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Probability Theory and Related Fields

A Tagged Particle Process in the Boltzmann-Grad Limit for the Broadwell Modell*

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Summary. We study a tagged particle process for a model dynamical system in which identical particles move deterministically with discrete velocities, initially starting from a random configuration. We pass to the Boltzmann-Grad limit so that the tagged particle process converges to a nontrivial process (for short times). We can show that recollisions are vanishing in this limit, and this fact may have one expect that the limiting process would be Markovian. Nevertheless it is not Markovian, for which claim we give intuitive reasoning as well as a mathematical proof.

0. Introduction

The model dynamical system we shall examine in this article is a modification of that studied in the previous paper [2]. The original dynamical system consists of "hard" squares in R^2 whose diagonals are of length ε and parallel (or orthogonal) to the coordinate axes. The particle (i.e., the square) moves in one of the four directions pointed by its corners with the unit modulus of speed. The collisions are defined in a trivial way (not to violate the above constraints); there are two distinct types of collisions, one called "head-on" and the other "side-to-side". In the present paper we call this dynamical system *Model I*. Now let us modify it as follows: keep the head-on collision unchanged; but suppress the effect of the side-to-side collision by interchanging the label of two colliding particles after a collision (see Fig. 1). This modified dynamical system, which we call *Model II*, is though very much unphysical still interesting from a mathematical viewpoint because of the fact mentioned below.

As in [2] we discuss the Grad limit (or Boltzmann-Grad limit): set $\varepsilon = 1/n$ and consider the n-particle system whose phase is randomly distributed by a symmetric density function, f_n say, which is chaotic with the limiting one-particle density f, so

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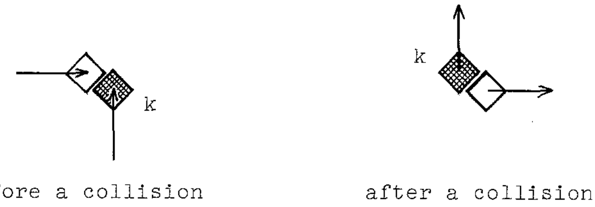

Fig. 1. In the modified dynamics the velocities are kept unchanged while positions are exchanged between colliding particles at the time of a side-to-side collision

that the m-particle-marginal density at time t, denoted by $u_{n|m}(t)$, converges to a limiting probability density (for short times). It should be noted that our modification of the dynamics does not change the marginal densities (because of the symmetry of f_n). Our interest here is in the behavior of a tagged particle; more specifically the Markovian nature of its limiting process. The crucial difference between the two dynamics is the difference for the probability of recollisions: though persistent in the Model I it is negligible in the Model II. Accordingly, while the limiting process must be non-Markovian for the former, one may expect that it would be Markovian for the latter. On the contrary, it is *non-Markovian even for the Model II* (in any short time intervals) unless f corresponds to the local equilibrium state, which claim to prove is our main purpose of this paper. (In the case of hard sphere dynamics the circumstance stands quite differently: there the tagged particle motion converges to McKean's non-linear Markov process (at least for short times) [1, 3].)

We have observed in [2] that the Boltzmann equation (the Broadwell model), which reads

$$\begin{aligned} &\frac{\partial}{\partial t} u(t,q,v)+v\cdot\frac{\partial}{\partial q} u(t,q,v) \\ &\quad =4\{u(t,q,w)u(t,q,-w)-u(t,q,v)u(t,q,-v)\} \end{aligned} \tag{0.1}$$

(see the first part of Sect. 1 for the notations used here), formally appears in the Grad limit but does not in the actual limit, i.e., the limit of $u_{n|1}(t)$, which agress with the density of the tagged particle distribution (for both dynamics), does not solve the seemingly associated Boltzmann equation (0.1). The present conclusion mentioned above must be connected with this fact. Indeed the former implies the latter (cf. [1, 3]), while the converse implication would not so definitely be proclaimed. Anyway it is the series expansion of $u_{n|1}(t)$ obtained from the BBGKY hierarchy – the same machinery that brought the latter to the surface – that we employ to prove the former. The series expansion may be described by means of a sort of perturbed system in which extra particles are added (or born) in a random fashion by the sides of existing particles alongside the action of the flow subject to our dynamics backward in time. In this perturbed system recollisions persist in the Grad limit for the Model II as well as Model I which fact essentially proves both claims. So as it might be, this persistence of recollisions does not so convincingly or directly account for the present claim as for the other one.

The reasoning of why the limiting process is not Markovian is somewhat different between the Models I and II: it is much subtler for the latter than for the former, as might be suggested by the somewhat simpler fact that *in the equilibrium situation the limiting process is Markovian for the Model II (during the time interval* $[0, 1/3]$*), but not for Model I*, provided that the particles move in the flat torus R^2/Z^2 rather than the whole R^2 so that equilibrium densities can be at once probabilities.

We shall prove the main claim in Sect. 1. The tagged particle in the Model I will be briefly dealt with in Sect. 2. In Sect. 3 we shall advance another way of reasoning to our result, which is more intuitive than the proof given in Sect. 1.

1. A Tagged Particle in the Model II

In this section we first give a precise description of the Model I and hence of the Model II, then formulate the main result, and lastly prove it.

1.1. Description of Model I

We shall use the same notation as in [2]. Let Λ be a square (a closed domain) in R^2 whose four vertices are $(\pm 1, 0)$, $(0, \pm 1)$. A particle is a square in R^2 which is a translation of the shrunken square $\frac{\varepsilon}{2}\Lambda$ $(\varepsilon > 0)$; thus a particle located at q is a square whose vertices are $q + (\pm\varepsilon/2, 0)$, $q + (0, \pm\varepsilon/2)$. Each particle moves with a constant speed v where

$$v \in S := \{(\pm 1, 0), (0, \pm 1)\}$$

between successive collisions. A collision between two particles takes place when they properly contact each other with their sides, i.e., they come into such positions q and q_1 that

$$l := \frac{1}{\varepsilon}(q_1 - q) \in \partial\Lambda \setminus \{(\pm 1, 0), (0, \pm 1)\}.$$

For the extremal case $l \in \{(\pm 1, 0), (0, \pm 1)\}$ the whole system is stopped and sent to the extra state ∂ at the moment of contact. Let v and v_1 be the velocities of two particles before the collision. For a possible collision it holds that $l \cdot v > 0$. The velocities v^*, v_1^* after collision are defined by

$$v^* = \sigma\iota v, \quad v_1^* = \sigma\iota v_1 \quad \text{if} \quad v \cdot v_1 = -1 \tag{HC}$$

$$v^* = \sigma\iota v, \quad v_1^* = -\sigma\iota v_1 \quad \text{if} \quad v \cdot v_1 = 0 \tag{SC}$$

where ι denotes the rotation operator which rotates a two-dimensional vector by $\pi/2$ around the origin counterclockwise and

$$\sigma = 1 \quad \text{or} \quad -1 \quad \text{according as} \quad (\iota v) \cdot l < 0 \quad \text{or} \quad (\iota v) \cdot l > 0.$$

