bar{\mathbf{S}}_t=(\bar{S}_t,\bar{S}_{t,1},\bar{S}_{t,2})$ on a proper probability space $(\bar{\Omega},\bar{\mathscr{F}},\bar{P})$ in the following manner: (i) Under $\bar{P}$, the distributions of $\bar{\mathbf{S}}^{l'}_t$ and $\bar{\mathbf{S}}_t$ are $\tilde{P}^{l'}$ and $\tilde{P}$, respectively, and (ii) $\bar{\mathbf{S}}^{l'}_t$ converges almost surely to $\bar{\mathbf{S}}_t$ in the space $[C((0,\infty),\mathscr{C})]^3$ as $l'\to\infty$. Define $\bar{S}^{l'}_{t,3}(=S^{l'}_{t,3}\cdot\,;\bar{S}^{l'}_{\cdot})$ and $\bar{S}_{t,3}=S_{t,3}(\cdot\,;\bar{S}_{\cdot})$, then an equality $\bar{S}^{l'}_t=\bar{S}^{l'}_{t,1}-\sqrt{2}\,\bar{S}^{l'}_{t,2}+\bar{S}^{l'}_{t,3}$ holds and Proposition 6.4 proves that $\bar{S}^{l'}_{t,3}$ converges almost surely to $\bar{S}_{t,3}$ in the space $C((0,\infty),\mathscr{C})$. Hence we obtain $\bar{S}_t=\bar{S}_{t,1}-\sqrt{2}\,\bar{S}_{t,2}+\bar{S}_{t,3}$. However Proposition 6.3 implies that $\bar{S}_{t,2}$ has a representation $\bar{S}_{t,2}(x)=\int_0^t\int_{\mathbb{R}}q_y(t-u,x,y)\,d\bar{w}_u(y)\,dy$ with a cylindrical Brownian motion $\bar{w}_t$ which is defined on the probability space $(\bar{\Omega},\bar{\mathscr{F}},\bar{P})$ or its proper extension if necessary. Therefore $\bar{S}_t$ is a solution of the integral Eq. (5.3) with w_t replaced by $\bar{w}_t$. This shows that the distribution of $\bar{S}_t$ is just given by P appearing in the statement of Theorem 6.1, which completes the proof of the theorem.

The following lemma which will be used in Sect. 8 is a consequence of the combination of Proposition 6.2, Lemma 6.5 and Lemma 6.6.

Lemma 6.7. $\sup_{l;\,l\geqq\bar{l}}E[\|S^l_t\|^2_{L^2_l}]<\infty$, $\quad\bar{l}\in\mathbb{N},\ t>0$.

6.2 Galerkin approximation

In this paragraph we state the results on a finite-dimensional approximation to the solution S_t^l of the SPDE (6.1) with fixed $l \in \mathbb{N}$. Define a sequence of L_l^2-valued processes $S_t^{(N)} = \sum_{n=1}^{N} a_n^{(N)}(t)\, e_n$, $N \in \mathbb{N}$, by solving the following finite-dimensional stochastic differential equation (SDE):

$$\begin{aligned} da_n^{(N)}(t) &= -\theta_n\, a_n^{(N)}(t)\, dt - \kappa_n \langle V'(S_t^{(N)}(\cdot)), e_n\rangle_l\, dt - \sqrt{2\kappa_n}\, dw_t^n, \\ a_n^{(N)}(0) &= \langle S^l, e_n\rangle_l, \qquad n = 1, 2, \ldots, N, \end{aligned} \tag{6.7}$$

where $\theta_n \equiv \theta_n^l = \kappa_n^2 + \gamma\kappa_n$ and κ_n, e_n, w_t^n are the same as those introduced in the paragraph 6.1.2. Then we can prove the following.

Theorem 6.2. (i) *There exists a unique solution* $S_t = S_t^l$ *of the* SPDE (6.1) *satisfying* $S_t \in C([0, \infty), L_l^2)$ (a.s.).

(ii) *We have a uniform estimate:*

$$E[\|S_t^{(N)}\|_{L_l^2}^2] \leqq \text{const.} + \|S^l\|_{L_l^2}^2, \tag{6.8}$$

with a positive const. *independent of* N, t, S^l *and*

$$\lim_{N \to \infty} \sup_{0 \leqq t \leqq T} E[\|S_t^{(N)} - S_t\|_{L_l^2}^2] = 0, \qquad T > 0. \tag{6.9}$$

We only give the sketch of the proof; cf. [Fu]. First for the construction of solutions of (6.1) it is shown that $\{S_t^{(N)}\}_{N \in \mathbb{N}}$ forms a Cauchy sequence in the space $L^2(\Omega \to L_l^2)$ for each $t > 0$. This is accomplished by deriving estimates on $E[\{a_n^{(N)}(t)\}^2]$ and $E[\{a_n^{(N)}(t) - a_n^{(N')}(t)\}^2]$, $1 \leqq n \leqq N < N'$, like

$$E[\{a_n^{(N)}(t)\}^2] \leqq C\kappa_n^{-1} + \langle S^l, e_n\rangle_l^2,$$

and

$$E[\|S_t^{(N)} - S_t^{(N')}\|_{L_l^2}^2] \leqq 2 e^{2\gamma_0^2 t} \sum_{n=N+1}^{N'} \{C\kappa_n^{-1} + \langle S^l, e_n\rangle_l^2\},$$

with some $C > 0$. Denote by S_t^* the limit of $\{S_t^{(N)}\}$ and define $S_t = S_t^l \in C([0, \infty), L_l^2)$ (a.s.) by the RHS of (6.2) with $S_u^l(y)$ replaced by $S_u^*(y)$. We can then prove that S_t is a modification of S_t^* and therefore from definition it gives a solution of the SPDE (6.1). During the course, (6.8) and (6.9) are naturally shown. The uniqueness of solutions, on the other hand, is a consequence of

$$\|S_{t,3}^l(\cdot\,; S.) - S_{t,3}^l(\cdot\,; \bar{S}.)\|_{L_l^2} \leqq C \int_0^t (t-u)^{-1/2} \|S_u - \bar{S}_u\|_{L_l^2}\, du, \qquad 0 \leqq t \leqq T,$$

which follows by using the estimates (5.4) and (6.4).

7. Integral Equations

In this section we analyze parabolic equations with measurable coefficients as a preparation for Sects. 8 and 9. The technique employed here is not novel.

7.1 Existence and uniqueness results

Let $\mathscr{B}([0, \infty) \times \mathbb{R})$ be a class of all measurable functions on $[0, \infty) \times \mathbb{R}$ and let $\mathscr{B}_b = \mathscr{B}_b([0, \infty) \times \mathbb{R})$ be its subclass of bounded functions. We associate with each $c = c(u, x) \in \mathscr{B}_b$ an operator $\mathscr{L}_u \equiv \mathscr{L}_{u,x} = -A + c(u, x)\Delta_x$, $x \in \mathbb{R}$, and consider backward equation for $Z_{u,t} = Z_{u,t}(x, \varphi; c)$, $0 \leqq u \leqq t < \infty$:

$$\begin{aligned} \frac{\partial}{\partial u} Z_{u,t} &= -\mathscr{L}_u Z_{u,t}, \qquad u \in [0, t], \\ Z_{t,t} &= \varphi \in C_0^\infty(\mathbb{R}). \end{aligned} \tag{7.1}$$

More precisely, we consider the corresponding integral equation:

$$Z_{u,t} = e^{-A(t-u)} \varphi + \int_u^t e^{-A(v-u)} \{c(v, \cdot) \Delta Z_{v,t}\}\, dv. \tag{7.2}$$

Before dealing with (7.2), we discuss the following auxiliary integral equation for $Y_{u,t} = Y_{u,t}(x, \varphi; c)$, $0 \leqq u \leqq t < \infty$, $\varphi \in C_0^\infty(\mathbb{R})$ on the space $\mathscr{C}_{-r}$, $r > 0$:

$$Y_{u,t} = e^{-A(t-u)}(\Delta \varphi) + \int_u^t \Delta e^{-A(v-u)} \{c(v, \cdot)\, Y_{v,t}\}\, dv. \tag{7.3}$$

We put $\mathbf{D}_T = \{(u, t) \in \mathbb{R}^2;\, 0 \leqq u \leqq t \leqq T\}$ and $\mathring{\mathbf{D}}_T = \{(u, t) \in \mathbf{D}_T;\, u \neq t\}$, $T > 0$.

Lemma 7.1. (i) *For every $r > 0$ and $T > 0$, there exists a solution of* (7.3) *satisfying* $Y_{u,t} \in C(\mathbf{D}_T, \mathscr{C}_{-r})$.

(ii) *The uniqueness of solutions of* (7.3) *holds in the class of measurable functions $Y_{u,t}$ satisfying*

$$\sup_{0 \leqq u \leqq t \leqq T} |||Y_{u,t}|||_{-r} (t-u)^{1-\varepsilon} < \infty, \qquad T > 0, \tag{7.4}$$

with some $0 < \varepsilon < 1$.

Proof. Denote by $Q_{u,t} = Q_{u,t}(Y)$ the second term in the RHS of (7.3) for given measurable $Y = Y_{u,t}$ satisfying the condition (7.4). Then, noting $\|c\|_\infty < \infty$, the bound (5.4) can be used for the derivation of the following three estimates: There exists $C_T>0$ depending only on T such that

$$|||Q_{u,t}(Y)|||_{-r} \leqq C_T \int_u^t (v-u)^{-1/2} |||Y_{v,t}|||_{-r}\, dv, \quad 0\leqq u \leqq t \leqq T, \tag{7.5}$$

$$|||Q_{u_1,t}(Y)-Q_{u_2,t}(Y)|||_{-r} \leqq C_T(u_2-u_1)^{1/2} \sup_{0\leqq v\leqq \tau\leqq T} |||Y_{v,\tau}|||_{-r},$$

$$0\leqq u_1<u_2\leqq t\leqq T,\ Y\in C(\mathbf{D}_T, \mathscr{C}_{-r}), \tag{7.6}$$

$$|||Q_{u,t_1}(Y)-Q_{u,t_2}(Y)|||_{-r} \leqq C_T(t_2-t_1)^{1/2} \sup_{t_1\leqq v\leqq t_2} |||Y_{v,t_2}|||_{-r}$$

$$+C_T \int_u^{t_1} (v-u)^{-1/2} |||Y_{v,t_1}-Y_{v,t_2}|||_{-r}\, dv,$$

$$0\leqq u\leqq t_1<t_2\leqq T,\ Y\in C(\mathbf{D}_T, \mathscr{C}_{-r}). \tag{7.7}$$

We now use a usual method of iteration in order to construct a solution of (7.3) satisfying $Y_{u,t}\in C(\mathbf{D}_T, \mathscr{C}_{-r})$, $T>0$. Indeed, Lemma 6.2 shows $e^{-A(t-u)}\Delta\varphi\in C(\mathbf{D}_T, \mathscr{C}_{-r})$, while the two estimates (7.6) and (7.7) prove that $Y\in C(\mathbf{D}_T, \mathscr{C}_{-r})$ implies $Q_{u,t}(Y)\in C(\mathbf{D}_T, \mathscr{C}_{-r})$. Therefore the iterative scheme is accomplished in the space $C(\mathbf{D}_T, \mathscr{C}_{-r})$. The limit, whose existence is shown by (7.5) noting that $Q_{u,t}$ is linear in Y, gives a desirable solution. The uniqueness assertion (ii) is also a consequence of (7.5). □

We return to the Eq. (7.2). Let $\mathscr{C}^2_{-r}$, $r>0$, be the space consisting of all functions $Z\in\mathscr{C}_{-r}$ such that Z', $Z''\in\mathscr{C}_{-r}$ equipped with a norm defined by $|||Z|||_{2,-r}=|||Z|||_{-r}+|||Z'|||_{-r}+|||Z''|||_{-r}$.

Lemma 7.2. (i) *There exists a solution of* (7.2) *such that* $Z_{u,t}\in C(\mathbf{D}_T, \mathscr{C}^2_{-r})$, $r>0$, $T>0$. *The uniqueness of solutions holds in the class of functions* $Z_{u,t}$ *satisfying*

$$\sup_{0\leqq u\leqq t\leqq T} |||Z_{u,t}|||_{2,-r}(t-u)^{1-\varepsilon}<\infty, \quad T>0, \tag{7.8}$$

with some $0<\varepsilon<1$.

(ii) *The solutions* $Z_{u,t}$ *of* (7.2) *and* $Y_{u,t}$ *of* (7.3) *are tied up by the relation:* $\Delta Z_{u,t}=Y_{u,t}$.

Proof. Denote by $P_{u,t}=P_{u,t}(Z)$ the second term in the RHS of (7.2) for given $Z_{u,t}\in C(\mathbf{D}_T, \mathscr{C}^2_{-r})$, $T>0$. Then we can show similar estimates on $P_{u,t}$ to (7.5)–(7.7) by using the bound (5.4) again. For example, (7.5) is replaced by

$$|||P_{u,t}(Z)|||_{2,-r} \leqq C_T \int_u^t (v-u)^{-1/2} |||Z_{v,t}|||_{2,-r}\, dv. \tag{7.9}$$

Therefore, noting $e^{-A(t-u)}\varphi\in C(\mathbf{D}_T, \mathscr{C}^2_{-r})$, the proof of the assertion (i) is concluded quite similarly to that of Lemma 7.1. For the proof of (ii) we set $\bar{Z}_{u,t}$ the RHS of (7.2) with $\Delta Z_{v,t}$ replaced by the solution $Y_{v,t}$ of (7.3). Then we see easily that $\bar{Z}_{u,t} \in C(\mathbf{D}_T, \mathscr{C}^2_{-r})$, $T>0$, satisfies $\Delta \bar{Z}_{u,t} = Y_{u,t}$ and therefore it is a solution of (7.2). The conclusion now follows from the uniqueness of its solutions. □

Now consider the forward integral equation corresponding to (7.2) for $\tilde{Z}_t = \tilde{Z}_t(x; \eta, c)$, $0 \leqq t < \infty$, $\eta \in \mathbf{H}_e$, $c \in \mathscr{B}_b$:

$$\tilde{Z}_t = e^{-tA}\eta + \int_0^t \Delta e^{-A(t-u)}\{c(u,\cdot)\tilde{Z}_u\}\,du, \qquad t \geqq 0. \tag{7.10}$$

Let $\mathscr{T}''$ be the class of all $\tilde{Z}_t \in C((0,\infty), \mathscr{C}_e)$ satisfying $\sup_{0 \leqq t \leqq T} t^{1/8} |||\tilde{Z}_t|||_r < \infty$ for every $T, r > 0$.

Lemma 7.3. *There exists a unique solution of* (7.10) *in the class* $\mathscr{T}''$.

Proof. Remind Lemma 5.3 to see that $e^{-tA}\eta \in C((0,\infty), \mathscr{C}_e)$ and

$$|||e^{-tA}\eta|||_r \leqq \sqrt{KK_1}\, t^{-1/8} |\eta|_{2r}, \qquad 0 < t \leqq T. \tag{7.11}$$

Denote by $\tilde{Q}_t(\tilde{Z}.) \equiv \tilde{Q}_t(x; \tilde{Z}.)$ the second term in the RHS of (7.10) for given $\tilde{Z}_t \in \mathscr{T}''$. Then the estimate (5.4) proves for $0 \leqq t_1 < t_2 \leqq T$:

$$\begin{aligned} |||\tilde{Q}_{t_1}(\tilde{Z}.) - \tilde{Q}_{t_2}(\tilde{Z}.)|||_r &\leqq K \|c\|_\infty \sup_{0 \leqq t \leqq T} t^{1/8} |||\tilde{Z}_t|||_r \\ &\cdot \left[\int_{t_1}^{t_2} u^{-1/8}(t_2-u)^{-1/2}\,du + \int_0^{t_1} 2u^{-1/8}\{(t_1-u)^{-1/2} - (t_2-u)^{-1/2}\}\,du \right], \end{aligned} \tag{7.12}$$

and this especially implies $\tilde{Q}_t(\tilde{Z}.) \in C([0,\infty), \mathscr{C}_r)$. The estimate (5.4) also proves

$$|||\tilde{Q}_t(\tilde{Z}.)|||_r \leqq K \|c\|_\infty \int_0^t (t-u)^{-1/2} |||\tilde{Z}_u|||_r\,du, \qquad 0 \leqq t \leqq T,\ \tilde{Z} \in \mathscr{T}''. \tag{7.13}$$

We can therefore complete the proof from these bounds similarly as before. □

Lemma 7.4. (i) *For every T, r and $C>0$, we have*

$$\sup\{t^{1/8} |||\tilde{Z}_t(\cdot;\eta,c)|||_r;\, 0 \leqq t \leqq T, \eta \in \mathbf{H}_e : |\eta|_{2r} \leqq C, c \in \mathscr{B}_b : \|c\|_\infty \leqq C\} < \infty. \tag{7.14}$$

(ii) *For each $\eta \in \mathbf{H}_e$ and r, $C>0$, the family $\{\tilde{Z}_t(\cdot;\eta,c);\ \|c\|_\infty \leqq C\}$ is relatively compact in the space $C((0,\infty), \mathscr{C}_r)$.*

Proof. The uniform estimate (7.14) follows by using the bounds (7.11) and (7.13). For the proof of (ii), we see from (5.4) that $\left|\left|\left| \frac{\partial}{\partial x} \tilde{Q}_t(\cdot; \tilde{Z}.) \right|\right|\right|_r$, $0 \leqq t \leqq T$, is bounded by

$$K \|c\|_\infty \{\sup_{0 \leqq u \leqq T} u^{1/8} |||\tilde{Z}_u|||_r\} \int_0^t u^{-1/8}(t-u)^{-3/4}\,du$$

and this proves with the help of (7.14) the equicontinuity of the family $\{\tilde{Q}_t(x; \tilde{Z}.(\cdot;\eta,c));\ \|c\|_\infty \leqq C\}$ in x. The equicontinuity in $t \in (0,\infty)$ follows from

(7.12) and (7.14). We therefore see that the family $\{\tilde{Z}_t(\cdot\,;\eta, c);\ \|c\|_\infty \leqq C\}$ is relatively compact in the space $C((0,\infty),\mathscr{C})$. However, it is now easy to obtain the conclusion by noting the uniform estimate (7.14) which holds for every $r>0$. □

Finally in this paragraph, we notice the continuity of $\tilde{Z}_t(\cdot\,;\eta, c)$ in $\eta\in\mathbf{H}_e$ and $c\in\mathscr{B}_b$. For this purpose, we define for $c, \bar{c}\in\mathscr{B}_b$:

$$d(c,\bar{c}) = \sup_{0\leqq t\leqq T,\, x\in\mathbb{R}} |c(t,x)-\bar{c}(t,x)|(1+|x|)^{-1}, \qquad T>0. \tag{7.15}$$

Lemma 7.5. *For every T, r and $C>0$, there exists a positive constant M such that*

$$\sup_{0\leqq t\leqq T} t^{1/8} |||\tilde{Z}_t(\cdot\,;\eta, c)-\tilde{Z}_t(\cdot\,;\bar{\eta}, c)|||_r \leqq M|\eta-\bar{\eta}|_{2r},$$

$$\eta, \bar{\eta}\in\mathbf{H}_e,\quad c\in\mathscr{B}_b\colon \|c\|_\infty\leqq C, \tag{7.16}$$

and

$$\sup_{0\leqq t\leqq T} |||\tilde{Z}_t(\cdot\,;\eta, c)-\tilde{Z}_t(\cdot\,;\eta, \bar{c})|||_r \leqq M\, d(c,\bar{c}),$$

$$\eta\in\mathbf{H}_e\colon |\eta|_{2r}\leqq C,\quad c, \bar{c}\in\mathscr{B}_b\colon \|c\|_\infty,\ \|\bar{c}\|_\infty\leqq C. \tag{7.17}$$

Proof. The estimate (7.16) follows by using (7.11) and (7.13). To prove (7.17), assume $\|c\|_\infty, \|\bar{c}\|_\infty\leqq C$. Then (5.4) can be used to show

$$|||\tilde{Z}_t(\cdot\,;\eta, c)-\tilde{Z}_t(\cdot\,;\eta, \bar{c})|||_r$$

$$\leqq K\int_0^t (t-u)^{-1/2}\{C|||\tilde{Z}_u(\cdot\,;\eta, c)-\tilde{Z}_u(\cdot\,;\eta, \bar{c})|||_r$$

$$+a d(c,\bar{c})|||\tilde{Z}_u(\cdot\,;\eta, \bar{c})|||_{r'}\}\,du, \tag{7.18}$$

where $0<r'<r$ and $a=\sup_{y\in\mathbb{R}}\{1+|y|\}\,\theta(y, r-r')<\infty$. The recursive usage of (7.18) gives (7.17) with the help of (7.14). □

7.2 Construction of Fundamental Solutions

A function $Z_{u,t}(x,y)\equiv Z_{u,t}(x,y;c)$ is called a fundamental solution of the integral Eq. (7.2) if its unique solution $Z_{u,t}(x,\varphi)\equiv Z_{u,t}(x,\varphi;c)$ can be represented by $Z_{u,t}(x,\varphi)=\int_{\mathbb{R}}\varphi(y)Z_{u,t}(x,y)\,dy$.

For the construction of the fundamental solution we define $\{Z_{u,t}^{(n)}(x, y) \equiv Z_{u,t}^{(n)}(x, y; c)\}_{n=0}^{\infty}$ inductively by

$$Z_{u,t}^{(0)}(x, y) = q(t-u, y-x),$$
$$Z_{u,t}^{(n+1)}(x, y) = \int_u^t dv \int_{\mathbb{R}} q(v-u, z-x)\, c(v, z)\, \Delta_z Z_{v,t}^{(n)}(z, y)\, dz, \tag{7.19}$$
$$n = 0, 1, 2, \ldots.$$

Here q is the function introduced in Sect. 5. Take L^* such that $0 < L^* < L_1$; see (5.2) for L_1.

Lemma 7.6. *An absolutely-converging series*

$$Z_{u,t}(x, y) \equiv Z_{u,t}(x, y; c) = \sum_{n=0}^{\infty} Z_{u,t}^{(n)}(x, y) \tag{7.20}$$

gives a fundamental solution of (7.2) *and has the following bounds:*

$$\left|\frac{\partial^k}{\partial x^k} Z_{u,t}(x, y; c)\right| \leqq K^*(t-u)^{-\frac{1+k}{4}} e^{-L^*\rho},$$
$$k = 0, 1, 2, 3,\ 0 \leqq u \leqq t \leqq T,\ \|c\|_\infty \leqq C, \tag{7.21}$$

for every $T, C > 0$, where $\rho = \left\{\frac{|x-y|^4}{t-u}\right\}^{1/3}$ and K^* is a constant depending only on T and C.

Proof. The inductive method can be used to establish

$$\left|\frac{\partial^k}{\partial x^k} Z_{u,t}^{(n)}(x, y)\right| \leqq \frac{a_{1,k} a_2^n}{\Gamma\left(\frac{3}{4} - \frac{k}{4} + \frac{n}{2}\right)} (t-u)^{-\frac{1+k}{4} + \frac{n}{2}} e^{-L^*\rho},$$
$$0 \leqq u \leqq t \leqq T, \tag{7.22}$$

for $(n, k) = (0, 2)$ and for $n \in \mathbb{N}$, $k = 0, 1, 2, 3$. Here

$$a_{1,k} = K_1 \Gamma(1/4) \Gamma(1-k/4)/\Gamma(1/2),$$
$$a_2 = K_1 \Gamma(1/2) \|c\|_\infty F,$$
$$F = \int_{\mathbb{R}} e^{-(L_1 - L^*)|z|^{4/3}} dz,$$

and K_1 is the constant appearing in (5.2) (see [Fu] for detail). The estimate (7.22) proves the absolute convergence of the series (7.20) itself and its derivatives in x up to the third order and the bounds (7.21) on the function $Z_{u,t}(x, y)$. Now it is easy to see that $Z_{u,t}(x, y)$ constructed in this manner is the fundamental solution of (7.2); use the uniqueness result in Lemma 7.2 taking $\varepsilon = 1/2$. □

Next we prove the continuity property of $Z_{u,t}(x, y; c)$ in c.

Lemma 7.7. *The following estimates hold:*

$$\left|\frac{\partial^k}{\partial x^k} Z_{u,t}(x, y; c) - \frac{\partial^k}{\partial x^k} Z_{u,t}(x, y; \bar{c})\right|$$
$$\leq K^{**} d(c, \bar{c}) \{1+|x|+|y|\} (t-u)^{-\frac{1+k}{4}} e^{-L^* \rho},$$
$$k=0, 1, 2, 3, \ 0 \leq u \leq t \leq T; \ \|c\|_\infty, \|\bar{c}\|_\infty \leq C,$$

where the constant K^{**} depends only on T and C.

Proof. The conclusion follows by showing

$$\left|\frac{\partial^k}{\partial x^k} Z^{(n)}_{u,t}(x, y; c) - \frac{\partial^k}{\partial x^k} Z^{(n)}_{u,t}(x, y; \bar{c})\right|$$
$$\leq \frac{d(c, \bar{c})\, a_{1,k}\, \tilde{a}_2^n\, C^{n-1}}{\Gamma\left(\frac{3}{4} - \frac{k}{4} + \frac{n}{2}\right)} \{1+|x|+|y|\} (t-u)^{-\frac{1+k}{4}+\frac{n}{2}} e^{-L^* \rho},$$

for $n \in \mathbb{N}$ and $k=0, 1, 2, 3$. Here $\tilde{a}_2 = 2K_1 \Gamma(1/2) F_T$ and

$$F_T = F + T^{1/4} \int |z| \exp\{-(L_1 - L^*)|z|^{4/3}\}\, dz.$$

Use the induction again (see [Fu] if necessary). □

Under an appropriate smoothness assumption on the coefficient c $Z_{u,t}(x, \varphi)$ actually solves the backward Eq. (7.1). Moreover, if we put

$$Z_{u,t}(\eta, y) \equiv Z_{u,t}(\eta, y; c) = \int \eta(x) Z_{u,t}(x, y; c)\, dx, \qquad \eta \in \mathbf{H}_r, \ r>0, \tag{7.23}$$

then $Z_{u,t}(\varphi, y)$ gives a solution of the corresponding forward equation:

$$\begin{aligned} \frac{\partial}{\partial t} Z_{u,t} &= \mathscr{L}^*_{t,y} Z_{u,t}, \qquad t \in [u, \infty), \\ Z_{u,u} &= \varphi \in C_0^\infty(\mathbb{R}), \end{aligned} \tag{7.24}$$

where $\mathscr{L}^*_{t,y} = -A + \Delta_y\{c(t, y)\cdot\}$. See Eidel'man [7].

Corollary 7.1. *For every $\eta \in \mathbf{H}_e$ and $c \in C_b([0, \infty) \times \mathbb{R})$, we have*

$$Z_{0,t}(\eta, x; c) = \tilde{Z}_t(x; \eta, c). \tag{7.25}$$

Proof. First we prove (7.25) for $\eta \in C_0^\infty(\mathbb{R})$ and $c \in C^\infty(\mathbb{R}) \cap \mathscr{B}_b$. Indeed, in this case, both sides of (7.25) are solutions of the PDE (7.24) with $u=0$ and $\varphi = \eta$ satisfying $\sup_{0 \leq t \leq T} |||Z_t|||_r < \infty$ for every T, $r>0$. However, it is known the uniqueness of such solutions; see Eidel'man [7]. Now the conclusion for general η and c follows from the continuity of the both sides of (7.25) in $\eta \in \mathbf{H}_e$ and $c \in C_b([0, \infty) \times \mathbb{R})$; see Lemmas 7.5 and 7.7. □

8. A Formula of Integration by Parts

The solution S_t^ε of the scaled TDGL Eq. (1.3) determines a Markov process on the space $\mathbf{H}_e$. The (formal) generator $\mathscr{G}^\varepsilon$ of this process is given by

$$\mathscr{G}^\varepsilon \Psi(S) = \sum_{i=1}^{k} \frac{\partial \psi}{\partial \alpha_i}(\langle S, \varphi_1\rangle, \ldots, \langle S, \varphi_k\rangle)\{-\varepsilon^2\langle S, \Delta^2\varphi_i\rangle + \langle U'(S(\cdot)), \Delta\varphi_i\rangle\}$$

$$+\varepsilon \sum_{i,j=1}^{k} \frac{\partial^2 \psi}{\partial\alpha_i\,\partial\alpha_j}(\langle S, \varphi_1\rangle, \ldots, \langle S, \varphi_k\rangle)\langle -\Delta\varphi_i, \varphi_j\rangle, \qquad S\in\mathbf{H}_e, \quad (8.1)$$

for $\Psi\in\mathscr{D}$ having the form (2.2); especially $\Psi(S_t^\varepsilon) - \int_0^t \mathscr{G}^\varepsilon \Psi(S_u^\varepsilon)\,du$ is a martingale for every $\Psi\in\mathscr{D}$. We denote $E_S[\Psi(S_t^\varepsilon)]$ by $T_t^\varepsilon \Psi(S)$ or simply by $\Psi_t^\varepsilon(S)$, where $E_S[\cdot]$ stands for an expectation with respect to the probability distribution of the process S_t^ε starting from $S\in\mathbf{H}_e$. Assume the profile function $\lambda=\lambda(\cdot)\in\Lambda$ is given and let $\mu_{\lambda(\cdot),\varepsilon}$ be the scaled $(U, \lambda(\cdot))$-Gibbs distribution constructed in Sect. 3.

We shall say as usual a real valued function Φ on the space $\mathbf{H}_e$ Fréchet differentiable if $\Phi(S+\delta\eta)$, $\delta\in\mathbb{R}$, is differentiable at $\delta=0$ for every S, $\eta\in\mathbf{H}_e$ and the derivative has an expression $\frac{d}{d\delta}\Phi(S+\delta\eta)|_{\delta=0} = \langle D\Phi(\cdot, S), \eta\rangle$, $\eta\in\mathbf{H}_e$, with some $D\Phi(\cdot, S)\in\mathbf{H}_e^*$. The purpose of the present section is to prove the following formula.

Theorem 8.1. (i) *For every $t>0$ and $\Psi\in\mathscr{D}$, Ψ_t^ε is Fréchet differentiable on $\mathbf{H}_e$ and the following formula holds:*

$$E^{\mu_{\lambda(\cdot),\varepsilon}}[\mathscr{G}^\varepsilon \Psi(S_t^\varepsilon)] = E^{\mu_{\lambda(\cdot),\varepsilon}}[\langle \Delta\lambda(\cdot), D\Psi_t^\varepsilon(\cdot, S)\rangle], \qquad (8.2)$$

where the LHS *is an expectation with respect to the distribution of the process S_t^ε having initial distribution $\mu_{\lambda(\cdot),\varepsilon}$ and the* RHS *is an integration with respect to $\mu_{\lambda(\cdot),\varepsilon}(dS)$ over the space $\mathbf{H}_e$.*

(ii) *The Fréchet derivative $D\Psi_t^\varepsilon(x, S)$ of Ψ_t^ε can be expressed explicitly as follows:*

$$D\Psi_t^\varepsilon(x, S) = \sum_{i=1}^{k} E_S\left[\frac{\partial\psi}{\partial\alpha_i}(\langle S_t^\varepsilon, \varphi_1\rangle, \ldots, \langle S_t^\varepsilon, \varphi_k\rangle)\, Z_t^\varepsilon(x, \varphi_i; S_\cdot^\varepsilon)\right]. \qquad (8.3)$$

Here, for $\varphi\in C_0^\infty(\mathbb{R})$ and $S_\cdot\in\mathscr{B}([0,\infty)\times\mathbb{R})$, we denote by $Z_t^\varepsilon(x, \varphi; S_\cdot)$ the solution $Z_{0,t}^\varepsilon(x, \varphi; V''(S_\cdot))$ of the integral Eq. (7.2) with $c(u,x)=V''(S_u(x))$ and with A replaced by $A^\varepsilon=\varepsilon^2\Delta-\gamma\Delta$.

The proof of the theorem will be completed after the following three main steps: We may assume $\varepsilon=1$ without loss of generality. The first step is to prove the corresponding formula for the finite-dimensional process $S_t^{(N)}\in L_l^2=$

$L^2([-l, l])$ which was constructed in Sect. 6. Then it is derived the formula for the solution S_t^l of the SPDE (6.1) by taking the limit of $N\to\infty$ and finally we take the limit of $l\to\infty$.