We shall call collisions of the type (HC) [resp. (SC)] *head-on* [resp. *side-to-side*]. The multiple (i.e., triple or higher order) collision is undefined and when the system comes into a configuration of the multiple contact it is sent to ∂. This virtually defines the dynamical system of n particles whose phase space is

$$\Omega_n = \Omega_n^{(\varepsilon)} := \{\mathbf{x} = (x_1; \ldots; x_n) : \varepsilon^{-1}(q_k - q_j) \notin \Lambda \setminus \partial\Lambda \text{ if } k \neq j\}$$

where $x_k = (q_k, v_k)$, $q_k \in R^2$, $v_k \in S$. The boundary of Ω_n is given by

$$\partial\Omega_n = \{\mathbf{x} = (x_1; \ldots; x_n) \in \Omega_n : \varepsilon^{-1}(q_k - q_j) \in \partial\Lambda \text{ for some } k \neq j\}.$$

Let $T_t\mathbf{x} = T_t^{(\varepsilon)}\mathbf{x}$, $t \in R$ be the left continuous version of the trajectory in $\Omega_n \cup \{\partial\}$ drawn by the system starting at $\mathbf{x} \in \Omega_n$ at time zero (the left continuity is asserted as long as the system is in Ω_n). Let dv stand for a discrete measure on the velocity space S which charges each point with unit mass and put $d\mathbf{x} = dx_1 dx_2 \ldots dx_n$, $dx_j = dq_j dv_j$. Then the n-particle phases which eventually (in past or future) lead to multiple or corner-to-corner collisions form a $d\mathbf{x}$-null set, and the flow T_t preserves the measure $d\mathbf{x}$.

1.2. Model II and the Statement of Theorem

We modify the Model I (the dynamical system described above) in such a way as stated in Introduction: thus, whenever the particle labeled "k" makes a side-to-side collision, in the modified system we trace its partner particle after the collision and label the latter "k" until it undergoes the next side-to-side collision (otherwise no change). We call the new dynamical system obtained in this way *Model II*. Let $X^n(t, \mathbf{x}) = X^n(t) = (X_1^n(t), \ldots, X_n^n(t))$ be (the left continuous version of) the trajectory drawn in $\Omega_n^{(\varepsilon)}$ by the n-particle system that starts from $\mathbf{x}$ and evolves according to the new dynamics of the Model II. We shall pursue the first particle as a tagged particle. Set $\varepsilon = 1/n$. Let f_n be a probability density on $\Omega_n^{(1/n)}$, i.e., $f_n \geqq 0$ and $\int f_n d\mathbf{x} = 1$, which is symmetric, i.e., invariant under the permutation of particles, and consider the motion of the first particle $X_1^n(\cdot) = X_1^n(\cdot, \mathbf{x})$ as a stochastic process defined on the probability space $(\Omega_n^{(1/n)}, f(\mathbf{x})d\mathbf{x})$. The probability density for $(X_1^{(n)}(t), \ldots, X_m^{(n)}(t))$ $(m \leqq n)$, which exists, is denoted by $u_{n|m}(t)$; $u_{n|m}(0)$ agrees with $f_{n|m}$ the m-particle-marginal of f_n.

Let us assume the following assumptions (AI) and (AII).

(AI) $\operatorname*{ess\,sup}_{\Omega_m^{(1/n)}} f_{n|m} \leqq CM^m$ for $m \leqq n$, $n = 1, 2, \ldots$ with some constants C and M which are independent of m and n;

(AII) there exists a continuous function f on Ω_1^0 such that $\operatorname*{ess\,sup}_K |f_{n|m} - f^{m\otimes}| \to 0$ $(n \to \infty)$ for $m = 1, 2, \ldots$ and for every compact set K of $\Omega_m^0 \setminus d_m$,

where $f^{m\otimes}$ is the m-fold outer product of f, $\Omega_m^0 = \Omega_m^{(0)} = (R^2 \times S)^m$ and

$$d_m = \{\mathbf{x} \in \Omega_m^0 : q_j = q_k \text{ for at least one pair } j \neq k\}.$$

Then it is shown that there exists $\lim u_{n|1}(t, x) =: u^{(1)}(t, x)$ $(0 \leqq t \leqq 1/8M)$, which is continuous in (t, x), and the sequence of processes $X_1^n(t)$, $0 \leqq t \leqq t_0$ is weakly convergent with respect to the Skorohod topology of $D[[0, t_0], \Omega_1]$ if $t_0 < 1/8M$ (cf. [2] and Sect. 7 of [3]). As a little reflection might convince us the conditional probability

$P^{(f_n)}$ [the j-th particle or its descendants make a head-on collision with the first particle in the time interval (t, t_0)| the j-th and the first particle makes a head-on collision between them at time t]

converges to zero in the Boltzmann-Grad limit (cf. Proposition 2 of Appendix in [2]). From this one may expect that in the limit the changes of the velocity of the first particle in the past would not affect those in the future if the present velocity is known, so that the limit process would be Markovian. The fact, on the contrary, is that the limit process of the first particle is not generally Markovian as asserted by

Theorem. *In addition to* (AI) *and* (AII) *assume*

(AIII) $f(q,v)f(q,-v) \neq f(q,w)f(q,-w)$ *for some* $x=(q,v)\in\Omega_1^0$.

Then for any $T \leqq 1/8M$ *the limit process of* $X_1^n(t)$, $0 \leqq t < T$, *is not Markovian.*

Remark 1. If $U_1^0(t)f := f\circ T_{-t}^0$ (T_t^0 denotes the free motion) is Maxwellian for all t, i.e., for all t and $x\in\Omega_1^0$

$$f(q-tv,v)f(q+tv,-v)=f(q-tw,w)f(q+tw,-w)$$

(which obviously contradicts (AIII), then the limit process of $X_1^n(t)$ is Markovian. This condition holds if and only if $u(t,x) := f(q-tv,v)$ is a solution (a Maxwellian solution) of a weak version of the Boltzmann equation (0.1).

Remark 2. Set $v_k = i^k(1,0)$, $k=0,1,2,3$, and $f_k(q) := f(q,v_k)$, and write $q=(q^1,q^2)$ for $q\in R^2$. Then $U_1^0(t)f$ is Maxwellian for every t if f is of the form

$$f_0(q)=g(q^1+q^2)h(q^1-q^2)\alpha(q^2), \quad f_2(q)=\tilde g(q^1+q^2)\tilde h(q^1-q^2)/\alpha(q^2),$$
$$f_1(q)=g(q^1+q^2)\tilde h(q^1-q^2)\beta(q^1), \quad f_3(q)=\tilde g(q^1+q^2)h(q^1-q^2)/\beta(q^1)$$

where g, $\tilde g$, h, $\tilde h$, α and β are continuous functions on R and α and β are positive. Conversely the Maxwellian solution is necessarily given by f of this form, provided that $f>0$. In fact let f be positive and $U_1(t)f$ Maxwellian for all t in a neighbourhood of zero, and assume that f_k's are smooth, which gives rise to no loss of generality because of the linearity of the relation to be satisfied by ($\log f_k$)'s. Taking the logarithms of the both sides of the assumed identity, and then differentiating them by t on the one hand and independently of it by q^1 on the other hand two times each, we observe that the function

$$H_t(q) := \log f_0(q^1-t,q^2)+\log f_2(q^1+t,q^2)$$

satisfies the wave equation $(\partial/\partial q^1)^2 H_t = (\partial/\partial q^2)^2 H_t$ (for each t). Differentiating this wave equation by t and by q^1 once more each together with simple manipulation of subtracting shows that $(\partial/\partial q^1)\log f_0(q)$ also satisfies the wave equation, so that f_0 must be of the required form.

Remark 3. The Theorem above implies that the process $(P^{(f)}, X(s), s\geqq t)$ is not necessarily determined by $u^{(1)}(t)$ (which fact follows also from the fact that $u^{(1)}(t)$ does not solves the Boltzmann equation), but for the converse implication we would need to closely examine the mechanism that brings the latter fact into being.

1.3. Proof of the Theorem (Divided into Three Steps)

Step 1. Let $\hat{T}_t = \hat{T}_t^{(\varepsilon)}$ denote the flow for the Model II so that $\hat{T}_t \mathbf{x} = X^n(t, \mathbf{x})$, and $\hat{U}_n(t)$, $\hat{\mathscr{U}}_n$, $\hat{M}_{k,\Delta}^{(\varepsilon)}$ etc. the corresponding operators, some of them that are relevant to this proof being to be defined later (cf. [2] or [3] for the definitions of the original operators). We denote by $(P^{(f_n)}, X^n(t))$ and $(P^{(f)}, X(t))$, respectively, the deterministic process with the initial density f_n and the limit process of $X_1^n(t)$ (the first component of $X^n(t)$); by $E^{(f_n)}$ and $E^{(f)}$ the corresponding expectations; and finally, for a sequence $g_m \in C(\Omega_m^{(1/n)})$ $(m = 1, \ldots, n)$, let the expression $\mathbb{L}_t^n(\{g_m\}_{m=1}^n)$ stand for the sequence whose m-th entry $(m = 1, \ldots, n)$ is

$$\sum_{k=0}^{n-m} (n-m)_k \varepsilon^k (\hat{\mathscr{U}}^{(1/n)} \hat{K}^{(1/n)})^k (\hat{U}_{m+k}^{(1/n)}(\cdot) g_{m+k})(t, x_1, \ldots, x_m) \tag{1.1}$$

(this sum is nothing but the series expansion for $u_{n|m}(t)$ if $\{g_m\} = \{f_{n|m}\}$; an alternative expression for (1.1) will be given later which may be taken as the definition in the present paper). Then for $\phi \in C_0(\Omega_1^0)$ and $0 < s < t < t_0$

$$\begin{aligned} H_t^s\{\phi\}(x) &:= E^{(f)}[(\phi/u^{(1)}(s))(X(s)); X(t) \in dx]/dx \\ &= \lim_{n\to\infty} E^{(f_n)}[(\phi/u^{(1)}(s))(X_1^n(s)); X_1^n(t) \in dx]/dx \\ &= \lim_{n\to\infty} [\mathbb{L}^n(\{(\phi/u^{(1)}(s)) u_{n|m}(s)\}_{m=1}^n)]_1, \qquad (x \in \Omega_1^0) \end{aligned}$$

where $[\{\cdot\}]_1$ stands for the first entry of the sequence $\{\cdot\}$ (cf. [2, 3]). For a Borel set A of $x = (q, v)$

$$\begin{aligned} &\int_A H_t^s\{H_s^0\{\phi f\}\}(x)dx \\ &= \int (E^{(f)}[\phi(X(0)); X(s) \in dy]/dy) \frac{1}{u^{(1)}(s,y)} P^{(f)}[X(s) \in dy, X(t) \in A] \\ &= \int E^{(f)}[\phi(X(0)); X(s) \in dy] P^{(f)}[X(t) \in A \mid X(s) = y]. \end{aligned}$$

If $(P^{(f)}, X(t))$ were Markovian, the last integral above must have been equal to

$$E^{(f)}[\phi(X(0)); X(t) \in A] = \int_A H_t^0\{\phi f\}(x)dx.$$

Such equality fails to hold except for special f's. In fact we shall prove

$$\begin{aligned} &\lim_{t\downarrow 0} t^{-4}[H_t^{t/2}\{H_{t/2}^0\{\phi f\}\}(x) - H_t^0\{\phi f\}(x)] \\ &= \frac{5\sqrt{2}}{9}\{\phi(q,v)F(v) + \frac{1}{4}F(w) + \frac{1}{4}F(-w)] + \frac{1}{4}\phi(q,-v)[F(w) + F(-w)] \\ &\qquad - \frac{1}{2}\phi(q,w)[F(v) + F(w)] - \frac{1}{2}\phi(q,-w)[F(v) + F(-w)]\} \end{aligned} \tag{1.2}$$

where

$$F(v) = F_q(v) := f(w)f(-w)f(-v)[f(v)f(-v) - f(w)f(-w)]$$

(the argument q is suppressed from f and F). If we take this relation as granted, the proof of Theoem is ready. In fact it proves that if $F(v) \neq 0$ for some x, then the limiting process $X(t)$ is not Markovian in any short times. Even when $f(v)f(-v)f(w)f(-w)=0$ for x satisfying (AIII), we have $F^{(t)}(v) \neq 0$ for very sufficiently small t where $F^{(t)}(v)$ is defined as above but with $u^{(1)}(t)$ in place of f and hence the same conclusion.

If the condition of Remark 1 is satisfied (i.e., $U_1^0(t)f$ is Maxwellian for all t), then any collision in the addition-backward flow evolution effects no change of $f^{(m+k)\otimes}(\hat{T}^{(\varepsilon)}_{-t+s_k}\hat{M}^{(\varepsilon)}_{k,\Delta}\mathbf{x})$ that persists in the Grad limit. This combined with the fact that the set of parameters Δ that cause the first – not the other – particle a head-on collision other than those forced by the operations of adding particles in the perturbed flow is negligible in the Grad limit enables us to follow the arguments of [1] and [3] to see that $X(t)$ is Markovian. (Note that in the series expansion of $H_t^0\{\phi f\}$ the argument of ϕ is always the phase of the first particle.)