8.1 A Formula for $S_t^{(N)}$

The N-dimensional process $S_t^{(N)}=\sum_{n=1}^{N} a_n^{(N)}(t)\, e_n\in L_l^2$ was constructed by solving the SDE (6.7). We shall fix $N\in\mathbb{N}$ throughout this paragraph. The generator of the diffusion process $a_t=\{a_n^{(N)}(t)\}_{n=1}^N\in\mathbb{R}^N$ is given by

$$\mathscr{A}\equiv\mathscr{A}^{(N)}=\sum_{n=1}^{N}\left[\kappa_n\frac{\partial^2}{\partial a_n^2}-\left\{\kappa_n^2 a_n+\kappa_n\left\langle U'\left(\sum_{m=1}^{N} a_m e_m(\cdot)\right), e_n(\cdot)\right\rangle_l\right\}\frac{\partial}{\partial a_n}\right].$$

Define a finite measure $\tilde{\mu}^{(N)}$ on $\mathbb{R}^N$ by

$$\tilde{\mu}^{(N)}(da)=\exp\left\{-\int_{-l}^{l} U\left(\sum_{n=1}^{N} a_n e_n(x)\right)dx-\frac{1}{2}\sum_{n=1}^{N}\kappa_n a_n^2\right\}da,\qquad da=\prod_{n=1}^{N} da_n,$$

then $\tilde{\mu}^{(N)}$ is a reversible measure for the process a_t:

$$\begin{aligned}\int\mathscr{A}f(a)\,g(a)\,\tilde{\mu}^{(N)}(da)&=\int f(a)\,\mathscr{A}g(a)\,\tilde{\mu}^{(N)}(da)\\&=-\sum_{n=1}^{N}\kappa_n\int\frac{\partial f}{\partial a_n}\frac{\partial g}{\partial a_n}\,\tilde{\mu}^{(N)}(da),\end{aligned}\tag{8.4}$$

for every $f, g\in C_b^2(\mathbb{R}^N)$. We set $f_t(a)\equiv\tilde{T}_t^{(N)}f(a)=E_a[f(a_t)]$, $a\in\mathbb{R}^N$, $t\geqq 0$, $f\in C_b(\mathbb{R}^N)$, the expectation with respect to the distribution of a_t starting from a and

$$\tilde{\mu}_{\lambda(\cdot)}^{(N)}(da)=\exp\left\{\sum_{n=1}^{N}\langle\lambda, e_n\rangle_l\, a_n\right\}\tilde{\mu}^{(N)}(da).$$

Lemma 8.1. $E^{\tilde{\mu}_{\lambda(\cdot)}^{(N)}}[\mathscr{A}^{(N)}f(a_t)]=E^{\tilde{\mu}_{\lambda(\cdot)}^{(N)}}\left[\sum_{n=1}^{N}\langle\lambda, \Delta e_n\rangle_l\frac{\partial f_t}{\partial a_n}(a)\right]$, $f\in C_0^2(\mathbb{R}^N)$.

Proof. Since $\tilde{T}_t^{(N)}\mathscr{A}^{(N)}f=\mathscr{A}^{(N)}\tilde{T}_t^{(N)}f$ for $f\in C_0^2(\mathbb{R}^N)$, the conclusion follows by taking $f=f_t$ and $g=\exp\left\{\sum_{n=1}^{N}\langle\lambda, e_n\rangle_l\, a_n\right\}$ in (8.4). □

Define a mapping $\not{p}_N:\mathbb{R}^N\to\Pi_N L_l^2$ by $\not{p}_N a=\sum_{n=1}^{N} a_n e_n$, $a=\{a_n\}_{n=1}^N$, and set $(\not{p}^N\Psi)(a)=\Psi(\not{p}_N a)$, $a\in\mathbb{R}^N$, for functions Ψ on $\Pi_N L_l^2$ (or L_l^2). Here Π_N is the orthogonal projection of the space L_l^2 onto its subspace spanned by $\{e_n\}_{n=1}^N$.

Let $\mathscr{D}_l$ be the class of tame functions on L_l^2, which was introduced in Sect. 2. We introduce an operator $\mathscr{G}^{(N)}$ by

$$\mathscr{G}^{(N)}\Psi(S)=\sum_{i=1}^{k}\frac{\partial\psi}{\partial\alpha_i}(\langle S,\varphi_1\rangle_l,\ldots,\langle S,\varphi_k\rangle_l)\cdot\{-\langle S,\Delta^2\Pi_N\varphi_i\rangle_l+\langle U'(S(\cdot)),\Delta\Pi_N\varphi_i\rangle_l\}$$
$$+\sum_{i,j=1}^{k}\frac{\partial^2\psi}{\partial\alpha_i\,\partial\alpha_j}(\langle S,\varphi_1\rangle_l,\ldots,\langle S,\varphi_k\rangle_l)\langle-\Delta\Pi_N\varphi_i,\varphi_j\rangle_l,\qquad S\in L_l^2,$$

for $\Psi\in\mathscr{D}_l$ having the form (2.2). The operators $\mathscr{G}^{(N)}$ and $\mathscr{A}^{(N)}$ are linked by the following relation:

$$\not{p}^N\mathscr{G}^{(N)}\Psi=\mathscr{A}^{(N)}\not{p}^N\Psi,\qquad \Psi\in\mathscr{D}_l. \tag{8.5}$$

Let $\mu_{\lambda(\cdot)}^{(N)}=\tilde{\mu}_{\lambda(\cdot)}^{(N)}\circ\not{p}_N^{-1}$ be an image measure on $\Pi_N L_l^2$ of $\tilde{\mu}_{\lambda(\cdot)}^{(N)}$ under the mapping $\not{p}_N$. The derivative $D^{(N)}\Phi(x,S)$, $x\in[-l,l]$, $S\in\Pi_N L_l^2$, of a function $\Phi=\Phi(S)$ on the space $\Pi_N L_l^2$ (or L_l^2) is defined by

$$D^{(N)}\Phi(x,S)=\sum_{n=1}^{N}e_n(x)\frac{\partial(\not{p}^N\Phi)}{\partial a_n}(\not{p}_N^{-1}S),$$

when the RHS exists, where $\not{p}_N^{-1}:\Pi_N L_l^2\to\mathbb{R}^N$ is an inverse mapping of $\not{p}_N$. We set $\Psi_t^{(N)}(S)\equiv T_t^{(N)}\Psi(S)=E_S[\Psi(S_t^{(N)})]$, $S\in\Pi_N L_l^2$, $\Psi\in\mathscr{D}_l$, which will be sometimes considered as a function on the space L_l^2 by putting $\Pi_N S$ instead of S in the RHS. Then we have the following proposition as an immediate consequence of Lemma 8.1 and (8.5).

Proposition 8.1. *For every* $\Psi\in\mathscr{D}_l$,

$$E^{\mu_{\lambda(\cdot)}^{(N)}}[\mathscr{G}^{(N)}\Psi(S_t^{(N)})]=E^{\mu_{\lambda(\cdot)}^{(N)}}[\langle\lambda(\cdot),\Delta D^{(N)}\Psi_t^{(N)}(\cdot,S)\rangle_l]. \tag{8.6}$$

Before concluding this paragraph we give useful representations of $\mu_{\lambda(\cdot)}^{(N)}$ and $\Delta D^{(N)}\Psi_t^{(N)}(x,S)$, respectively.

Lemma 8.2. *With some positive constant* c_N, *we have*

$$\mu_{\lambda(\cdot)}^{(N)}(dS)=c_N\exp\{\langle\lambda,S\rangle_l-\langle U(S),1\rangle_l\}(\mu\circ\Pi_N^{-1})(dS),\qquad S\in\Pi_N L_l^2,$$

where $\mu=\mu_{-l,0;l,0}$; *see Sect.* 3.

Proof. The conclusion follows from the Wiener's representation of the Brownian motion: μ is the distribution of $\tilde{S}(x;\{a_n\})=\sum_{n=1}^{\infty}a_n e_n(x)$, $x\in[-l,l]$, which is realized on a probability space $\left(\mathbb{R}^\infty,\prod_{n=1}^{\infty}\sqrt{\kappa_n/2\pi}\exp\{-\kappa_n a_n^2/2\}\,da_n\right)$. □

Consider the following backward integral equation for $Y_{u,t}^{(N)} = Y_{u,t}^{(N)}(x, \varphi; S.)$, $0 \leqq u \leqq t < \infty$, with given $\varphi \in C_0^\infty(-l, l)$ and $S. \in C([0, \infty), L_l^2)$:

$$Y_{u,t}^{(N)} = e^{-A^l(t-u)} \Pi_N(\Delta \varphi) + \int_u^t \Delta e^{-A^l(v-u)} \Pi_N\{V''(S_v) Y_{v,t}^{(N)}\} \, dv, \tag{8.7}$$

where A^l is the operator on L_l^2 introduced in Sect. 6.

Lemma 8.3. (i) *The integral Eq. (8.7) has a unique solution satisfying $Y_{\cdot,t}^{(N)} \in C([0, t], \Pi_N L_l^2)$ for each $t \geqq 0$.*

(ii) *For every $\Psi \in \mathscr{D}_l$, $\Delta D^{(N)} \Psi_t^{(N)}(x, S)$* is represented as

$$\sum_{i=1}^k E_S\left[\frac{\partial \psi}{\partial \alpha_i}(\langle S_t^{(N)}, \varphi_1\rangle_l, \ldots, \langle S_t^{(N)}, \varphi_k\rangle_l) \, Y_{o,t}^{(N)}(x, \varphi_i; S_\cdot^{(N)})\right]. \tag{8.8}$$

Proof. Through the mappings $\not{p}_N$ and $\not{p}_N^{-1}$, (8.7) is rewritten into an equivalent linear backward ODE on $\mathbb{R}^N$, for which the existence and uniqueness results are established easily. For the proof of (ii), let us denote the solution of SDE (6.7) starting at time u (instead of time 0) from the point $a \in \mathbb{R}^N$ by $a_{u,t}(a) = \{a_{u,t;n}(a)\}_{n=1}^N$, $0 \leqq u \leqq t < \infty$. Put $\bar{Y}_{u,t;n}(a; \varphi) = -\kappa_n \sum_{m=1}^N b_{u,t;m}^n \langle e_m, \varphi\rangle$ and $\bar{Y}_{u,t}(a; \varphi) = \sum_{n=1}^N \bar{Y}_{u,t;n}(a; \varphi) \, e_n$, where $b_{u,t;m}^n = \partial a_{u,t;m}(a)/\partial a_n$, $1 \leqq n, m \leqq N$. Then, deriving a backward ODE for the system $\{b_{u,t;m}^n\}$, we can verify that $\{\bar{Y}_{u,t;n}\}_{n=1}^N$ satisfies the same ODE as $\not{p}_N^{-1} Y_{u,t}^{(N)}(\cdot, \varphi; S_\cdot^{(N)})$, where $S_\cdot^{(N)}$ is the process determined by the SDE (6.7). Therefore we obtain $\bar{Y}_{u,t}(\not{p}_N^{-1} S; \varphi) = Y_{u,t}^{(N)}(\cdot, \varphi; S(N))$. However, since it is easy to see that $\Delta D^{(N)} \Psi_t^{(N)}(x, S)$ is given by (8.8) with $Y_{0,t}^{(N)}(x, \varphi_i; S_\cdot^{(N)})$ replaced by $\bar{Y}_{0,t}(\not{p}_N^{-1} S; \varphi_i)$, we have the conclusion. □

8.2 *A formula for S_t^l*

Let $\varphi \in C_0^\infty(-l, l)$ and $S. = S_t(x) \in \mathscr{B}^l = \mathscr{B}([0, \infty) \times [-l, l])$, the class of all measurable functions on $[0, \infty) \times [-l, l]$, be given and consider the following backward integral equation for $Y_{u,t}^l = Y_{u,t}^l(x, \varphi; S.)$, $0 \leqq u \leqq t < \infty$, in the space $\mathscr{C}_{-r,l}$, $r > 0$:

$$Y_{u,t}^l = e^{-A^l(t-u)}(\Delta \varphi) + \int_u^t \Delta e^{-A^l(v-u)} \{V''(S_v) Y_{v,t}^l\} \, dv, \qquad 0 \leqq u \leqq t \tag{8.9}$$

See Lemma 8.6 below for the existence and uniqueness result for this integral equation. We *define* $\Delta D \Psi_t^l(x, S)$, $x \in [-l, l]$, $S \in L_l^2$, for $\Psi \in \mathscr{D}_l$ by

$$\Delta D \Psi_t^l(x, S) = \sum_{i=1}^k E_S\left[\frac{\partial \psi}{\partial \alpha_i}(\langle S_t^l, \varphi_1\rangle_l, \ldots, \langle S_t^l, \varphi_k\rangle_l) \, Y_{0,t}^l(x, \varphi_i; S_\cdot^l)\right], \tag{8.10}$$

the expectation with respect to the distribution of the solution $S^l_\cdot = \{S^l_t; t \geqq 0\}$ of (6.1) having an initial condition $S^l_0 = S$. We shall prove the following proposition.

Proposition 8.2. *For every* $\Psi \in \mathscr{D}_l$, we have

$$E^{\mu_{\bar{\lambda}(\cdot)}^{l,l}(\cdot;0,0)}[\mathscr{G}\Psi(S^l_t)] = E^{\mu_{\bar{\lambda}(\cdot)}^{l,l}(\cdot;0,0)}[\langle \lambda(\cdot), \Delta D\Psi^l_t(\cdot, S)\rangle_l]. \tag{8.11}$$

The LHS of (8.11) is the expectation with respect to the distribution of S^l_t having initial distribution $\mu_{\lambda(\cdot)}^{-l,l}(\cdot; 0, 0)$, the local specification introduced in Sect. 3, and $\mathscr{G}\Psi(S)$ is defined by replacing $\langle\,,\rangle$ with $\langle\,,\rangle_l$ in (8.1). The proof of the proposition will be completed by taking the limit $N \to \infty$ of the both sides of (8.6). In this paragraph we denote the norm of the space L^2_l simply by $\|\cdot\|$.

Lemma 8.4. *We have the following estimates for each* $\Psi \in \mathscr{D}_l$ *with a sequence* $\{\beta_N = \beta_N(\Psi) \downarrow 0\}_{N=1}^{\infty}$ *and a positive constant* $C = C(\Psi)$:

$$|\mathscr{G}^{(N)}\Psi(S) - \mathscr{G}\Psi(S)| \leqq \beta_N\{1 + \|S\|\}, \qquad S \in L^2_l, \tag{8.12}$$

$$|\mathscr{G}\Psi(S)| \leqq C(1 + \|S\|), \qquad S \in L^2_l, \tag{8.13}$$

$$|\mathscr{G}\Psi(S_1) - \mathscr{G}\Psi(S_2)| \leqq C(1 + \|S_1\| + \|S_2\|)\,\|S_1 - S_2\|, \qquad S_1, S_2 \in L^2_l. \tag{8.14}$$

This lemma can be shown by simple calculations so that we omit the proof. Now we can prove the convergence of the LHS of (8.6).

Lemma 8.5. *For every* $\Psi \in \mathscr{D}_l$, *we have*

$$\lim_{N \to \infty} c_N^{-1} E^{\mu_{\lambda(\cdot)}^{(N)}}[\mathscr{G}^{(N)}\Psi(S^{(N)}_t)] = \Xi^l E^{\mu_{\lambda(\cdot)}^{-l,l}(\cdot;0,0)}[\mathscr{G}\Psi(S^l_t)], \tag{8.15}$$

where $\Xi^l = \Xi_{\lambda(\cdot)}^{-l,l}(0, 0)$; *see Sect.* 3.

Proof. First we note that Theorem 6.2 combined with Lemma 8.4 proves $\lim_{N \to \infty} E_{\Pi_N S}[\mathscr{G}^{(N)}\Psi(S^{(N)}_t)] = E_S[\mathscr{G}\Psi(S^l_t)]$ for every $S \in L^2_l$ and $\Psi \in \mathscr{D}_l$. Therefore we complete the proof from Lemma 8.2 by using Lebesgue's dominated convergence theorem. $\square$

The second task is to show the convergence of the RHS of (8.6). We discuss the integral Eq. (8.9).

Lemma 8.6. (i) *For every* $r > 0$ *and* $T > 0$, *there exists a solution of* (8.9) *satisfying* $Y^l_{u,t} \in C(\mathbf{D}_T, \mathscr{C}_{-r,l})$.

(ii) *The uniqueness of solutions of* (8.9) *holds in the class of measurable functions* $Y^l_{u,t}$ *satisfying*

$$\sup_{0 \leqq u \leqq t \leqq T} |||Y^l_{u,t}|||_{-r}(t-u)^{1-\varepsilon} < \infty, \qquad T > 0,$$

with some $0 < \varepsilon < 1$.

(iii) *The following uniform bound holds for every* $\varphi \in C_0^\infty(-\bar{l}, \bar{l})$, $\bar{l} \in \mathbb{N}$:

$$\sup\{|||Y_{u,t}^l(\cdot, \varphi; S.)|||_{-r}; l \geqq \bar{l}, (u,t) \in \mathbf{D}_T, S. \in \mathscr{B}^l\} < \infty. \tag{8.16}$$

Here $|||\cdot|||_{-r}$ *denotes the norm of the space* $\mathscr{C}_{-r,l}$; *see Sect.* 6.