The relation (1.2) will be proved through the following two steps.

Step 2. In this step we derive an expression, given by (1.3) below, to the difference in the bracket [ ] on the left hand side of (1.2). First let us introduce some notations. For $\mathbf{x}\in\Omega_m^{(\varepsilon)}$, $l\in\partial\Lambda$, $v\in S$ and $j=1,2,\dots,m$ put

$$C_{j,0}^{v,l}\mathbf{x}=(x_1;\dots;x_{j-1};q_j,v_j^*;x_{j+1};\dots;x_m;q_j-\varepsilon l,v^*)$$
$$C_{j,1}^{v,l}\mathbf{x}=(x_1;\dots;x_m;q_j+\varepsilon l,v),$$

if $v\cdot l<0$ and $v_j\cdot l>0$;

$$C_{j,0}^{v,l}\mathbf{x}=C_{j,1}^{v,l}\mathbf{x}=\partial \quad \text{if} \quad v\cdot l\geqq 0 \quad \text{or} \quad v_j\cdot l\leqq 0; \qquad \text{and} \quad C_{j,\sigma}^{v,l}\partial=\partial \ (\sigma=0,1)$$

(∂ is an extra point). For $k=1,2,\dots,n-m$, $x\in\Omega_m^{(\varepsilon)}$ and a set of multivariables $\Delta=(\mathbf{s},\mathbf{l},\mathbf{v},\boldsymbol{\sigma},\mathbf{j})$ where $\mathbf{l}\in(\partial\Lambda)^k$, $\boldsymbol{\sigma}\in\{0,1\}^k$, $\mathbf{v}=(v_{m+1},\dots,v_{m+k})\in S^k$ and

$$\mathbf{s}=(s_1,\dots,s_k)\in[0,\infty)^k \quad \text{with} \quad s_1<s_2<\dots<s_k$$
$$\mathbf{j}=(j_1,\dots,j_k) \quad \text{with} \quad 1\leqq j_p\leqq m+p-1 \ (p=1,\dots,k),$$

put $\hat{M}^{(\varepsilon)}_{0,\Delta}\mathbf{x}=\mathbf{x}$ and

$$\hat{M}^{(\varepsilon)}_{k,\Delta}\mathbf{x}=C^{v_{m+k},l_k}_{j_k,\sigma_k}\hat{T}^{(m+k)}_{-s_k+s_{k-1}}\dots C^{v_{m+1},l_1}_{j_1,\sigma_1}\hat{T}^{(m)}_{-s_1+s_0}\mathbf{x}$$

where $s_0=0$. Then, by writing $|\boldsymbol{\sigma}|=\Sigma_j\sigma_j$, the sum in (1.1) can be written as

$$g_m(\hat{T}_{-t}\mathbf{x})+\sum_{k=1}^{n-m}\sum_{\boldsymbol{\sigma}}\sum_{j_k=1}^{m+k-1}\dots\sum_{j_1=1}^{m}(-1)^{|\boldsymbol{\sigma}|}(n-m)_k\varepsilon^k\sqrt{2}^k$$
$$\times\int_0^t ds_k\int_0^{s_k}\dots\int_0^{s_2}ds_1\int_{(\partial\Lambda)^k\times S^k}g_{m+k}(\hat{T}_{-t+s_k}\hat{M}^{(\varepsilon)}_{k,\Delta}\mathbf{x})d\mathbf{l}d\mathbf{v}.$$

The mapping $\mathbf{x}\to\hat{M}^{(\varepsilon)}_{k,\Delta}\mathbf{x}$ is made up of successive applications of operation of the time reversed flow $\hat{T}^{(\varepsilon)}_{-t}$ and operation $C^{v,l}_{j,\sigma}$ of adding a new particle beside the j-th particle. Correspondingly we shall be concerned with a particle-history of an evolving system during a time interval $[0,t]$, determined by t, Δ and $\mathbf{x}\in\Omega_m^{(\varepsilon)}$, which starts at $\mathbf{x}$ at time zero and ends in $T^{(\varepsilon)}_{-t+s_k}\hat{M}^{(\varepsilon)}_{k,\Delta}\mathbf{x}$ at time t. In this system new particles are added according to the applications of $C^{v,l}_{j,\sigma}$ at times $s_1,\dots,s_k$ so that the number of particles increases and the system evolves by the time-reversed flow $\hat{T}^{(\varepsilon)}_{-s}$ of $m+j$ particles during each time interval $[s_j, s_{j+1}]$, $j=0,\dots,k$ $(s_0=0, s_{k+1}=t)$.

Put

$$\mathscr{T}^t_{m,k}=\{\Delta=(\mathbf{s},\mathbf{v},\mathbf{l},\mathbf{j},\boldsymbol{\sigma}):\mathbf{s}=(s_1,\dots,s_k)\text{ etc., } 1\leqq j_1\leqq m, s_k\leqq t\}$$

(m indicates the number of components of $\mathbf{x}$ to which $\hat{M}_{k,\Delta}$ should be applied, and k the number of the particles added by $\hat{M}_{k,\Delta}$). Given $0<s<t$ and $x\in\Omega^0_1$ we shall use the following abbreviations: for $\Delta\in\mathscr{T}^{t-s}_{1,k}$ and $\Delta'\in\mathscr{T}^s_{k+1,k'}$

$$X^\Delta\{x\}=\lim_{n\to\infty}\hat{T}^{(1/n)}_{-(t-s)+s_k}\hat{M}^{(1/n)}_{k,\Delta}x$$

$$Y^{\Delta'}\{x\}=\lim_{n\to\infty}\hat{T}^{(1/n)}_{-s+s'_{k'}}\hat{M}^{(1/n)}_{k',\Delta'}x\quad(\text{for }\Delta'\in\mathscr{T}^s_{1,k'})$$

$$Y^{\Delta'}X^\Delta\{x\}=\lim_{n\to\infty}\hat{T}^{(1/n)}_{-s+s'_{k'}}\hat{M}^{(1/n)}_{k',\Delta'}\hat{T}^{(1/n)}_{-(t-s)+s_k}\hat{M}^{(1/n)}_{k,\Delta}x$$

and

$$X^0\{x\}=T^0_{-(t-s)}x,\qquad Y^0\{x\}=T^0_{-s}x,\qquad Y^0X^\Delta\{x\}=\lim\hat{T}^{(1/n)}_{-t+s_k}\hat{M}^{(1/n)}_{k,\Delta}x.$$

Also we write as follows: for a function g^Δ of Δ

$$\int_{\mathscr{T}^t_{m,k}}d\Delta g^\Delta=\begin{cases}\sqrt{2}^k\sum_{\mathbf{v}}\sum_{\mathbf{j}}\sum_{\boldsymbol{\sigma}}(-1)^{|\boldsymbol{\sigma}|}\int_{(\partial\Lambda)^k}d\mathbf{l}\int_0^t ds_k\int_0^{s_k}\dots\int_0^{s_2}ds_1 g^\Delta & \text{if } k\geqq 1\\ g^0 & \text{if } k=0\end{cases}$$

where the sum runs over all $\mathbf{v}=(v_{m+1},\dots,v_{m+k})$, etc. Then

$$\lim_{n\to\infty}[\mathbb{L}^n_{t-s}(\{g_k\}^n_{k=1})]_1=\sum_{k=0}^\infty\int_{\mathscr{T}^{t-s}_{1,k}}d\Delta g_k(X^\Delta_1).$$

Now, setting

$$\psi_s(x)=H^0_s\{\phi f\}(x)/u^{(1)}(s,x)$$

and making use of $\mathbb{L}^n_t=\mathbb{L}^n_{t-s}\circ\mathbb{L}^n_s$, we have

$$\begin{aligned}I_{s,t}:&=H^s_t\{H^0_s\{\phi f\}\}-H^0_t\{\phi f\}\\ &=\lim_{n\to\infty}[\mathbb{L}^n_{t-s}(\{\psi_s u_{n|k}(s)\}^n_{k=1})-\mathbb{L}^n_{t-s}(\mathbb{L}^n_s(\{\phi f_{n|k}\}^n_{k=1}))]_1\\ &=\sum_{k,k'=0}^\infty\int_{\mathscr{T}^{t-s}_{1,k}}d\Delta\int_{\mathscr{T}^s_{k+1,k'}}d\Delta'\{\psi_s(X^\Delta_1)-\phi(Y^{\Delta'}_1X^\Delta)\}f^{(k+k'+1)\otimes}(Y^{\Delta'}X^\Delta)\end{aligned}$$

where $X^\Delta_1=X^\Delta_1\{x\}$ denotes the phase of the first particle (i.e., the particle which starts at x) in $X^\Delta\{x\}$ (and similarly for Y^Δ_1). Here and below x is suppressed. Recalling $u^{(1)}(s,x)=H^0_s\{f\}(x)$, we have

$$\psi_s(X^\Delta_1)-\phi(Y^{\Delta'}_1X^\Delta)=[H^0_s\{\phi f\}(X^\Delta_1)-\phi(Y^{\Delta'}_1X^\Delta)H^0_s\{f\}(X^\Delta_1)]/u^{(1)}(s,X^\Delta_1)$$

and, substituting the expression

$$H^0_s\{\chi f\}(X^\Delta_1)=\sum_{k''=0}^\infty\int_{\mathscr{T}^s_{1,k''}}d\Delta''\chi(Y^{\Delta''}_1\{X^\Delta_1\})f^{(k''+1)\otimes}(Y^{\Delta''}\{X^\Delta_1\})$$

with $\chi=\phi$ and 1,

$$I_{s,t}=\sum_{k,k',k''=0}^{\infty}\int_{\mathcal{T}^{t-s}_{1,k}} d\Delta \int_{\mathcal{T}^{s}_{k+1,k'}} d\Delta' \int_{\mathcal{T}^{s}_{1,k''}} d\Delta'' u^{(1)}(s,X^{\Delta}_1)^{-1}$$
$$\times[\phi(Y^{\Delta''}_1\{X^{\Delta}_1\})-\phi(Y^{\Delta'}_1X^{\Delta})]f^{(k''+1)\otimes}(Y^{\Delta''}\{X^{\Delta}_1\})f^{(k+k'+1)\otimes}(Y^{\Delta'}X^{\Delta}) \tag{1.3}$$

Step 3. We must compute the limit of $t^{-4}I_{t/2,t}$ as $t\downarrow 0$. Put $s=t/2$. The m-th order term of $I_{s,t}=I_{t/2,t}$ as a function of t is well represented by the sum of those terms on the right-hand side of (1.3) for which $k+k'+k''=m$. If the effect of interactions diminishes to zero in the Boltzmann-Grad limit, then such a sum vanishes for each m. This is the case for $m\leqq 2$. The sum is zero also for $m=3$. To see this we have merely to consider the contribution to the sum by those $(\Delta,\Delta',\Delta'')$ for which $Y^{\Delta'}X^{\Delta}$ involves an interaction of the head-on collision. But this last condition for $(\Delta,\Delta',\Delta'')$ entails another condition that $k+k'=3$ (i.e., $k''=0$) and then these two conditions together imply $Y^{\Delta''}_1\{X^{\Delta}_1\}=T^0_{-s}X^{\Delta}_1=Y^{\Delta'}_1X^{\Delta}$, showing that the corresponding contribution vanishes.

Let us now compute the fourth order term. Thus we look at those terms on the right-hand side of (1.3) for which

$$k+k'+k''=1\,. \tag{1.4}$$

Given a Δ, we denote by $A(\Delta',\Delta'')$ the integrand of the triple integral in (1.3). If $k=0$, then $A(\Delta',\Delta'')+A(\Delta'',\Delta')=0$ and therefore the corresponding integral over Δ' and Δ'' under (1.4) vanishes.

Let us consider the case $k=3$. Let the expression $\Delta'\in\mathcal{H}$ denote that Δ' adds extra particles only beside the first particle or its descendants born by the application of Δ'. If $k'=1$, $k''=0$ and $\Delta'\notin\mathcal{H}$, then $A(\Delta',\Delta'')=0$. Thus the integral of $A(\Delta',\Delta'')$ over Δ' and Δ'' under (1.5) with $k=3$ is reduced to that over

$$k'=0 \quad\text{and}\quad k''=1\,;\qquad\text{or}\quad k'=1,\qquad k''=0\quad\text{and}\quad \Delta'\in\mathcal{H}\,. \tag{1.5}$$

If $\Delta'\in\mathcal{T}^s_{m,k}$ then we can naturally consider Δ' as a member of $\mathcal{T}^s_{m+j,k}$ $(j\geqq 1)$. Since $k=3$, one can easily ascertain that if $\Delta''\in\mathcal{T}^s_{1,1}$, then $Y^{\Delta''}_1X^{\Delta}=Y^{\Delta''}_1\{X^{\Delta}_1\}$. Noting this relation we see that the integral of $A(\Delta',\Delta'')$ over (1.4) can be written

$$\int_{\mathcal{T}^s_{1,1}} d\Delta''[\phi(Y^{\Delta''}_1\{X^{\Delta}_1\})-\phi(Y^0_1X^{\Delta}))f^{2\otimes}(Y^{\Delta''}_1\{X^{\Delta}_1\})f^{4\otimes}(Y^0X^{\Delta})$$
$$+(\phi(Y^0\{X^{\Delta}_1\})-\phi(Y^{\Delta''}_1\{X^{\Delta}_1\}))f(Y^0\{X^{\Delta}_1\})f^{5\otimes}(Y^{\Delta''}X^{\Delta})]$$
$$=\int_{\mathcal{T}^s_{1,1}} d\Delta''[\phi(Y^{\Delta''}_1\{X^{\Delta}_1\})-\phi(Y^0\{X^{\Delta}_1\})]$$
$$\times[f^{2\otimes}(Y^{\Delta''}\{X^{\Delta}_1\})f^{4\otimes}(Y^0X^{\Delta})-f(Y^0\{X^{\Delta}_1\})f^{5\otimes}(Y^{\Delta''}X^{\Delta})]\,. \tag{1.6}$$