Proof. The proof of (i) and (ii) is completed in a quite parallel manner to that of Lemma 7.1. Actually we can derive similar estimates to (7.5)–(7.7) by using (6.5) instead of (5.4). Note that the constant C_T appearing in these estimates, especially in (7.5), can be taken independent of l. Therefore we get from Lemma 6.2 (ii) (see also its proof):

$$|||Y_{u,t}^l|||_{-r} \leqq M e^{\delta T} |||\Delta\varphi|||_{-r} + C_T \int_u^t (v-u)^{-1/2} |||Y_{v,t}^l|||_{-r}\, dv, \qquad 0 \leqq u \leqq t \leqq T.$$

The recursive usage of this inequality proves (8.16). □

Every solution of (8.9) satisfies $Y_{u,t}^l(\pm l)=0$ and therefore $Y_{u,t}^l \in C(\mathbf{D}_T, \hat{\mathscr{C}}_{-r,l})$. We prepare the following estimates on $\{Y_{u,t}^l\}$:

Lemma 8.7. *For every* $T>0$, *there exists a positive constant* C_T *such that*

$$\|Y_{u,t}^l\| \leqq C_T \|\Delta\varphi\|, \tag{8.17}$$

$$\|Y_{u,t}^l\|_\infty \leqq C_T\{1+(t-u)^{-1/8}\}\|\Delta\varphi\|, \qquad 0 \leqq u \leqq t \leqq T. \tag{8.18}$$

Proof. Since the operator norms of $e^{-A^l t}$ and $\Delta e^{-A^l t}$ on the space L_l^2 are bounded by 1 and $\sup_{\kappa>0} \kappa \exp\{-t(\kappa^2+\gamma\kappa)\}$ $(\leqq 1/\sqrt{2t})$ respectively, the bound (8.17) follows easily from the equation (8.9). To show (8.18), we notice that the operator norms of $e^{-A^l t}$ and $\Delta e^{-A^l t}$: $L_l^2 \to (C([-l,l]), \|\cdot\|_\infty)$ are bounded by $Ct^{-1/8}$ and $Ct^{-5/8}$, respectively, with C independent of t and $l \in \mathbb{N}$. In fact, this is shown by using the Fourier series expansions. We can therefore estimate $\|Y_{u,t}^l\|_\infty$ by (8.9) with the help of (8.17). □

To make initial values clear, we shall denote by $S_t^{(N)}(S)$ and $S_t^l(S)$ the processes determined by (6.7) respectively (6.1) such that $S_0^{(N)}(S)=S$ and $S_0^l(S)=S$.

Lemma 8.8. *For every* $0 \leqq u \leqq t < \infty$, $\varphi \in C_0^\infty(-l,l)$ *and* $S \in L_l^2$, $Y_{u,t}^{(N)} = Y_{u,t}^{(N)}(\cdot, \varphi; S_\cdot^{(N)}(\Pi_N S))$ *converges to* $Y_{u,t}^l = Y_{u,t}^l(\cdot, \varphi; S_\cdot^l(S))$ *as* $N \to \infty$ *in the following sense:* $\lim_{N\to\infty} E[\|Y_{u,t}^{(N)} - Y_{u,t}^l\|^2]=0$.

Proof. By the Eqs. (8.7) and (8.9) we see that $\|Y_{u,t}^{(N)} - Y_{u,t}^l\|$ is bounded by the sum of $I_1 \equiv \|(\Pi_N - I)\Delta\varphi\|$, I_2, I_3 and I_4 defined as follows:

$$I_2 = \int_u^t \|\Delta e^{-A^l(v-u)} \Pi_N\{V''(S_v^{(N)})(Y_{v,t}^{(N)} - Y_{v,t}^l)\}\|\, dv$$

$$I_3 = \int_u^t \|\Delta e^{-A^l(v-u)} \Pi_N[\{V''(S_v^{(N)}) - V''(S_v^l)\} Y_{v,t}^l]\|\, dv$$

$$I_4 = \int_u^t \|\Delta e^{-A^l(v-u)}(-\Delta)^\beta(-\Delta)^{-\beta}(I-\Pi_N)\{V''(S_v^l)\, Y_{v,t}^l\}\|\, dv, \qquad 0<\beta<1.$$

For deriving a bound on I_3, use $\|\Delta e^{-A^l(v-u)}\|_{L_l^2\to L_l^2}\leqq 1/\sqrt{2(v-u)}$ and (8.18). Then (6.9) proves $E[\{I_3\}^2]\to 0$ as $N\to\infty$. For I_4, note that the operator norms of $(-\Delta)^{1+\beta}e^{-A^l t}$ and $(-\Delta)^{-\beta}(I-\Pi_N)$: $L_l^2\to L_l^2$ are bounded by const. $t^{-(1+\beta)/2}$ and $\kappa_N^{-\beta}$, respectively. Therefore we see $I_4\to 0$ as $N\to\infty$ from (8.17). Finally, giving an estimate on I_2, we arrive at

$$E[\|Y_{u,t}^{(N)}-Y_{u,t}^l\|^2]\leqq \delta^{(N)}+C\int_u^t (v-u)^{-1/2}E[\|Y_{v,t}^{(N)}-Y_{v,t}^l\|^2]\,dv$$

with some $\delta^{(N)}\downarrow 0$ as $N\to\infty$ and C which depend only on T, l and φ. From this inequality, it is easy to complete the proof of the lemma. □

Lemma 8.9. $\lim_{N\to\infty} c_N^{-1}E^{\mu_{\lambda(\cdot)}^{(N)}}[\langle\lambda(\cdot),\Delta D^{(N)}\Psi_t^{(N)}(\cdot,S)\rangle_l]$

$$=\Xi^l E^{\mu_{\lambda(\cdot)}^{-l,l}(\cdot;0,0)}[\langle\lambda(\cdot),\Delta D\Psi_t^l(\cdot,S)\rangle_l],\qquad \Psi\in\mathscr{D}_l.$$

Proof. Recall the expressions (8.8) and (8.10). Then, using (6.9), (8.17) and Lemma 8.8, we can prove that $\Delta D^{(N)}\Psi_t^{(N)}(\cdot,\Pi_N S)$ converges to $\Delta D\Psi_t^l(\cdot,S)$ as $N\to\infty$ in the space L_l^2 for every $t>0$ and $S\in L_l^2$. Therefore the proof is completed from Lemma 8.2; use the Lebesgue's dominated convergence theorem noting a uniform bound $\sup_{N,S}\|\Delta D^{(N)}\Psi_t^{(N)}(\cdot,\Pi_N S)\|<\infty$. This is obtained by deriving a uniform estimate on $\|Y_{u,t}^{(N)}\|$ similarly to (8.17). □

Now Proposition 8.2 follows as a combination of Proposition 8.1 with Lemmas 8.5 and 8.9.

8.3 Convergence of $Y_{u,t}^l$

Before completing the proof of Theorem 8.1, we need to examine the convergence of the solution $Y_{u,t}^l$ of the integral Eq. (8.9) as $l\to\infty$. We regard $Y_{u,t}^l\in\mathscr{C}$ by setting $Y_{u,t}^l(x)=0$ for $x\in\mathbb{R}\setminus[-l,l]$ as before.

Lemma 8.10. *For every $T>0$, $r>0$ and $\varphi\in C_0^\infty(-\bar{l},\bar{l})$, $\bar{l}\in\mathbb{N}$, the family of functions $\{Y_{u,t}^l(\cdot,\varphi;S_\cdot);\, l\geqq\bar{l},\, S_\cdot\in\mathscr{B}^l\}$ is relatively compact in the space $C(\mathring{\mathbf{D}}_T,\mathscr{C}_{-r})$ equipped with the usual topology of uniform convergence on each compact subset of $\mathring{\mathbf{D}}_T$.*

Proof. First we see that $\{Y_{u,t}^l\}$ satisfies

$$\sup\left\{\left|\frac{\partial}{\partial x}Y_{u,t}^l(x,\varphi;S_\cdot)\right|;\, x\in[-l,l],\, l\geqq\bar{l},\,(u,t)\in\mathbf{D}_T,\,|u-t|>\varepsilon,\, S_\cdot\in\mathscr{B}^l\right\}<\infty,$$

for every $\varepsilon>0$. Indeed, a bound on the derivative of the first term in the RHS of (8.9) is obtained from the estimates (5.4) combined with (6.4); while, as for the second term denoted by $Q_{u,t}^l$ in the RHS of (8.9), we may use (6.5) and then (8.16). Especially, $\{Y_{u,t}^l(x)\}$ are equicontinuous in $x\in\mathbb{R}$ for $(u,t)\in\mathring{\mathbf{D}}_T$.

Next we show the equicontinuity of $\{Y^l_{u,t}(x)\}$ in $(u,t)\in\mathring{\mathbf{D}}_T$. Actually the first term in the RHS of (8.9) is equicontinuous in (u,t), since we see

$$|e^{-A^l t'}\Delta\varphi(x)-e^{-A^l t}\Delta\varphi(x)|\leqq K'|||\Delta\varphi|||_{-r}\,\theta(x,r)\int_t^{t'} u^{-1}\,du. \tag{8.19}$$

For the second term $Q^l_{u,t}$, using (6.5), similar estimates to (7.6) and (7.7) can be derived with a constant C_T independent of l and $S_.$; we may just replace $Y_{v,t}$ by $Y^l_{v,t}$ in the RHS's. The equicontinuity of $\{Q^l_{u,t}\}$ in u follows immediately from (8.16) and the estimate like (7.6). On the other hand, the equicontinuity in t follows by using the estimate corresponding to (7.7) recursively noting (8.16) and (8.19).

Now we have shown that the family $\{Y^l_{u,t}\}$ is relatively compact in the space $C(\mathring{\mathbf{D}}_T,\mathscr{C})$, since (8.16) proves the uniform-boundedness of $\{Y^l_{u,t}\}$. The proof of the lemma therefore can be completed by noting the following fact: Generally if functions $\bar{Y}^n_{u,t}$ converge to $\bar{Y}_{u,t}$ as $n\to\infty$ in the space $C(\mathring{\mathbf{D}}_T,\mathscr{C})$ and a uniform estimate $\sup_n\sup_{(u,t)\in\mathbf{D}_T}|||\bar{Y}^n_{u,t}|||_{-r'}<\infty$ holds for $r'>0$ and $T>0$, then this convergence also holds in the space $C(\mathring{\mathbf{D}}_T,\mathscr{C}_{-r})$, $0<r<r'$. □

Lemma 8.11. *Let $\{S^l_t\in\mathscr{B}^l\}_{l=1}^\infty$ and $S_t\in\mathscr{B}([0,\infty)\times\mathbb{R})$ be functions satisfying that $(\Pi_l^{-1}S^l_t)(x)$ converges to $S_t(x)$ a.e.-$(t,x)\in[0,\infty)\times\mathbb{R}$ as $l\to\infty$, where Π_l^{-1} is a mapping of $\mathscr{B}^l$ into $\mathscr{B}$ defined similarly to in Sect. 6. Then $Y^l_{u,t}=Y^l_{u,t}(x,\varphi;S^l_.)$ converges to the solution $Y_{u,t}=Y_{u,t}(x,\varphi;V''(S_.))$ of (7.3) with $c(u,x)=V''(S_u(x))$ in the space $C(\mathring{\mathbf{D}}_T,\mathscr{C}_{-r})$, $r>0$, as $l\to\infty$ for every $\varphi\in C_0^\infty(\mathbb{R})$.*

Proof. Take an arbitrary subsequence $\{l'\}$ of $\{l\}$ such that $Y^{l'}_{u,t}$ converges to some $\bar{Y}_{u,t}$ in the space $C(\mathring{\mathbf{D}}_T,\mathscr{C}_{-r})$, $r>0$. This is possible from the preceding lemma. Note that $\sup\{|||\bar{Y}_{u,t}|||_{-r};(u,t)\in\mathbf{D}_T\}<\infty$ follows from (8.16). We shall prove that $\bar{Y}_{u,t}$ is a solution of the integral Eq. (7.3) with $c(u,x)=V''(S_u(x))$ and this completes the proof because of the uniqueness of its solutions. To this end, we may only show the following three equalities for every test function $\eta\in C_0^\infty(\mathbb{R})$:

$$\lim_{l'\uparrow\infty}\langle Y^{l'}_{u,t},\eta\rangle=\langle\bar{Y}_{u,t},\eta\rangle, \tag{8.20}$$

$$\lim_{l'\uparrow\infty}\langle\Pi_{l'}^{-1}e^{-A^{l'}(t-u)}(\Delta\varphi),\eta\rangle=\langle e^{-A(t-u)}(\Delta\varphi),\eta\rangle, \tag{8.21}$$

and

$$\lim_{l'\uparrow\infty}\int_u^t\langle V''(\Pi_{l'}^{-1}S^{l'}_v)\,Y^{l'}_{v,t},\,\Pi_{l'}^{-1}e^{-A^{l'}(v-u)}\Delta\eta\rangle\,dv$$

$$=\int_u^t\langle V''(S_v)\,\bar{Y}_{v,t},\,e^{-A(v-u)}\Delta\eta\rangle\,dv. \tag{8.22}$$

However (8.20) is trivially shown and (8.21) follows by using Lemma 6.3. Finally, (8.22) follows from Lemma 6.3 and (8.16), since $\Pi_{l'}^{-1}S^{l'}_v(x)$ converges to $S_v(x)$ a.e.-$(v,x)\in[u,t]\times\mathbb{R}$. □

8.4 The proof of Theorem 8.1

For $\Psi \in \mathscr{D}$ having the form (2.2), we *define* $\Delta D\Psi_t(x, S)$, $x \in \mathbb{R}$, $S \in \mathscr{C}_e$, by

$$\Delta D\Psi_t(x, S) = \sum_{i=1}^{k} E_S\left[\frac{\partial \psi}{\partial \alpha_i}(\langle S_t, \varphi_1\rangle, \ldots, \langle S_t, \varphi_k\rangle)\, Y_{0,t}(x, \varphi_i; S.)\right],$$

where $S. = \{S_t\}$ is the solution of the SPDE (1.1) with initial value S and $Y_{0,t}$ is the solution of the integral Eq. (7.3) with $c(u, x) = V''(S_u(x))$. We are abusing the notation somewhat here. First we show the following result:

Proposition 8.3. *For every* $\Psi \in \mathscr{D}$, *we have*

$$E^{\mu_{\lambda(\cdot)}}[\mathscr{G}\Psi(S_t)] = E^{\mu_{\lambda(\cdot)}}[\langle \lambda(\cdot), \Delta D\Psi_t(\cdot, S)\rangle]. \tag{8.23}$$

Proof. Let P^l and P be the distributions on $C([0, \infty), \mathscr{C})$ of solutions $S^l_.$ and $S.$ of (6.1) and (1.1) with initial distributions $\mu^{-l,l}_{\lambda(\cdot)}(\cdot\,; 0, 0)$ and $\mu_{\lambda(\cdot)}$, respectively. Remind Proposition 3.2 and Theorem 6.1. Then Skorohod's representation theorem can be applied to construct stochastic processes $\{\bar{S}^l_t\}$ and $\bar{S}_t$ on a proper probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ in the following manner: (i) Under $\bar{P}$, the probability distributions on the space $C([0, \infty), \mathscr{C})$ of $\bar{S}^l_.$ and $\bar{S}.$ are P^l and P, respectively. (ii) $\bar{S}^l_.$ converges almost surely to $\bar{S}.$ in the space $C((0, \infty), \mathscr{C})$ as $l \to \infty$. (iii) $\bar{S}^l_0$ converges almost surely to $\bar{S}_0$ in the space $\mathscr{C}_r$, $r > 0$. Now we may assume $\Psi \in \mathscr{D}_{\bar{l}}$ with some $\bar{l} \in \mathbb{N}$, because $\mathscr{D} = \bigcup_{l \in \mathbb{N}} \mathscr{D}_l$. Then the LHS of (8.11), which can be expressed as $E^{\bar{P}}[\mathscr{G}\Psi(\bar{S}^l_t)]$, converges to $E^{\bar{P}}[\mathscr{G}\Psi(\bar{S}_t)]$ as $l \to \infty$. Here we should note that (8.14) implies $\mathscr{G}\Psi \in C(\mathscr{C})$ and (8.13) shows the uniform integrability of $|\mathscr{G}\Psi(\bar{S}^l_t)|$ with respect to $\bar{P}$ because of Lemma 6.7. On the other hand, the convergence of the RHS of (8.11) is shown by noting Lemma 8.11 and the uniform bound (8.16). □

We write simply $\Psi_t(S) = \Psi^\varepsilon_t(S)|_{\varepsilon=1}$. In order to prove its Fréchet differentiability, we denote the solution of the SPDE (1.1) with the initial value $S \in \mathbf{H}_e$ by $S_t(S) \equiv S_t(x; S)$. Set $D^\delta S_t(x) \equiv D^\delta S_t(x; S, \eta) = \{S_t(x; S + \delta\eta) - S_t(x; S)\}/\delta$ for $0 < \delta < 1$ and $\eta \in \mathbf{H}_e$. Then we see that $D^\delta S_t$ is a solution of the integral Eq. (7.10) with $c(u, x) = V''(X^\delta_u(x))$, where X^δ is some random element of $\mathscr{B}([0, \infty) \times \mathbb{R})$. Since $D^\delta S_t \in \mathscr{T}''$ (a.s.), the uniqueness result for (7.10) implies $D^\delta S_t(x; S, \eta) = \tilde{Z}_t(x; \eta, V''(X^\delta_.))$.

Lemma 8.12. *The function* $D^\delta S_t$ *converges to* $\tilde{Z}_t(\cdot\,; \eta, V''(S.(S)))$ *as* $\delta \downarrow 0$ *in the space* $C((0, \infty), \mathscr{C}_r)$, $r > 0$, *with probability one.*

Proof. Lemma 7.4 proves the relative compactness of the family $\{D^\delta S_t\}_{0<\delta<1}$. Take an arbitrary limit $\bar{Z}_t$ of $D^\delta S_t$ in the space $C((0, \infty), \mathscr{C}_r)$. Then the uniform estimate (7.14) verifies that $\bar{Z}_t \in \mathscr{T}''$. However, since $D^\delta S_t$ satisfies

$$D^\delta S_t = e^{-tA}\eta + \int_0^t \Delta e^{-A(t-u)}\{V''(S_u(\cdot\,; S))\, D^\delta S_u(\cdot)\}\, du$$

$$+ \frac{\delta}{2}\int_0^t \Delta e^{-A(t-u)}[V'''(\tilde{X}_u(\cdot))\{D^\delta S_u(\cdot)\}^2]\, du,$$

with some $\tilde{X}_u(y) \in \mathscr{B}([0, \infty) \times \mathbb{R})$, it is easy to see that $\bar{Z}_t = \tilde{Z}_t(\cdot\,; \eta, V''(S.(S)))$ by taking the limit. This gives the conclusion. □

For $0 < \delta < 1$, $\{\Psi_t(S + \delta\eta) - \Psi_t(S)\}/\delta$ is expressed as

$$E\left[\sum_{i=1}^{k} \frac{\partial \psi}{\partial \alpha_i}(\langle S_t(S), \varphi_1\rangle, \ldots, \langle S_t(S), \varphi_k\rangle)\langle D^\delta S_t(\cdot\,; S, \eta), \varphi_i\rangle\right] + R_\delta,$$

with the remainder term R_δ having an estimate:

$$|R_\delta| \leqq \frac{\delta}{2} \sum_{i,j=1}^{k} \left\| \frac{\partial^2 \psi}{\partial \alpha_i \partial \alpha_j} \right\|_\infty E[|\langle D^\delta S_t(\cdot\,; S, \eta), \varphi_i\rangle \langle D^\delta S_t(\cdot\,; S, \eta), \varphi_j\rangle|].$$

Therefore Lemma 8.12 and Corollary 7.1 with the help of (7.14) show that $\Psi_t(S)$ is Fréchet differentiable on $\mathbf{H}_e$ and the equality (8.3) with $\varepsilon = 1$ holds. Note that the RHS of (8.3) belongs to the space $\mathbf{H}_e^*$; see Sect. 7. We have an equality $\langle \lambda(\cdot), Y_{0,t}\rangle = \langle \Delta\lambda(\cdot), Z_{0,t}\rangle$ by using integration by parts, since $\lambda(\cdot) \in \Lambda$ and $\Delta Z_{0,t} = Y_{0,t}$; see Lemma 7.2. Now the formula (8.2) with $\varepsilon = 1$ follows from (8.3) and Proposition 8.3. This completes the proof of Theorem 8.1 when $\varepsilon = 1$.

9. Basic Estimates

The purpose of this section is twofold. We derive energy estimates for solutions of parabolic equations and then give L^p-estimates on the fundamental solutions, cf. Fritz [9, 10].

9.1 Energy Inequalities

Let $q^\varepsilon(t, x - y) \equiv q^\varepsilon(t, x, y)$, $\varepsilon > 0$, be the fundamental solution of the parabolic operator $\frac{\partial}{\partial t} + A^\varepsilon$, $A^\varepsilon = \varepsilon^2 \Delta^2 - \gamma\Delta$. For $\eta \in \mathbf{H}_r$, $r > 0$, we define a function $\omega = \omega(x; \eta) \in \bigcap_{r' > r} \mathbf{H}_{r'}$ by $\omega(0) = 0$ and $\nabla\omega = \eta$. For given $\eta_0 \in \mathbf{H}_e$ and $f = f_t(x) \in C_b([0, \infty) \times \mathbb{R})$, we define two functions $\eta_t^\varepsilon(x)$ and $\zeta_t^\varepsilon(x)$ by

$$\eta_t^\varepsilon(x) = \int_{\mathbb{R}} \eta_0(y)\, q^\varepsilon(t, x, y)\, dy + \int_0^t du \int_{\mathbb{R}} q_{yy}^\varepsilon(t - u, x, y)\, f_u(y)\, dy \tag{9.1}$$

and

$$\zeta_t^\varepsilon(x) = \int_{\mathbb{R}} \omega(y; \eta_0)\, q^\varepsilon(t, x, y)\, dy - \int_0^t du \int_{\mathbb{R}} q_y^\varepsilon(t - u, x, y)\, f_u(y)\, dy. \tag{9.2}$$

Lemma 9.1. (i) *For every $t \geqq 0$, $\eta_t^\varepsilon, \zeta_t^\varepsilon \in \mathbf{H}_e$ and $\nabla\zeta_t^\varepsilon = \eta_t^\varepsilon$.*

(ii) *Assume* $f \in C_b^\infty([0, \infty)\times \mathbb{R})$. *Then we have* $\zeta_t^\varepsilon \in C^\infty(\mathbb{R})$ *and* $\frac{d^k}{dx^k}\zeta_t^\varepsilon \in \mathbf{H}_e$ *for every* $t>0$ *and* $k=0, 1, 2, \ldots$.

This lemma is easily shown so that we omit the proof. Now we give the first energy inequality assuming $\gamma_0 \equiv \|V''\|_\infty < \gamma$.

Lemma 9.2. *We suppose the condition:*

$$|f_t(x)| \leqq \gamma_0 |\eta_t^\varepsilon(x)|, \qquad t \geqq 0,\ x \in \mathbb{R},\ 0<\varepsilon<1. \tag{9.3}$$

Then there exist positive constants r_0, C_1 *and* C_2, *which are independent of* ε, r *and* t, *such that*

$$\int_0^t |\eta_u^\varepsilon|_r^2\, du \leqq C_2 |\omega(\cdot\,;\eta_0)|_r^2\, e^{C_1 r t}, \qquad 0<\varepsilon<1,\ 0<r\leqq r_0,\ t\geqq 0.$$

Proof. We first assume $f \in C_b^\infty([0, \infty)\times \mathbb{R})$ without imposing (9.3). In this case, ζ_t^ε is a solution of the parabolic equation $\frac{\partial}{\partial t}\zeta_t^\varepsilon = -A^\varepsilon \zeta_t^\varepsilon + \nabla f$. Therefore, using this equation, simple calculations show

$$\begin{aligned} e^{C_1 r t}\frac{d}{dt}\{e^{-C_1 r t}|\zeta_t^\varepsilon|_r^2\} \\ \leqq -(2\gamma - 4Mr)|\eta_t^\varepsilon|_r^2 + Mr|f_t|_r^2 + 2\int_{\mathbb{R}} |\eta_t^\varepsilon(x)|\,|f_t(x)|\,\theta(x,r)\,dx, \\ 0<\varepsilon<1,\ 0<r<1,\ t>0, \end{aligned}$$

where $C_1 = M(2+\gamma)$. Here we have applied integration by parts in the variable x with the help of Lemma 9.1 and used the estimates $2|\zeta_t^\varepsilon(x)|\,|f_t(x)| \leqq |\zeta_t^\varepsilon(x)|^2 + |f_t(x)|^2$ and

$$\left|\frac{\partial^k}{\partial x^k}\theta(x,r)\right| \leqq Mr\,\theta(x,r), \qquad k=1, 2, 3, 4,\ 0<r<1, \tag{9.4}$$

with some $M>0$. Hence we obtain

$$\begin{aligned} &\int_0^t \{(2\gamma-4Mr)|\eta_u^\varepsilon|_r^2 - Mr|f_u|_r^2 - 2\int_{\mathbb{R}} |\eta_u^\varepsilon(x)|\,|f_u(x)|\,\theta(x,r)\,dx\}\, e^{-C_1 r u}\, du \\ &\leqq |\omega(\cdot\,;\eta_0)|_r^2. \end{aligned} \tag{9.5}$$

Now we can remove the assumption $f \in C_b^\infty([0, \infty)\times \mathbb{R})$ by the usual approximation method and (9.5) still holds for general f. The conclusion follows from the condition (9.3) by taking r_0; $0<r_0<1$ in such a way that $C_2^{-1} = 2\gamma - 2\gamma_0 - 4Mr_0 - Mr_0\gamma_0^2 > 0$. □

The second task is to give estimates on the fundamental solution $Z_{u,t}^\varepsilon(x, y) = Z_{u,t}^\varepsilon(x, y; c)$, $\varepsilon>0$, of the Eq. (7.1) with $\mathscr{L}_u$ replaced by $\mathscr{L}_{u,c}^\varepsilon = -A^\varepsilon + c(u, x)\Delta_x$ for given $c = c(u, x) \in C_b \equiv C_b([0, \infty)\times \mathbb{R})$. This fundamental solution can be constructed in a similar manner to the case of $\varepsilon=1$; see Sect. 7. We define $Z^{\varepsilon}_{u,t}(\eta, y)$ by (7.23) with $Z_{u,t}(x, y; c)$ replaced by $Z^{\varepsilon}_{u,t}(x, y; c)$.

Lemma 9.3. *There exist positive constants r_0, C_1 and C_2, which are independent of ε, r, t, c and η, such that the following three estimates hold for every $0<\varepsilon<1$, $0<r\leqq r_0$, $t\geqq 0$ and $c\in C_b$; $\|c\|_\infty\leqq\gamma_0$:*

$$\int_0^t |Z^{\varepsilon}_{0,u}(\eta, \cdot\,; c)|_r^2\, du \leqq C_2\, e^{C_1 rt}\, |\omega(\cdot\,; \eta)|_r^2, \qquad \eta\in \mathbf{H}_r, \tag{9.6}$$

$$\int_0^t |Z^{\varepsilon}_{0,u}(\nabla\eta, \cdot\,; c)|_r^2\, du \leqq C_2\, e^{C_1 rt}\, |\eta|_r^2, \qquad \eta\in C_b^\infty(\mathbb{R}), \tag{9.7}$$

and

$$\int_0^t |Z^{\varepsilon}_{0,u}(x, \cdot\,; c)|_r^2\, du \leqq C_2\, r^{-1}\, e^{C_1 rt}\, \theta(x, r), \qquad x\in\mathbb{R}. \tag{9.8}$$

Proof. To complete the proof, we may assume $c\in C_0^\infty([0,\infty)\times\mathbb{R})$. Indeed, after proving the concluding estimates for such smooth c's, we can generalize them for $c\in C_b$ by using approximation method with the help of Lemma 7.7. Remind that $\tilde\eta^{\varepsilon}_t \equiv Z^{\varepsilon}_{0,t}(\eta, \cdot\,; c)$ solves the forward Eq. (7.24) with $\mathscr{L}^*_{t,y}$ replaced by $\mathscr{L}^{\varepsilon *}_{t,y} = -A^{\varepsilon}+\Delta_y\{c(t, y)\,\cdot\}$ if the coefficient c is smooth. Consider the following equation:

$$\frac{\partial}{\partial t}\tilde\zeta^{\varepsilon}_t = -A^{\varepsilon}\,\tilde\zeta^{\varepsilon}_t + \nabla\{c(t, y)\,\nabla\tilde\zeta^{\varepsilon}_t\}, \qquad t>0,$$

$$\tilde\zeta^{\varepsilon}_0 = \omega(\cdot\,; \eta)+a, \qquad a\in\mathbb{R}. \tag{9.9}$$

Then we have $\nabla\tilde\zeta^{\varepsilon}_t = \tilde\eta^{\varepsilon}_t$ and simple calculations show

$$\frac{d}{dt}|\tilde\zeta^{\varepsilon}_t|_r^2 \leqq -(2\gamma-2\gamma_0-4Mr-Mr\gamma_0)\,|\tilde\eta^{\varepsilon}_t|_r^2 + C_1 r\,|\tilde\zeta^{\varepsilon}_t|_r^2,$$

$$0<\varepsilon<1, \quad 0<r<1, \quad t>0,$$

with $C_1 = M(1+\gamma+\gamma_0)$, where M is the same constant appearing in (9.4). Therefore, taking r_0; $0<r_0<1$ in such a way that $C_2^{-1} = 2\gamma-2\gamma_0-4Mr_0-Mr_0\gamma_0>0$, we obtain

$$\int_0^t |\tilde\eta^{\varepsilon}_u|_r^2\, du \leqq C_2\, e^{C_1 rt}\, |\omega(\cdot\,; \eta)+a|_r^2, \qquad 0<r\leqq r_0. \tag{9.10}$$

Now the estimate (9.6) follows by taking $a=0$, while (9.7) follows by taking $\nabla\eta$ instead of η and $a=\eta(0)$ in (9.10), since $\omega(\cdot\,; \nabla\eta)+\eta(0)=\eta$. The third estimate (9.8) is a consequence of (9.7); we may take an approximating sequence $\{\eta_n\in C_b^\infty(\mathbb{R})\}_{n=1}^\infty$ such that $\nabla\eta_n$ converges to δ_x in a proper sense. $\square$

Lemma 9.4. *There exist positive constants r_0, C_1 and C_2, which are independent of ε, r, t, c, $\bar{c}$ and η, such that*

$$\int_0^t |Z^\varepsilon_{0,u}(\eta,\cdot\,;c)-Z^\varepsilon_{0,u}(\eta,\cdot\,;\bar{c})|_r^2\,du$$

$$\leqq C_2\,e^{C_1 rt}\int_0^t |(c-\bar{c})(u)\,Z^\varepsilon_{0,u}(\eta,\cdot\,;c)|_r^2\,du,$$

for every $0<\varepsilon<1$, $0<r\leqq r_0$, $t\geqq 0$, c, $\bar{c}\in C_b$; $\|c\|_\infty$, $\|\bar{c}\|_\infty\leqq\gamma_0$ *and* $\eta\in\mathbf{H}_r$.

Proof. As in the proof of Lemma 9.3, we may assume c, $\bar{c}\in C_0^\infty([0,\infty)\times\mathbb{R})$. We define $\tilde{\zeta}^\varepsilon_t$ as in the proof of Lemma 9.3 and also introduce $\bar{\zeta}^\varepsilon_t$, the solution of (9.9) with c replaced by $\bar{c}$. We assume the initial values are same; $\tilde{\zeta}^\varepsilon_0=\bar{\zeta}^\varepsilon_0=\omega(\cdot\,;\eta)$. Set $\tilde{\eta}^\varepsilon_t=Z^\varepsilon_{0,t}(\eta,\cdot\,;c)$ and $\bar{\eta}^\varepsilon_t=Z^\varepsilon_{0,t}(\eta,\cdot\,;\bar{c})$. Then we have

$$\frac{d}{dt}|\tilde{\zeta}^\varepsilon_t-\bar{\zeta}^\varepsilon_t|_r^2\leqq -(2\gamma-2\gamma_0-4Mr-Mr\gamma_0)\,|\tilde{\eta}^\varepsilon_t-\bar{\eta}^\varepsilon_t|_r^2$$

$$+rM(1+\gamma+\gamma_0)\,|\tilde{\zeta}^\varepsilon_t-\bar{\zeta}^\varepsilon_t|_r^2+I,\qquad 0<\varepsilon<1,\ 0<r<1,\ t>0,$$

where

$$I=-2\int(c_t-\bar{c}_t)\,\tilde{\eta}^\varepsilon_t\{(\tilde{\eta}^\varepsilon_t-\bar{\eta}^\varepsilon_t)\,\theta(x,r)+(\tilde{\zeta}^\varepsilon_t-\bar{\zeta}^\varepsilon_t)\,\nabla\theta(x,r)\}\,dx.$$

Each term in I is estimated as follows:

$$2\,|(c_t-\bar{c}_t)\,\tilde{\eta}^\varepsilon_t(\tilde{\eta}^\varepsilon_t-\bar{\eta}^\varepsilon_t)|\leqq(\gamma-\gamma_0)\,|\tilde{\eta}^\varepsilon_t-\bar{\eta}^\varepsilon_t|^2+(\gamma-\gamma_0)^{-1}\,|(c_t-\bar{c}_t)\,\tilde{\eta}^\varepsilon_t|^2,$$

$$2\,|(c_t-\bar{c}_t)\,\tilde{\eta}^\varepsilon_t(\tilde{\zeta}^\varepsilon_t-\bar{\zeta}^\varepsilon_t)|\leqq|\tilde{\zeta}^\varepsilon_t-\bar{\zeta}^\varepsilon_t|^2+|(c_t-\bar{c}_t)\,\tilde{\eta}^\varepsilon_t|^2.$$

We therefore get the conclusion by taking $C_1=M(2+\gamma+\gamma_0)$ and r_0; $0<r_0<1$ such that $C_2=\{Mr_0+(\gamma-\gamma_0)^{-1}\}\{\gamma-\gamma_0-4Mr_0-Mr_0\gamma_0\}^{-1}>0$. □

We may assume the constants r_0, C_1 and C_2 appearing in Lemmas 9.2–9.4 are common.