If $Y^{\Delta''}_1\{X^{\Delta}_1\}\neq Y^0\{X^{\Delta}_1\}$ (i.e., Δ'' represents the addition of a particle in a position of the head-on collision with $\sigma''_1=0$), then the operation of Δ'' does not cause a new interaction in $Y^{\Delta''}X^{\Delta}$ so that the second factor of the integrand in the last integral vanishes. Therefore the integral itself vanishes. Consequently the integral of $A(\Delta',\Delta'')$ is zero if $k=3$.

As for the case $k=2$, we divide it into the following subcases.

(i) $k'=0$, $k''=2$.
(ii.1) $k'=2$, $k''=0$ and $\Delta' \in \mathscr{H}$.
(ii.2) $k'=2$, $k''=0$ and Δ' adds one particle beside the first particle and the other beside one of the remaining two particles in X^Δ.
(ii.3) $k'=2$, $k''=0$ and Δ' adds no particle beside the first particle.
(iii.1) $k'=k''=1$ and $\Delta' \in \mathscr{H}$.
(iii.2) $k'=k''=1$ and $\Delta' \notin \mathscr{H}$.

The integral over (ii.3) trivially vanishes. By the same reasoning as advanced for (1.6) the integral over (iii.1) vanishes. To treat the case (ii.2) we introduce some notational convention. Namely we denote by Δ_1' the part of Δ' which adds a particle beside the first particle and by Δ_2' the other part of Δ', and write $\Delta'=(\Delta_1', \Delta_2')$ for convenience. Then making Δ' run over (ii.2) amounts to making Δ_1' and Δ_2' run independently of each other over

$$\Delta_1' \in \mathscr{T}_{3,1}^s \cap \mathscr{H} \quad \text{and} \quad \Delta_2' \in \mathscr{T}_{3,1}^s \backslash \mathscr{H}.$$

Then the integral of $A(\Delta', \Delta'')$ over (ii.2) and (iii.2) is reduced to

$$\int_{\mathscr{T}_{1,1}^s} d\Delta'' \int_{\mathscr{T}_{3,1}^s \backslash \mathscr{H}} d\Delta' [\phi(Y^0\{X_1^\Delta\}) - \phi(Y_1^{\Delta''}\{X_1^\Delta\})]$$
$$\times [f(Y^0\{X_1^\Delta\}) f^{5\otimes}(Y^{(\Delta'', \Delta')}X^\Delta) - f^{2\otimes}(Y^{\Delta''}\{X_1^\Delta\}) f^{4\otimes}(Y^{\Delta'}X^\Delta)],$$

where $\mathscr{T}_{3,1}^s \cap \mathscr{H}$ naturally is identified with $\mathscr{T}_{1,1}^s$. This vanishes for every Δ as is easily seen.

Finally the integral over (i) and (ii.1) equals

$$a^\Delta := \int_{\mathscr{T}_{1,2}^s} d\Delta' [\phi(Y_1^{\Delta'}\{X_1^\Delta\}) - \phi(Y^0\{X_1^\Delta\})]$$
$$\times [f^{3\otimes}(Y^{\Delta'}\{X_1^\Delta\}) f^{3\otimes}(Y^0 X^\Delta) - f(Y^0\{X_1^\Delta\}) f^{5\otimes}(Y^{\Delta'}X^\Delta)]$$

($\mathscr{T}_{3,2}^s \cap \mathscr{H}$ is identified with $\mathscr{T}_{1,2}^s$), which this time does not vanish for appropriate Δ's (see Fig. 2).

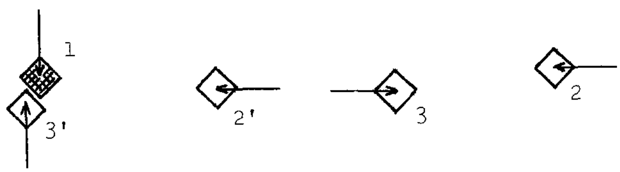

Fig. 2. Here is illustrated a typical configuration, which contributes to the integral of a^Δ with $\boldsymbol{\sigma}=(1,1)$, $\mathbf{j}=(1,2)$, $\mathbf{v}=(-v,v)$, and $\boldsymbol{\sigma}'=(1,0)$, $\mathbf{j}'=(1,1)$, $\mathbf{v}'=(-v,v)$, just after the last addition operation is made

For $k=1$ the computation is carried out similarly. It is readily seen that there is no contribution to $I_{s,t}$ from subcases of $k=1$ other than the following three:

1) $k'=2$, $k''=1$ and Δ' is decomposed as $\Delta'=(\Delta_1', \Delta_2')$ where Δ_1' adds a particle beside the first particle and Δ_2' beside the other one of X^Δ;

2.1) $k'=1$, $k''=2$ and $\Delta'\notin\mathscr{H}$;

2.2) $k'=3$, $k''=0$ and Δ' is decomposed as $\Delta'=(\Delta'_1,\Delta'_2)$ with $\Delta'_1\in\mathscr{T}^s_{2,2}\cap\mathscr{H}$ and $\Delta'_2\in\mathscr{T}^s_{2,1}\backslash\mathscr{H}$.