9.2 L^p-Estimates on the Fundamental Solutions

Here we shall show the following type of estimate:

Lemma 9.5. *For every* $1<p<7/3$, *there exist positive constants* $\gamma_1^{(p)}$ *and* $C_2^{(p)}$ *such that, if* $\gamma_0<\gamma_1^{(p)}\wedge\gamma$, *then we have*

$$\int_0^t du\int_{\mathbb{R}}|Z^\varepsilon_{0,u}(x,y;c)|^p\,\theta(y,r)\,dy$$

$$\leqq C_2^{(p)}\,r^{-(p-1)/2}\,t^{(7-3p)/4}\,e^{C_1 rt}\,\theta(x,r),\tag{9.11}$$

for every $0<\varepsilon<1$, $0<r\leqq(p-1)\,r_0/2$, $t\geqq 0$, $x\in\mathbb{R}$ *and* $c\in C_b$; $\|c\|_\infty\leqq\gamma_0$. *Here* r_0 *and* C_1 *are the constants appearing in Lemma* 9.3.

To give the proof of this lemma, we regard $Z^{\varepsilon}_{u,t}(x, y; c)$ as a perturbation from $q^{\varepsilon}(t-u, x-y)=Z^{\varepsilon}_{u,t}(x, y; 0)$. We may consider that $Z^{\varepsilon}_{u,t}(x, y; c)$ is defined for every $-\infty<u<t<\infty$, x, $y\in\mathbb{R}$, by extending $c\in C_b([0,\infty)\times\mathbb{R})$ to $c\in C_b((-\infty,\infty)\times\mathbb{R})$ properly. Define an operator Q^{ε}_c on the space $\mathbf{L}^p=L^p(\mathbb{R}\times\mathbb{R})$, $p>1$, by

$$(Q^{\varepsilon}_c h)(u, x)=\int_u^{\infty} dt \int_{\mathbb{R}} Z^{\varepsilon}_{u,t}(x, y; c)\, h(t, y)\, dy, \qquad (u, x)\in\mathbb{R}\times\mathbb{R},\ h\in\mathbf{L}^p,$$

and set $\bar{Q}^{\varepsilon}=Q^{\varepsilon}_0$ (Q^{ε}_c with $c\equiv 0$) and $R^{\varepsilon}_c=c\,\Delta\,\bar{Q}^{\varepsilon}$ $(=(\mathscr{L}^{\varepsilon}_{u,c}-\mathscr{L}^{\varepsilon}_{u,0})\bar{Q}^{\varepsilon})$. Let $\mathbf{L}^p_T$, $T\in\mathbb{R}$, be the class of $h\in\mathbf{L}^p$ such that $h(t,\cdot)\equiv 0$ for $t\geqq T$. Note that, if $p>5/4$, $(Q^{\varepsilon}_c\cdot)(u, x)$ determines a continuous operator of $\mathbf{L}^p_T\to\mathbb{R}$ for each fixed (u, x); use Lemma 7.6.

Lemma 9.6. (i) *There exist positive constants $C_{1,p}$, $1<p\leqq 3$, and $C_{2,p}$, $p>1$, which depend only on p, such that*

$$\int_0^t du \int_{\mathbb{R}} \{q^{\varepsilon}(u, x)\}^p\, dx \leqq C_{1,p}\, t^{(3-p)/2}, \qquad 1<p\leqq 3, \tag{9.12}$$

$$\|\Delta\,\bar{Q}^{\varepsilon} h\|_{\mathbf{L}^p} \leqq C_{2,p}\, \|h\|_{\mathbf{L}^p}, \qquad p>1, \tag{9.13}$$

for every $0<\varepsilon<1$ and $t\geqq 0$.

(ii) *Assume $\gamma_0 C_{2,p}<1$ and $\|c\|_{\infty}\leqq\gamma_0$. Then $(1-R^{\varepsilon}_c)^{-1}=\sum_{n=0}^{\infty}\{R^{\varepsilon}_c\}^n$ exists as an operator on $\mathbf{L}^p$ and its operator norm is bounded by $(1-\gamma_0 C_{2,p})^{-1}$. If $h\in\mathbf{L}^p_T$ with some $T\in\mathbb{R}$, then $(1-R^{\varepsilon}_c)^{-1}\, h(t, y)\in\mathbf{L}^p_T$.*

Proof. We notice that $q^{\varepsilon}(t, x)$ is given by a convolution $\{q^{(1)}_{\varepsilon^2 t} * q^{(2)}_t\}(x)$ of fundamental solutions $q^{(1)}_t$ and $q^{(2)}_t$ of $\frac{\partial}{\partial t}+\Delta^2$ respectively $\frac{\partial}{\partial t}-\gamma\Delta$. Therefore (9.12) follows by using Hausdorff-Young's inequality and by the facts; $\|q^{(2)}_t\|^p_{L^p(\mathbb{R})}=p^{-1/2}(2\pi t\gamma)^{(1-p)/2}$ and $\sup_{t>0}\|q^{(1)}_t\|_{L^1(\mathbb{R})}<\infty$. For the proof of (9.13), we need to give estimates on singular integrals. However, these can be derived by similar argument in the proof of Lemma 5 of Fritz [10]. The assertion (ii) follows by using Neumann series expansion. □

Proof of Lemma 9.5. Assume $\gamma_0 C_{2,\tilde{p}}<1$ with $\tilde{p}=2(p-1)^{-1}$. Then, for every $h\in\mathbf{L}^{\tilde{p}}_t$, we have

$$\begin{aligned}
|Q^{\varepsilon}_c h(0, x)| &= |\bar{Q}^{\varepsilon}(1-R^{\varepsilon}_c)^{-1}\, h(0, x)| \\
&= \left|\int_0^t du \int_{\mathbb{R}} q^{\varepsilon}(u, x-y)\{(1-R^{\varepsilon}_c)^{-1}\, h\}(u, y)\, dy\right| \\
&\leqq (C_{1,p'})^{1/p'}\, t^{(7-3p)/4}(1-\gamma_0 C_{2,\tilde{p}})^{-1}\, \|h\|_{\mathbf{L}^{\tilde{p}}},
\end{aligned} \tag{9.14}$$

with $p'=2(3-p)^{-1}$. Here we have used Lemma 9.6 and Hölder's inequality. The first equality in (9.14) is a consequence of

$$\bar{Q}^\varepsilon\{R_c^\varepsilon\}^n h(u,x)=\int_u^T dt \int_{\mathbb{R}} Z_{u,t}^{\varepsilon,(n)}(x,y)\,h(t,y)\,dy,$$

$$n=0,1,2,\ldots,(u,x)\in\mathbb{R}\times\mathbb{R},$$

where $\{Z_{u,t}^{\varepsilon,(n)}\}_{n=0}^\infty$ are the functions defined by (7.19) with q replaced by q^ε. Note that $\tilde{p}>5/4$. Now consider functions $h_{t,x}^\varepsilon\in\mathbf{L}^{\tilde{p}}$, $t>0$, $x\in\mathbb{R}$, defined by $h_{t,x}^\varepsilon(u,y)=\operatorname{sign}\{Z_{0,u}^\varepsilon(x,y;c)\}\,|Z_{0,u}^\varepsilon(x,y;c)|^{p-1}\,\theta(y,r)$ for $0\leqq u\leqq t$ and $=0$, otherwise. Then we see $Q_c^\varepsilon h_{t,x}^\varepsilon(0,x)=$ "the LHS of (9.11)" and (9.8) verifies

$$\|h_{t,x}^\varepsilon\|_{\mathbf{L}^p}=\left\{\int_0^t |Z_{0,u}^\varepsilon(x,\cdot\,;c)|_{r\tilde{p}}^2\,du\right\}^{1/\tilde{p}}\leqq (C_2/r\tilde{p})^{1/\tilde{p}}\,e^{C_1rt}\,\theta(x,r),$$

if $0<r\tilde{p}\leqq r_0$. Therefore we obtain the conclusion from (9.14) by taking $\gamma_1^{(p)}=C_{2,\tilde{p}}^{-1}$ and $C_2^{(p)}=(C_{1,p'})^{1/p'}(1-\gamma_0C_{2,\tilde{p}})^{-1}(C_2/\tilde{p})^{1/\tilde{p}}$. □

10. Compactness Argument

We need to investigate, for every $\Psi\in\mathscr{D}$, the compactness of the family $\{\Psi_t^\varepsilon(S)\}_{0<\varepsilon<1}$ introduced in Sect. 8 and their Fréchet derivatives $\{D\Psi_t^\varepsilon(\cdot,S)\}_{0<\varepsilon<1}$ as continuous functions of $S\in\mathbf{H}_{e,w}$. For this purpose, however, it is more convenient to treat their Laplace transforms defined by $R_a\Psi^\varepsilon(S)=\int_0^\infty e^{-at}\,\Psi_t^\varepsilon(S)\,dt$ and $R_aD\Psi^\varepsilon(x,S)=\int_0^\infty e^{-at}D\,\Psi_t^\varepsilon(x,S)\,dt$, $a>0$. We also study the compactness of $\{\Psi_t^\varepsilon(S)\}_{0<\varepsilon<1}$ regarding as functions of t. Denote by $S_t^\varepsilon(\cdot\,;S)$ the solution of the scaled TDGL Eq. (1.3) to make its initial value $S\in\mathbf{H}_e$ clear as before. We assume $\gamma_0\equiv\|V''\|_\infty<\min(\gamma,\gamma_1)$ taking $\gamma_1\equiv\sup_{2<p<7/3}\gamma_1^{(p)}$, where $\gamma_1^{(p)}$ is the constant appearing in Lemma 9.5.

10.1 Compactness of $\{R_a\Psi^\varepsilon;0<\varepsilon<1\}$

The following Lemma 10.1 is an immediate consequence of Lemma 9.2.

Lemma 10.1. *For every* $0<\varepsilon<1$, $0<r\leqq r_0$, $t\geqq 0$ *and* $S,\bar{S}\in\mathbf{H}_e$, *we have*

$$\int_0^t |S_u^\varepsilon(\cdot\,;S)-S_u^\varepsilon(\cdot\,;\bar{S})|_r^2\,du\leqq C_2\,|\omega(\cdot\,;S-\bar{S})|_r^2\,e^{C_1rt}.$$

Lemma 10.2. *For every* $a_0>0$, *there exist positive constants* $C=C(a_0,\Psi)$ *and* $\tilde{r}=\tilde{r}(a_0)$ *such that*

$$|R_a\Psi^\varepsilon(S)-R_a\Psi^\varepsilon(\bar{S})|\leqq C\,|\omega(\cdot\,;S-\bar{S})|_{\tilde{r}},\quad 0<\varepsilon<1,\ a\geqq a_0,\ S,\bar{S}\in\mathbf{H}_e.$$

Proof. Since the function Ψ of the form (2.2) has a bound:

$$|\Psi(S_1)-\Psi(S_2)| \leqq C_r(\Psi)\,|S_1-S_2|_r, \qquad r>0, \tag{10.1}$$

with some constant $C_r(\Psi)>0$, we obtain from Lemma 10.1

$$\begin{aligned}|R_a\,\Psi^\varepsilon(S)-R_a\,\Psi^\varepsilon(\bar{S})| &\leqq C_r(\Psi)\,E\left[a\int_0^\infty e^{-at}\,dt\int_0^t |S_u^\varepsilon(\cdot\,;S)-S_u^\varepsilon(\cdot\,;\bar{S})|_r\,du\right]\\ &\leqq C_r(\Psi)\,C(a,r)\,|\omega(\cdot\,;S-\bar{S})|_r,\end{aligned}$$

where $C(a,r)=a\sqrt{C_2}\int_0^\infty \sqrt{t}\exp\{-t(a-C_1 r/2)\}\,dt$, $a>0$, $0<r<2a/C_1$. Therefore the conclusion follows by taking $\tilde{r}$; $0<\tilde{r}<r_0\wedge 2a_0/C_1$ and $C=C_{\tilde{r}}(\Psi)\sup\{C(a,\tilde{r});\,a\geqq a_0\}<\infty$. □

We regard $R_a\,\Psi^\varepsilon(S)$ as real-valued functions of $(a,S)\in(0,\infty)\times\mathbf{H}_{e,w}$. Remind that the set $B(\{b_r\})=\{S\in\mathbf{H}_{e,w};\,|S|_r\leqq b_r,\,r>0\}$ is compact in $\mathbf{H}_{e,w}$ for every sequence $\{b_r>0;\,r>0\}$.

Proposition 10.1. *For every $0<a_0<a_1<\infty$ and $\{b_r>0;\,r>0\}$, the family of functions $\{R_a\,\Psi^\varepsilon(S);\,0<\varepsilon<1\}$ restricted on $[a_0,a_1]\times B(\{b_r\})$ is relatively compact in the space $C([a_0,a_1]\times B(\{b_r\}))$ having the usual uniform-convergence topology.*

Proof. Since the uniform boundedness of the family $\{R_a\,\Psi^\varepsilon(S)\}$ follows from $|R_a\,\Psi^\varepsilon(S)|\leqq\|\psi\|_\infty/a$ and the equicontinuity in a follows from $\left|\frac{\partial}{\partial a}R_a\,\Psi^\varepsilon(S)\right|\leqq\|\psi\|_\infty/a^2$, the proof is completed if one can show the equicontinuity in S of this family. For this purpose, note that the fundamental system of neighborhoods of $0\in\mathbf{H}_{e,w}$ consists of all subsets U of $\mathbf{H}_{e,w}$ having the forms:

$$U\equiv U_\alpha(\eta_1,\dots,\eta_n)=\{S\in\mathbf{H}_{e,w};\,|\langle S,\eta_i\rangle|<\alpha,\,i=1,2,\dots,n\}, \tag{10.2}$$

with $n\in\mathbb{N}$, $\alpha>0$ and $\eta_i\in\mathbf{H}_e^*$, $i=1,2,\dots,n$. Therefore, with the help of Lemma 10.2, we may only prove that for every $\delta>0$ there exists a set U of the form (10.2) such that $\sup\{|\omega(\cdot\,;S)|_{\tilde{r}};\,S\in U\cap B(\{2b_r\})\}<\delta$. However, this is not difficult. □

10.2 Compactness of $\{R_a D\,\Psi^\varepsilon;\,0<\varepsilon<1\}$

The first task is to show the relative-compactness in the space $\mathbf{H}_{-\bar{r}}$ of the family $\{R_a D\,\Psi^\varepsilon(\cdot,S);\,0<\varepsilon<1,\,a\geqq a_0,\,S\in\mathbf{H}_e\}$, $a_0>0$, with some $\bar{r}=\bar{r}(a_0)>0$. The following lemma gives a criterion for the relative-compactness of a subset in the Hilbert space $\mathbf{H}_{-r}$, $r>0$. The proof is easy and omitted.

Lemma 10.3. *A subset $\mathscr{E}$ is relatively compact in the space $\mathbf{H}_{-r}$ if it satisfies the following two conditions:*

(a) $\sup\{|\xi|_{-r'}; \xi\in\mathscr{E}\}<\infty$ *with some* $r'>r$.

(b) *For every bounded interval* I *of* $\mathbb{R}$, *the family of restrictions* $\{\xi|_I; \xi\in\mathscr{E}\}$ *is relatively compact in the space* $L^2(I, dx)$.

Lemma 10.4. *For every* $a_0>0$, *there exists a positive constant* $\bar{r}=\bar{r}(a_0)$ *such that* $\{R_a D\Psi^\varepsilon(\cdot, S); 0<\varepsilon<1, a\geqq a_0, S\in\mathbf{H}_e\}$ *is relatively compact in the space* $H_{-\bar{r}}$.

Proof. We show that two conditions (a) and (b) of Lemma 10.3 hold for $\mathscr{E}=\{R_a D\Psi^\varepsilon(\cdot, S); 0<\varepsilon<1, a\geqq a_0, S\in\mathbf{H}_e\}$. First, using Theorem 8.1 (ii) and then (9.6), we obtain for every $\eta\in\mathbf{H}_r$:

$$\begin{aligned}
&|\langle R_a D\Psi^\varepsilon(\cdot, S), \eta\rangle| \\
&\leqq C_r(\Psi)\int_0^\infty a e^{-at}\,dt\, E_S\left[\int_0^t |Z^\varepsilon_{0,u}(\eta,\cdot; S^\varepsilon_\cdot)|_r\,du\right] \\
&\leqq C_r(\Psi)\, C(a,r)\,|\omega(\cdot;\eta)|_r, \quad 0<\varepsilon<1, 0<r<r_0\wedge 2a_0/C_1, a\geqq a_0, \qquad (10.3)
\end{aligned}$$

where $C_r(\Psi)$ and $C(a, r)$ are the same constants in the proof of Lemma 10.2. However, it holds

$$|\omega(\cdot;\eta)|_r \leqq C_{r,r'}\,|\eta|_{r'}, \qquad 0<r'<r, \tag{10.4}$$

with some constant $C_{r,r'}$ and therefore we obtain for every $0<r'<r_0\wedge 2a/C_1$:

$$|R_a D\Psi^\varepsilon(\cdot, S)|_{-r'} \leqq \inf_{r;r>r'}\{C_r(\Psi)\, C(a,r)\, C_{r,r'}\}, \qquad 0<\varepsilon<1,\ a\geqq a_0,\ S\in\mathbf{H}_e. \tag{10.5}$$

Now take $\bar{r}=\bar{r}(a_0)$ such that $0<\bar{r}<r_0\wedge 2a_0/C_1$. Then, since $\sup\{C(a,r); a\geqq a_0\}<\infty$ for $r<2a_0/C_1$, we get the condition (a) with $r=\bar{r}$ from (10.5) by taking r': $\bar{r}<r'<r_0\wedge 2a_0/C_1$.

Secondly to show the condition (b), we see for every $\eta\in C_0^\infty(\mathbb{R})$:

$$|\langle R_a D\Psi^\varepsilon(\cdot, S), \nabla\eta\rangle| \leqq C_r(\Psi)\, C(a,r)\,|\eta|_r, \tag{10.6}$$

holds for $0<\varepsilon<1$, $0<r<r_0\wedge 2a_0/C_1$, $a\geqq a_0$ and $S\in\mathbf{H}_e$. This follows by a similar calculation as above using (9.7). Since it holds $|\eta|_r\leqq \|\eta\|_{L^2(I)}$, $r>0$, for η satisfying $\operatorname{supp}\eta\subset I$, (10.5) and (10.6) imply

$$\sup\left\{\sum_{k=0}^{1}\|\nabla^k R_a D\Psi^\varepsilon(\cdot, S)\|_{L^2(I)}; 0<\varepsilon<1, a\geqq a_0, S\in\mathbf{H}_e\right\}<\infty,$$

where ∇ is the derivative in the distribution's sense. This proves the condition (b) with the help of Rellich's theorem (see, e.g., Adams [1]). □

The second task in this paragraph is proving the equicontinuity of $\{R_a D\Psi^\varepsilon(\cdot, S); 0<\varepsilon<1\}$ as $\mathbf{H}_{-\bar{r}}$-valued functions of (a, S). The assumption $\gamma_0<\gamma_1$ will be used to show the following lemma.

Lemma 10.5. *For every* $a_0>0$, *there exist positive constants* $\bar{r}=\bar{r}(a_0)$, $\tilde{r}=\tilde{r}(a_0)$, $C=C(a_0, \Psi)$ *and* $0<\alpha<1$ *such that*

$$|R_a D\Psi^\varepsilon(\cdot, S)-R_a D\Psi^\varepsilon(\cdot, \bar{S})|_{-\bar{r}}\leqq C\{|\omega(\cdot; S-\bar{S})|^\alpha_{\tilde{r}}+|\omega(\cdot; S-\bar{S})|_{\tilde{r}}\},$$

holds for every $0<\varepsilon<1$, $a\geqq a_0$ *and* $S, \bar{S}\in\mathbf{H}_e$.

Proof. Positive constants $\bar{r}$ and $\tilde{r}$ will be chosen later. Theorem 8.1 (ii) shows for every $\eta \in \mathbf{H}_{\bar{r}}$:

$$|\langle R_a D\Psi^\varepsilon(\cdot, S) - R_a D\Psi^\varepsilon(\cdot, \bar{S}), \eta\rangle| = \left|\sum_{i=1}^{k} \int_0^\infty a e^{-at} dt \int_0^t \{I_{1,i}^\varepsilon(u) + I_{2,i}^\varepsilon(u)\} du\right|,$$

where

$$I_{1,i}^\varepsilon(t) = E[\{\Psi_i(S_t^\varepsilon) - \Psi_i(\bar{S}_t^\varepsilon)\} \langle Z_{0,t}^\varepsilon(\eta, \cdot; S_\cdot^\varepsilon), \varphi_i\rangle],$$
$$I_{2,i}^\varepsilon(t) = E[\Psi_i(\bar{S}_t^\varepsilon) \langle Z_{0,t}^\varepsilon(\eta, \cdot; S_\cdot^\varepsilon) - Z_{0,t}^\varepsilon(\eta, \cdot; \bar{S}_\cdot^\varepsilon), \varphi_i\rangle],$$
$$\Psi_i(S) = \frac{\partial \psi}{\partial \alpha_i}(\langle S, \varphi_1\rangle, \ldots, \langle S, \varphi_k\rangle), \qquad i = 1, 2, \ldots, k,$$

and $S_t^\varepsilon = S_t^\varepsilon(\cdot; S)$, $\bar{S}_t^\varepsilon = S_t^\varepsilon(\cdot; \bar{S})$. Estimation on a term including $I_{1,i}^\varepsilon$ goes as follows:

$$\left|\int_0^t I_{1,i}^\varepsilon(u)\, du\right| \leqq C_{\bar{r}}(\Psi_i)\, E\left[\left\{\int_0^t |S_u^\varepsilon - \bar{S}_u^\varepsilon|_{\bar{r}}^2\, du\right\}^{1/2} \left\{\int_0^t \langle Z_{0,u}^\varepsilon(\eta, \cdot; S_\cdot^\varepsilon), \varphi_i\rangle^2\, du\right\}^{1/2}\right]$$
$$\leqq C_{\bar{r}}(\Psi_i)\, C_2\, C_{\bar{r}',\bar{r}}\, |\varphi_i|_{-\bar{r}'}\, |\omega(\cdot; S - \bar{S})|_{\bar{r}}\, |\eta|_{\bar{r}} \exp\{C_1 t(\tilde{r} + \bar{r}')/2\},$$
$$0 < \varepsilon < 1,\ 0 < \tilde{r} \leqq r_0,\ 0 < \bar{r} < \bar{r}' \leqq r_0.$$

Here we have used Schwarz's inequality and (10.1) for the first line and then (9.6), (10.4) and Lemma 10.1 for the second line. To estimate the term including $I_{2,i}^\varepsilon$, we fix $2 < p < 7/3$ in such a way that $\gamma_0 < \gamma_1^{(p)}$ holds. Then we have

$$\left|\int_0^t I_{2,i}^\varepsilon(u)\, du\right| \leqq \left\|\frac{\partial \psi}{\partial \alpha_i}\right\|_\infty |\varphi_i|_{-\tilde{r}} \int_0^t E[|Z_{0,u}^\varepsilon(\eta, \cdot; S_\cdot^\varepsilon) - Z_{0,u}^\varepsilon(\eta, \cdot; \bar{S}_\cdot^\varepsilon)|_{\tilde{r}}]\, du$$
$$\leqq \left\|\frac{\partial \psi}{\partial \alpha_i}\right\|_\infty |\varphi_i|_{-\tilde{r}} \sqrt{t C_2}\, e^{C_1 \tilde{r} t/2}$$
$$\cdot E\left[\left\{\int_0^t |\{V''(S_u^\varepsilon(\cdot)) - V''(\bar{S}_u^\varepsilon(\cdot))\}\, Z_{0,u}^\varepsilon(\eta, \cdot; S_\cdot^\varepsilon)|_{\tilde{r}}^2\, du\right\}^{1/2}\right]$$
$$\leqq \left\|\frac{\partial \psi}{\partial \alpha_i}\right\|_\infty |\varphi_i|_{-\tilde{r}} \sqrt{t C_2}\, e^{C_1 \tilde{r} t/2}\, E[\{I_{3,i}^\varepsilon(t)\}^{1/q} \{I_{4,i}^\varepsilon(t)\}^{1/p}],$$
$$0 < \varepsilon < 1,\ 0 < \tilde{r} \leqq r_0, \tag{10.7}$$

with $q > 2$ such that $1/p + 1/q = 1/2$, where

$$I_{3,i}^\varepsilon(t) = \int_0^t du \int_{\mathbb{R}} \{V''(S_u^\varepsilon(x)) - V''(\bar{S}_u^\varepsilon(x))\}^q\, \theta(x, \tilde{r})\, dx,$$

and

$$I_{4,i}^\varepsilon(t) = \int_0^t du \int_{\mathbb{R}} |Z_{0,u}^\varepsilon(\eta, y; S_\cdot^\varepsilon)|^p\, \theta(y, \tilde{r})\, dy.$$

We have used Schwarz's inequality and then applied Lemma 9.4 for the second inequality in (10.7). We estimate further as follows using Lemma 10.1:

$$I^{\varepsilon}_{3,i}(t) \leqq (2\gamma_0)^{q-2}\gamma_*^2 C_2 |\omega(\cdot; S-\bar{S})|^2_{\tilde{r}} e^{C_1 \tilde{r} t},$$

where $\gamma_* = \|V'''\|_\infty$. On the other hand,

$$\begin{aligned} I^{\varepsilon}_{4,i}(t) &\leqq |\eta|^p_{\tilde{r}} \int_0^t du \int_{\mathbb{R}} |Z^{\varepsilon}_{0,u}(\cdot, y; S^{\varepsilon}_{\cdot})|^p_{-\bar{r}}\, \theta(y, \tilde{r})\, dy \\ &\leqq |\eta|^p_{\tilde{r}} \left\{ M\left(\frac{p}{p-2}(\bar{r}''-r)\right)\right\}^{-1+p/2} \\ &\quad \cdot \int_{\mathbb{R}} \theta(x, -p\bar{r}''/2)\, dx \int_0^t du \int_{\mathbb{R}} |Z^{\varepsilon}_{0,u}(x, y; S^{\varepsilon}_{\cdot})|^p\, \theta(y, \tilde{r})\, dy \\ &\leqq |\eta|^p_{\tilde{r}} \left\{ M\left(\frac{p}{p-2}(\bar{r}''-r)\right)\right\}^{-1+p/2} M(\tilde{r}-p\bar{r}''/2)\, C_2^{(p)}\, \tilde{r}^{-(p-1)/2}\, t^{(7-3p)/4}\, e^{C_1 t \tilde{r}}, \\ &\quad 0<\tilde{r} \leqq (p-1) r_0/2,\ 0<\bar{r}<\bar{r}''<2\tilde{r}/p, \end{aligned}$$

where $M(r) = \int \theta(x, r)\, dx < \infty$ if $r>0$. We have used Hölder's inequality for the second inequality and Lemma 9.5 for the third inequality. Now the combination of these estimates leads us to the conclusion by taking $\alpha = 2/q$, choosing $\tilde{r}$ and $\bar{r}$ in such a way that $0 < p\bar{r}/2 < \tilde{r} < \{a_0/C_1 \wedge (p-1) r_0/2\}$. □

We may assume that two $\bar{r}(a_0)$'s appearing in Lemmas 10.4 and 10.5 are common.

Proposition 10.2. *For every* $0<a_0<a_1<\infty$ *and* $\{b_r>0; r>0\}$, *the family of* $\mathbf{H}_{-\bar{r}(a_0)}$*-valued functions* $\{R_a D \Psi^{\varepsilon}(\cdot, S); 0<\varepsilon<1\}$ *restricted on* $[a_0, a_1] \times B(\{b_r\})$ *is relatively compact in the space* $C([a_0, a_1] \times B(\{b_r\}), \mathbf{H}_{-\bar{r}(a_0)})$ *equipped with the usual uniform-convergence topology.*

Proof. We apply Ascoli-Arzelà's theorem using Lemmas 10.4 and 10.5. We note that the equicontinuity in a follows from

$$\sup\left\{\left|\frac{\partial}{\partial a} R_a D \Psi^{\varepsilon}(\cdot, S)\right|_{-\bar{r}(a_0)};\ a \in [a_0, a_1],\ 0<\varepsilon<1,\ S \in \mathbf{H}_e\right\} < \infty,$$

which can be shown similarly to (10.5). Therefore the proof can be completed in a similar manner to that of Proposition 10.1. □

10.3 Compactness of $\{\Psi^{\varepsilon}_t(S); 0<\varepsilon<1\}$

We prepare the following.

Lemma 10.6. *For every* $\varphi \in C_0^\infty(\mathbb{R})$, *there exist positive constants* C_1 *and* $C_2 = C_2(\varphi)$ *such that*

$$E_S[\langle S^{\varepsilon}_t, \varphi\rangle^2] \leqq C_2 (1 + |S|^2_r)\{e^{C_1 r t} + t^2\}$$

holds for every $0<\varepsilon<1$, $0<r<1$, $t \geqq 0$ *and* $S \in \mathbf{H}_e$, *where* $S^{\varepsilon}_t = S^{\varepsilon}_t(\cdot; S)$.

Proof. Let $\{S^\varepsilon_{t,i}\}_{i=1}^3$ be three functions defined similarly to $\{S_{t,i}\}_{i=1}^3$, which were introduced in Sect. 5; we replace q by q^ε and, in addition, S_0 by S for $S^\varepsilon_{t,1}$ and $S_u(y)$ by $S^\varepsilon_u(y)$ for $S^\varepsilon_{t,3}$. We may derive estimates on $I^\varepsilon_i \equiv E_S[\langle S^\varepsilon_{t,i}, \varphi\rangle^2]$, $i=1$, 2, 3, individually. For I^ε_1, we see

$$|\langle S^\varepsilon_{t,1}, \varphi\rangle| = |\langle S, q^\varepsilon(t,\cdot)*\varphi\rangle| \\ \leqq |S|_r \, \|\varphi\|_{1,-r} \, \|q^{(1)}_{\varepsilon^2 t}\|_{1,-r} \, \|q^{(2)}_t\|_{1,-r},$$

by using Hausdorff-Young's inequality, where $\|\cdot\|_{1,-r}$ stands for the norm of the space $L^1(\mathbb{R}, \theta(x,-r)\,dx)$. However, it is not difficult to show $\|q^{(1)}_{\varepsilon^2 t}\|_{1,-r} \leqq K e^{Lrt}$, $\|q^{(2)}_t\|_{1,-r} \leqq K e^{Lrt}$, for every $0<\varepsilon<1$, $0\leqq r<1$ and $t>0$ with some K, $L>0$. Therefore $|\langle S^\varepsilon_{t,1}, \varphi\rangle|$ is bounded by $|S|_r \|\varphi\|_{1,-r} K^2 e^{2Lrt}$. The estimates on I^ε_2 and I^ε_3 can be derived similarly and we get the conclusion; cf. [Fu]. □

Proposition 10.3. (i) $\sup\{|\Psi^\varepsilon_t(S)|; 0<\varepsilon<1, t\geqq 0, S\in \mathbf{H}_e\}<\infty$.

(ii) *As functions of t, a family $\{\Psi^\varepsilon_\cdot(S); 0<\varepsilon<1, S\in\mathbf{H}_e\colon |S|_r\leqq b\}$ is relatively compact in the space $C([0,\infty))$ equipped with the usual compact-open topology for every $0<r<1$ and $b>0$.*

Proof. The assertion (i) is trivial. For the assertion (ii), we may only prove the equicontinuity in t of this family. Indeed this follows by showing

$$\sup\{|E_S[\mathcal{G}^\varepsilon \Psi(S^\varepsilon_t)]|; 0<\varepsilon<1, 0\leqq t\leqq T, |S|_r\leqq b\}<\infty, \qquad T>0, \tag{10.8}$$

since we have $\frac{\partial}{\partial t}\Psi^\varepsilon_t(S) = E_S[\mathcal{G}^\varepsilon \Psi(S^\varepsilon_t)]$, $t>0$. However, (10.8) can be proved by using Lemma 10.6. □

We conclude this section by making the definition of an operator D more restrictive. With fixed $\bar r>0$, the domain $\mathcal{D}(D)$ of D consists of all $\Psi\in C(\mathbf{H}_{e,w})$ which are Fréchet differentiable on $\mathbf{H}_e$ and satisfy $D\Psi\in C(\mathbf{H}_{e,w}, \mathbf{H}_{-\bar r})$. For $\Psi\in\mathcal{D}(D)$, we set $D\Psi(\cdot, S)=$ the Fréchet derivative of Ψ at S. The proof of the following lemma is not difficult so that we omit it.