The integral of $A(\Delta',\Delta'')$ over 1) is

$$b_1^\Delta := \int_{\mathscr{T}^s_{1,1}} d\Delta'_1 \int_{\mathscr{T}^s_{2,1}\backslash\mathscr{H}} d\Delta'_2 \int_{\mathscr{T}^s_{1,1}} d\Delta''[\phi(Y_1^{\Delta''}\{X_1^\Delta\})-\phi(Y_1^{\Delta'_1}\{X_1^\Delta\})]$$
$$\times f^{2\otimes}(Y^{\Delta''}\{X_1^\Delta\})f^{4\otimes}(Y^{(\Delta'_1,\Delta'_2)}X^\Delta).$$

Set $\mathscr{T}^{s,0}_{1,1}=\{\Delta\in\mathscr{T}^s_{1,1}:\sigma_1=0\}$ (the operation of $\Delta\in\mathscr{T}^{s,0}_{1,1}$ changes the velocity of the first particle) and $\mathscr{T}^{s,1}_{1,1}=\mathscr{T}^s_{1,1}\backslash\mathscr{T}^{s,0}_{1,1}$. Then

$$b_1^\Delta := \int_{\mathscr{T}^s_{2,1}\backslash\mathscr{H}} d\Delta'_2 \int_{\mathscr{T}^{s,1}_{1,1}\backslash\mathscr{H}} d\Delta'_1 \int_{\mathscr{T}^{s,0}_{1,1}} d\Delta''[\phi(Y_1^{\Delta''}\{X_1^\Delta\})-\phi(Y_1^{\Delta'_1}\{X_1^\Delta\})]$$
$$\times [f^{2\otimes}(Y^{\Delta''}\{X_1^\Delta\}f^{4\otimes}(Y^{(\Delta'_1,\Delta'_2)}X^\Delta)$$
$$-f^{2\otimes}(Y^{\Delta'_1}\{X_1^\Delta\})f^{4\otimes}(Y^{(\Delta'',\Delta'_2)}X^\Delta)].$$

The integral over 2.1) and 2.2) together is

$$b_2^\Delta := \int_{\mathscr{T}^s_{2,1}\backslash\mathscr{H}} d\Delta'_2 \int_{\mathscr{T}^s_{1,2}} d\Delta''[\phi(Y_1^{\Delta''}\{X_1^\Delta\})-\phi(Y^0\{X_1^\Delta\})]$$
$$\times [f^{3\otimes}(Y^{\Delta''}\{X_1^\Delta\})f^{3\otimes}(Y^{\Delta'_2}X^\Delta)-f(Y^0\{X_1^\Delta\})f^{5\otimes}(Y^{(\Delta'',\Delta'_2)}X^\Delta)].$$

Observe that the range of the inner integral above can be reduced to the set of $\Delta''=(\mathbf{s}'',\mathbf{l}'',\ldots)\in\mathscr{T}^s_{1,2}$ such that $\sigma''_1=1$, $\sigma''_2=0$ (i.e., the first particle changes its velocity by and only by the second addition of a particle in the operation of Δ''). Then a little reflection proves that $b_1^\Delta+b_2^\Delta=0$. Consequently

$$I_{s,t}=\int_{\mathscr{T}^{t-s}_{1,2}} u^{(1)}(s,X_1^\Delta)^{-1}\,d\Delta a^\Delta+O(t^5).$$

An elementary calculation shows that the integral above with $s=t/2$ equals $B(t)[-h(v)+\frac{1}{2}(h(w)+h(-w))]$ where

$$h(v)=h(q,v)=\{\tfrac{1}{2}[\phi(q,w)+\phi(q,-w)]-\phi(q,v)\}F(v)$$

and

$$B(t)=4A\int_0^{t/2} ds_2\int_0^{s_2} ds_1\int_0^{t/2} dt_2\int_0^{t_2} dt_1\, I\left\{t_2-t_1<\frac{t}{2}-s_1\right\}=\frac{5\sqrt{2}}{9}t^4.$$

(A is the same constant as in the proof of (1.9) of [2]). This proves the relation (1.2) and hence completes the proof of the Theorem.

2. A Tagged Particle in the Model I

We briefly present reasoning of why

for the Model I the limiting process of a tagged particle is not Markovian even in the equilibrium situation. (2.1)

Intuitively this is convincingly explained by the persistence of recollisions: if the tagged particle makes a side-to-side collision with another particle, then it will repeat side-to-side collisions with the same particle with a good probability that does not vanish in the Grad limit. Here must be noticed the crutial role played by the head-on collisions that the partner particle of the side-to-side collision makes. Indeed if we suppress the head-on collision (in a way analogous to the case of the side-to-side collision), then the recollisions of course still persist in the Grad limit but they do not affect the Markovian nature of the tagged particle process, since the effect of collisions on the tagged particle for this dynamical system is essentially the same as those for the dynamical system in which side-to-side collisions that do not involve the tagged particle are further suppressed. (Thus in this trivial situation the persistence of the recollisions does not contradict the Markov property of the tagged particle process.)

For the (mathematical) demonstration of (2.1) we can proceed as in the previous section, starting from the relation (1.3) in which X^{Δ}, Y^{Δ} etc. are understood to be defined with the original dynamics. It suffices to consider the sum with $k+k'+k''=3$ in (1.3). For $k=0$ or 3 the roles of Δ'' and Δ' are symmetrical and the corresponding integrals in (1.3) cancels each other. Let $k=1$ or 2. Then in the process resulting in $Y_1^{\Delta''}\{X_1^{\Delta}\}$ there can be no collision, while for $Y_1^{\Delta'}X^{\Delta}$ with appropriate Δ' and Δ, we may have (side-to-side) collisions which (inevitably) involve the first particle, to the effect that

$$\lim_{t\downarrow 0} t^{-3}[H_t^{t/2}H_{t/2}^{0}\{\phi f\}(x)-H_t^{0}\{\phi f\}(x)]$$
$$=C\{[\phi(q,v)-\phi(q,-w)]F_1(q,v)+[\phi(q,v)-\phi(q,w)]F_2(q,v)\}$$

where F_1 and F_2 are positive functions determined by f only and C is a positive constant. Therefore the limiting process is not Markovian, whether the system is in the equilibrium or not.

3. Intuitive Reasoning

For our model the Boltzmann equation does not emerge in the Grad limit, which fact, as is well understood, must imply that the two particles which are about to collide are correlated. The latter fact may strongly suggest our present claim that the limiting process is not Markovian, but does the correlation asserted therein alone really prove the claim? At a very heuristic level one would be able to affirmatively answer this question as follows. Since any particle of which we know only its present state is expected to have made no collision in the near past with high probability, the correlation of two particles being about to collide is almost the same as that measured under the condition that one of the two has made no collision in the near past. On the other hand one may well guess that these two particles might be hardly correlated if we know that one of them, say α, is right after a collision, because, conditioned on this event, the existence of the particle α at a given state, x say, does not seem able to have any influence on the probability that there is found a particle that is about to collide with another particle in the state x if any. (These together amount to saying that it is the duration of the particle's run without any collision that causes the correlation.) Therefore it would make a difference to the correlation at issue whether we have or have not certain knowledge about the past history of one of the two particles, implying what was to be proved.

What we shall actually do below is to compare the two conditional probabilities of the first particle's future collision, the one conditional on its making no collision in the near past and the other on its being right after a collision. We shall reduce this comparison to that between the corresponding conditional probabilities of having a particle, a "scatterer", that is running ahead of the first particle, reducing the chance of its future collision. Figure 3 illustrates how is produced such an effect that the conditioning on being right after a collision increases the probability of having a scatterer. For this to be compatible with the fact that in the equilibrium case the process is Markovian there must be another effect that counteracts it, so we shall need to estimate these effects with some accuracy.