Lemma 10.7. *The operator D defined as above is "closed" in the following sense: Let $\{\Psi^\varepsilon\in\mathcal{D}\}_{0<\varepsilon<1}$, $\Psi\in C(\mathbf{H}_{e,w})$ and $\Phi\in C(\mathbf{H}_{e,w}, \mathbf{H}_{-\bar r})$ be given and satisfy that Ψ^ε and $D\Psi^\varepsilon$ converge as $\varepsilon\downarrow 0$ to Ψ and Φ, respectively, uniformly on each compact ball $B(\{b_r\})$ of $\mathbf{H}_{e,w}$. Then we have $\Psi\in\mathcal{D}(D)$ and $D\Psi=\Phi$.*

11. The Proof of Main Theorem

We conclude the proof of Theorem 1.1 dividing it into three steps. We assume $\gamma_0<\min(\gamma,\gamma_1)$.

Step 1: Convergence of Ψ^ε and $D\Psi^\varepsilon$

We fix an increasing sequence of compact balls $\{B_n = B(\{b^{(n)}_r\})\}_{n=1}^\infty$ in $\mathbf{H}_{e,w}$ satisfying $\bigcup_{n=1}^\infty B_n = \mathbf{H}_{e,w}$. Take any $a_0\in(0,\infty)$ and subsequence $\{\varepsilon'\downarrow 0\}$ of $\{\varepsilon\}$. We set simply $\bar r = \bar r(a_0)$, the constant appearing in Proposition 10.2.

Lemma 11.1. (i) *There exist* $\tilde{\Psi}_a^{(1)}(S) \in C_b([a_0, \infty) \times \mathbf{H}_{e,w})$, $\tilde{\Psi}_a^{(2)}(\cdot, S) \in C_b([a_0, \infty) \times \mathbf{H}_{e,w}, \mathbf{H}_{-\tilde{r}})$ *and a subsequence* $\{\varepsilon'' \downarrow 0\}$ *of* $\{\varepsilon'\}$ *such that* $R_a \Psi^{\varepsilon''}(S)$ *and* $R_a D \Psi^{\varepsilon''}(\cdot, S)$ *converge to* $\tilde{\Psi}_a^{(1)}(S)$ *and* $\tilde{\Psi}_a^{(2)}(\cdot, S)$, *respectively, uniformly on* $[a_0, a_1] \times B_n$ *for every* a_1; $a_1 > a_0$ *and* $n \in \mathbb{N}$ *as* $\varepsilon'' \downarrow 0$.

(ii) $\tilde{\Psi}_a^{(1)} \in \mathscr{D}(D)$ *and* $D \tilde{\Psi}_a^{(1)}(\cdot, S) = \tilde{\Psi}_a^{(2)}(\cdot, S)$ *for every* $a \in [a_0, \infty)$,

Proof. The assertion (i) is a consequence of Propositions 10.1 and 10.2. We only remark that the boundedness of the limit functions $\tilde{\Psi}_a^{(1)}(S)$ and $\tilde{\Psi}_a^{(2)}(\cdot, S)$ follows from $|\tilde{\Psi}_a^{(1)}(S)| = \lim_{\varepsilon'' \downarrow 0} |R_a \Psi^{\varepsilon''}(S)| \leqq \|\psi\|_\infty / a$ and (10.5), respectively. For the proof of (ii), use Lemma 10.7 noting $R_a\{D \Psi^\varepsilon(\cdot, S)\} = D R_a \Psi^\varepsilon(\cdot, S)$. □

Lemma 11.2. *For every* $\lambda(\cdot) \in \Lambda$ *and* $a \in [a_0, \infty)$, *we have*

$$\lim_{\varepsilon'' \downarrow 0} E^{\mu_{\lambda(\cdot),\varepsilon''}}[R_a \Psi^{\varepsilon''}(S)] = \tilde{\Psi}_a^{(1)}(\rho), \tag{11.1}$$

$$\lim_{\varepsilon'' \downarrow 0} E^{\mu_{\lambda(\cdot),\varepsilon''}}[\langle \Delta \lambda(\cdot), R_a D \Psi^{\varepsilon''}(\cdot, S) \rangle] = \langle \Delta \lambda(\cdot), \tilde{\Psi}_a^{(2)}(\cdot, \rho) \rangle, \tag{11.2}$$

where $\rho \equiv \rho(\cdot) = \bar{\rho}(\lambda(\cdot))$ *and* $\bar{\rho}$ *is the mean spin function.*

Proof. Since $\tilde{\Psi}_a^{(1)} \in C_b(\mathbf{H}_{e,w})$ for every $a \geqq a_0$, Theorem 4.1 implies

$$\lim_{\varepsilon'' \downarrow 0} |E^{\mu_{\lambda(\cdot),\varepsilon''}}[\tilde{\Psi}_a^{(1)}(S)] - \tilde{\Psi}_a^{(1)}(\rho)| = 0.$$

Therefore, for the proof of (11.1), we may only show $I^{\varepsilon''} \equiv |E^{\mu_{\lambda(\cdot),\varepsilon''}}[\{R_a \Psi^{\varepsilon''} - \tilde{\Psi}_a^{(1)}\}(S)]| \to 0$ as $\varepsilon'' \downarrow 0$. However, $I^{\varepsilon''}$ is bounded by the sum of $I_{1,n}^{\varepsilon''} \equiv \sup_{S \in B_n} |\{R_a \Psi^{\varepsilon''} - \tilde{\Psi}_a^{(1)}\}(S)|$ and $I_{2,n}^{\varepsilon''} \equiv 2\|\psi\|_\infty \mu_{\lambda(\cdot),\varepsilon''}(B_n^c)/a$ for every $n \in \mathbb{N}$. Lemma 11.1 implies $\lim_{\varepsilon'' \downarrow 0} I_{1,n}^{\varepsilon''} = 0$, $n \in \mathbb{N}$. On the other hand, $I_{2,n}^{\varepsilon''}$ is bounded by $2\|\psi\|_\infty \{b_r^{(n)}\}^{-2} E^{\mu_{\lambda(\cdot),\varepsilon''}}[|S|_r^2]/a$, which converges to 0 as $n \to \infty$ uniformly in ε''; see Proposition 3.1 (iii). Therefore we obtain (11.1). The convergence (11.2) can be shown similarly. □

Lemma 11.3. *For every* $S \in \mathbf{H}_{e,w}$, $\Psi_t^{\varepsilon''}(S)$ *converges to some* $\Psi_t^{(1)}(S) \in C_b([0, \infty))$ *uniformly on each bounded interval of* $[0, \infty)$ *as* $\varepsilon'' \downarrow 0$ *and it holds* $\tilde{\Psi}_{at}^{(1)}(S) = R_a \Psi^{(1)}(S)$.

Proof. Let $\Psi_t^{(1)}(S) \in C_b([0, \infty))$ be an arbitrary limit of $\{\Psi_t^{\varepsilon''}(S)\}$; recall Proposition 10.3. Then, taking the limit, it holds $R_a \Psi^{(1)}(S) = \tilde{\Psi}_a^{(1)}(S)$, $a \in [a_0, \infty)$, and this proves from the uniqueness of the Laplace transform that the limit $\Psi^{(1)}(S)$ is determined uniquely. Therefore we get the conclusion. □

Step 2: Derivation of the Limit Equation

Lemma 11.4. *For every* $a > 0$, $0 < \varepsilon < 1$ *and* $\lambda(\cdot) \in \Lambda$,

$$a E^{\mu_{\lambda(\cdot),\varepsilon}}[R_a \Psi^\varepsilon(S)] = E^{\mu_{\lambda(\cdot),\varepsilon}}[\Psi(S)] + E^{\mu_{\lambda(\cdot),\varepsilon}}[\langle \Delta \lambda(\cdot), R_a D \Psi^\varepsilon(\cdot, S) \rangle].$$

Proof. Since $\Psi(S_t^\varepsilon)-\int_0^t \mathscr{G}^\varepsilon \Psi(S_u^\varepsilon)\,du$ is a martingale, we have,

$$\begin{aligned} aR_a\Psi^\varepsilon(S) &= a\int_0^\infty e^{-at}\Psi(S)\,dt + a\int_0^\infty e^{-at}\,dt\int_0^t E_S[\mathscr{G}^\varepsilon\Psi(S_u^\varepsilon)]\,du \\ &= \Psi(S)+\int_0^\infty e^{-at}E_S[\mathscr{G}^\varepsilon\Psi(S_t^\varepsilon)]\,dt. \end{aligned}$$

In this calculation, we have used integration by parts by noting the result of Lemma 10.6. Now the conclusion follows from Theorem 8.1. □

The following is an immediate consequence of the above Lemmas.

Proposition 11.1. *For every $a \geqq a_0$ and $\rho=\rho(\cdot)\in\bar{\rho}(\Lambda)$, we have*

$$aR_a\Psi^{(1)}(\rho)=\Psi(\rho)+\langle \Delta\{\bar{\lambda}(\rho(\cdot))\}, DR_a\Psi^{(1)}(\cdot,\rho)\rangle, \tag{11.3}$$

where $\bar{\lambda}$ is an inverse function of the mean spin function $\bar{\rho}$ and

$$\bar{\rho}(\Lambda)=\{\rho(\cdot)=\bar{\rho}(\lambda(\cdot));\ \lambda(\cdot)\in\Lambda\}.$$

Now we prepare the following lemma.

Lemma 11.5. $\lim_{\lambda\to\pm\infty}\bar{\rho}(\lambda)=\pm\infty.$

Proof. Consider self-adjoint operators $\tilde{H}_0=-\frac{1}{2}d^2/ds^2+\frac{\gamma}{2}s^2$ and $\tilde{H}_W=\tilde{H}_0+W$ defined on the space $L^2(\mathbb{R}, ds)$ for a bounded function W and let $\tilde{\Omega}_W$ be a positive and normalized eigenfunction of $\tilde{H}_W$ corresponding to its minimal eigenvalue $\tilde{\kappa}(W)$. Then Rayleigh-Ritz principle (Simon [23, p. 199]) proves $|\tilde{\kappa}(W)-\tilde{\kappa}(0)|\leqq\|W\|_\infty$. Using this estimation, similar argument employed by Reed and Simon [20, IV p. 251] shows

$$\begin{aligned} \tilde{\Omega}_W(s) &= e^{t\{\tilde{\mathscr{K}}(W)+\|W\|_\infty\}}\,e^{-t\{\tilde{H}_W+\|W\|_\infty\}}\,\tilde{\Omega}_W(s) \\ &\leqq e^{t\{\tilde{\mathscr{K}}(0)+2\|W\|_\infty\}}\,e^{-t\tilde{H}_0}\,\tilde{\Omega}_W(s). \end{aligned}$$

The RHS can be estimated further by noting $\|\tilde{\Omega}_W\|_{L^2}=1$ and using Mehler's formula and finally, by taking $t=1$, we obtain

$$\tilde{\Omega}_W(s)\leqq C_2\exp\{2\|W\|_\infty-C_1s^2\},\qquad s\in\mathbb{R}, \tag{11.4}$$

with some positive constants C_1 and C_2 which are independent of W and s. Now we look at the mean spin function $\bar{\rho}(\lambda)$. Since $\Omega_\lambda(s)=\tilde{\Omega}_{V(\cdot+\lambda/\gamma)}(s-\lambda/\gamma)$, we have the conclusion from (11.4) and

$$\begin{aligned} \left|\bar{\rho}(\lambda)-\frac{\lambda}{\gamma}\right| &= \left|\int_{\mathbb{R}} s\,\tilde{\Omega}^2_{V(\cdot+\lambda/\gamma)}(s)\,ds\right| \\ &\leqq C_2^2\,e^{4\|V\|_\infty}\int_{\mathbb{R}}|s|\,e^{-2C_1s^2}\,ds<\infty. \end{aligned}$$ □

Lemma 11.6. *The equality* (11.3) *holds for every $a\geqq a_0$ and $\rho\in C_b^2(\mathbb{R})$.*

Proof. First we note that Lemma 11.5 proves $\bar{\rho}(\Lambda)=\Lambda$, since the function $\bar{\rho}=\bar{\rho}(\lambda)$ is real analytic and strictly increasing in λ. For every $\rho\in C_b^2(\mathbb{R})$, we can take a sequence $\{\rho_n\in\Lambda\}_{n=1}^{\infty}$ in such a manner that (a) ρ_n converges to ρ in the space $\mathbf{H}_{e,w}$ (i.e., for every neighborhood U of ρ, $\rho_n\in U$ for all sufficiently large n) and (b) $\Delta\{\bar{\lambda}(\rho_n(\cdot))\}$ converges to $\Delta\{\bar{\lambda}(\rho(\cdot))\}$ in the space $\mathbf{H}_{\bar{r}}$. Therefore we obtain the conclusion from Proposition 11.1 because $R_a\Psi^{(1)}\in C(\mathbf{H}_{e,w})$ and $D R_a\Psi^{(1)}(\cdot,\cdot)\in C_b(\mathbf{H}_{e,w},\mathbf{H}_{-\bar{r}})$. □

Step 3: Identification of the Limit

See the book of Ladyzenskaya, Solonnikov and Ural'ceva [18, Theorem 8.1] for the following result on the nonlinear diffusion Eq. (1.4).

Theorem. *Assume the initial value* $\rho_0\in C_b^{2+\beta}(\mathbb{R})$, $0<\beta<1$.

(i) *There exists a classical solution* ρ_t *of* (1.4) *belonging to the class* $H^{2+\beta,1+\beta/2}(\mathbb{R}\times[0,T])$ *for every* $T>0$; *see* [18, p. 7] *for the definition of this class.*

(ii) *The classical solution of* (1.4) *satisfying the condition*

$$\sup\left\{\sum_{k=0}^{2}|\nabla^k\rho_t(x)|;\, t\in[0,T],\, x\in\mathbb{R}\right\}<\infty,\qquad T>0,$$

is unique.

We denote by $\rho_t(\rho)=\rho_t(\cdot;\rho)$ the unique solution of (1.4) with initial value $\rho\in C_b^{2+\beta}(\mathbb{R})$.

Lemma 11.7. $\Psi_t^{(1)}(\rho)=\Psi(\rho_t(\rho))$ *holds for every* $t\geqq 0$ *and* $\rho\in C_b^{2+\beta}(\mathbb{R})$.

Proof. Put $f(t)=R_a\Psi^{(1)}(\rho_t(\rho))$ with fixed $a\geqq a_0$ and $\rho\in C_b^{2+\beta}(\mathbb{R})$. Then, using Lemma 11.6, we have

$$\frac{d}{dt}f(t)=\left\langle D R_a\Psi^{(1)}(\cdot,\rho_t),\frac{\partial}{\partial t}\rho_t(\cdot)\right\rangle=a f(t)-\Psi(\rho_t)$$

and therefore

$$e^{-at}f(t)=R_a\Psi^{(1)}(\rho)-\int_0^t e^{-au}\Psi(\rho_u)\,du.$$

After letting $t\to\infty$ in this equality, we now obtain $R_a\Psi^{(1)}(\rho)=R_a\{\Psi(\rho_{\cdot}(\rho))\}$ for every $a\geqq a_0$ and $\rho\in C_b^{2+\beta}(\mathbb{R})$. This verifies that $\Psi_t^{(1)}(\rho)=\Psi(\rho_t(\rho))$ for a.e. t and consequently for every $t\geqq 0$, since the both sides are continuous in t. □

Theorem 11.1. (i) *Assume* $S\in C_b^{2+\beta}(\mathbb{R})$ *with some* $0<\beta<1$. *Then* $E_S[\Psi(S_t^\varepsilon)]$ *converges to* $\Psi(\rho_t(S))$ *uniformly in* $t\in[0,T]$, $T>0$, *as* $\varepsilon\downarrow 0$ *for every* $\Psi\in\mathcal{D}$.

(ii) *Assume* $\lambda(\cdot)\in\Lambda\cap C_b^{2+\beta}(\mathbb{R})$ *with some* $0<\beta<1$. *Then* $E^{\mu_{\lambda(\cdot)},\varepsilon}[\Psi(S_t^\varepsilon)]$ *converges to* $\Psi(\rho_t)$ *uniformly in* $t\in[0,T]$, $T>0$, *as* $\varepsilon\downarrow 0$ *for every* $\Psi\in\mathcal{D}$, *where* $\rho_t=\rho_t(\bar{\rho}(\lambda(\cdot)))$.

Proof. (i) is a consequence of Lemmas 11.3 and 11.7; note the limit function $\Psi(\rho_t(S))$ is independent of the subsequence $\{\varepsilon''\}$. For the proof of (ii), we first notice the relative compactness of the family $\{E^{\mu_{\lambda(\cdot)},\varepsilon}[\Psi(S_t^\varepsilon)]; 0<\varepsilon<1\}$ in the space $C([0,\infty))$; combine Propositions 3.1 and 10.3. Then the conclusion follows, since we see from (11.1), Lemmas 11.3 and 11.7:

$$\lim_{\varepsilon\downarrow 0} R_a E^{\mu_{\lambda(\cdot)},\varepsilon}[\Psi(S_\cdot^\varepsilon)] = R_a \Psi(\rho_\cdot), \qquad a \geqq a_0. \quad \square$$

It is not difficult to see that Theorem 11.1(i) implies the assertion of Theorem 1.1.

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