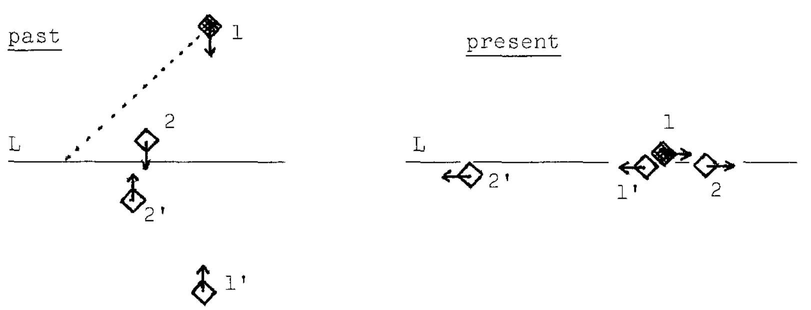

Fig. 3. Suppose that the particle 1 is observed running along a line, L say, at present. If it has made a head-on collision just before its coming into the present phase, then with a good probability there might take place a head-on collision between some two particles, 2 and 2′ say, on or nearly on L at some past time (as illustrated in the figure left) from which it ensues that one of them, say 2, either runs ahead of the particle 1 at present so as to play a scatterer in future or has already played it and disappeared from along L.

Through the rest of this section we shall make the arguments given above precise to provide another proof of the Theorem stated in Sect. 1. Here we are concerned only with the Model II and, for the sake of brevity, exclude the (suppressed) side-to-side collisions from our account (to do this can be justified: their effects cancels each other or, otherwise, are small enough to neglect). Suppose that the present time is t and consider the conditional probability that the first particle makes a (head-on) collision during $(t, t+\tau)$, $\tau > 0$, given its present phase, $x = (q, v)$ say, together with either the event

$\mathscr{A}_1$: the first particle has not collided during $[0, t]$,

or the event

$\mathscr{A}_2$: the first particle has just collided but not made any other collision during $[0, t)$,

(in the case of $\mathscr{A}_2$ the present phase x represents the post-collisional phase). In what follows we compare the probabilities p_1 and p_2:

$p_k := P^{(f)}$ [the first particle makes a (head-on) collision during $[t, t+\tau)|X(t)=x, \mathscr{A}_k]$, $(k=1,2)$

provided that both t and τ/t are small enough.

We must first discuss in terms of the n-particle process $X^{(n)}(t)$ our problem of estimating $p_1 - p_2$. For a moment we accordingly take p_k as defined with $(P^{(f_n)}, X^{(n)}(t))$ in stead of $(P^{(f)}, X(t))$.

The partner particle of the first particle's future collision, if any, may come out in front of the first particle in two ways: (A) without any collision during $(t, t+\tau)$ and (B) via a collision made with another particle during $(t, t+\tau)$. First we consider the contribution from the case (A) to the probability p_k in question ($k=1$ or 2), which we shall need to compute up to the order $O(t\tau^2)$. By letting, e.g., $v=(1,0)$, this contribution may be written as

$$(n-1) \int_{-\varepsilon}^{\varepsilon} dh \int_0^{2\tau} u_{n|1}(t+s, q+(s,h), \ -v)(1-b_k(s,h))ds \tag{3.1}$$

plus a negligible term, where $b_k(s,h)$ denotes the conditional probability, given $\mathscr{A}_k$ as well as $X_1^{(n)}(t)=x$, of the event $\mathscr{E}$ which (if $v=(1,0)$) is defined by

$\mathscr{E}$: presently there is a particle (ahead of the first particle) with the velocity $v=(1,0)$ in the region $\{q+(\xi, h+\eta): 0<\xi<s, \ -\varepsilon<\eta<\varepsilon\}$.

As for the second possibility (B), it makes little difference in comparison with the case (A) which of $\mathscr{A}_1$ or $\mathscr{A}_2$ we known occurs, since τ/t is supposed to be very small (to be more precise the effect to p_1-p_2 is similar but of smaller order).

Now, taking the limit of (3.1) as $n\to\infty$, we again consider p_k and b_k as defined with the limit process, and see that the difference p_1-p_2 can be estimated by computing the difference $b_2(s,h)-b_1(s,h)$.

Let us deal with $b_2(s,h)$ first. The particle appearing in the description of $\mathscr{E}$ can come out ahead of the first particle in the two ways (A) and (B) as before but with s replacing τ. If $\mathscr{A}_2$ is the case the existence of the first particle has little influence on each of these ways and, as in [2] (see a discussion leading to (2.12) therein), we can easily find that

$$b_2(s,h) = 2(1-4tf(-v))f(v)s + 8f(\imath v)f(-\imath v)ts + o(ts).$$

Here (and below) we omit the position arguments from f, for they are not significant. (For a more exact expression the first two terms on the right-hand side should be replaced by certain integrals over the starting positions (at time 0) of two particles; the errors, however, can be absorbed in the remainder term $o(ts)$ owing to the continuity of f. Alternatively one may argue that the same integrals should appear for the expression of $b_1(s,k)$ so that they cancels each other in the subtraction b_1-b_2.)

As for $b_1(s,h)$ we see that the possibility to the way of (B) is largely cut off by the condition that $\mathscr{A}_1$ is the case; after an elementary computation it turns out that its contribution to $b_1(s,h)$ is written as

$$\theta(h)f(\imath v)f(-\imath v)ts + o(ts), \qquad \text{with} \quad 2 \leqq \theta(h) \leqq 3.$$

By a similar computation we also observe that, given $X(t)=x$, the conditional probability of $\mathscr{A}_1$ is $(1-4tf(-v))+o(t)$ and that the contribution from (A) to the conditional probability of $\mathscr{A}_1 \cap \mathscr{E}$ given $X(t)=x$ is $f(v)s(2-[8+\theta(h)]tf(-v)) + o(ts)$. Combining these estimates we have

$$b_1(s,h)=(2-\theta(h)tf(-v))f(v)s+\theta(h)f(\imath v)f(-\imath v)ts+o(ts).$$

Consequently we have

$$b_1(s,h)-b_2(s,h)=-(8-\theta(h))(f(\imath v)f(\imath v)-f(v)f(-v))ts+o(ts),$$

which together with (3.1) shows that the chance of the first particle's collision in the near future under $\mathscr{A}_1$ is larger or smaller than that under $\mathscr{A}_2$ according as $f(\imath v)f(-\imath v)-f(v)f(-v)$ is positive or negative. Thus the past history of the tagged particle influences its future behavior even when its present state is given.

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