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Probability Theory and Related Fields

Generalized Stochastic Integrals and the Malliavin Calculus

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Summary. The paper first reviews the Skorohod generalized stochastic integral with respect to the Wiener process over some general parameter space T and it's relation to the Malliavin calculus as the adjoint of the Malliavin derivative. Some new results are derived and it is shown that every sufficiently smooth process $\{u_t, t\in T\}$ can be decomposed into the sum of a Malliavin derivative of a Wiener functional, and a process whose generalized integral over T vanishes. Using the results on the generalized integral, the Bismut approach to the Malliavin calculus is generalized by allowing non adapted variations of the Wiener process yielding sufficient conditions for the existence of a density which is considerably weaker than the previously known conditions.

Let e_i be a non-random complete orthonormal system on T, the Ogawa integral $\int u \overset{\circ}{\delta} W$ is defined as $\Sigma_i (e_i, u) \int e_i \, dW$ where the integrals are Wiener integrals. Conditions are given for the existence of an intrinsic Ogawa integral i.e. independent of the choice of the orthonormal system and results on it's relation to the Skorohod integral are derived.

The transformation of measures induced by $(W + \int u \, d\mu)$ with u non adapted is discussed and a Girsanov-type theorem under certain regularity conditions is derived.

1. Introduction

Different definitions of the stochastic integral of a non-adapted random process with respect to a Wiener process have been proposed by several authors (cf. e.g. [12, 14, 17]). In [17] Skorohod introduces a stochastic integral with respect to a Gaussian measure with orthogonal increments. This integral is constructed as a random linear functional on an abstract Hilbert space. Roughly speaking, the role of the adaptability property is replaced in this kind of stochastic integrals

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by some regularity conditions on the integrand processes. Also, this stochastic integral does not possess some of the more natural properties of the ordinary Ito integral. For instance, let $\int_T u \delta w$ denote this integral then

(i) It is possible to have $\int_T u \delta w = 0$ a.s. even though $E \int_T u^2 \mu(dt) > 0$.

(ii) Let $C(w)$ denote a random constant, then in general, $\int_T C(w) \delta w \neq C(w) \int_T dw$.

In a recent article [4] Gaveau and Trauber have shown that this generalized stochastic integral is equivalent to the dual operation of the differential in the Malliavin stochastic calculus. In the paper of Skorohod a stochastic derivative with respect to the fundamental Gaussian process is already introduced, and in [4] it is proved that it coincides with the Malliavin gradient.

The purpose of this paper is to further clarify the relation between the Skorohod integral and the Malliavin calculus and to show that this integral enables to derive conditions for the absolute continuity of the probability law of Wiener functionals under conditions which are weaker than those of the Malliavin approach. In Sects. 2 and 3 the basic properties of the Skorohod integral and the Malliavin operators are presented in the general context of a Gaussian measure with orthogonal increments. The special case where $T = [0, 1]$ is discussed in example 3.6 and it is pointed out that the Skorohod integral generalizes the forward and backward Ito integrals. The integration by parts formula of [4] is reviewed in Sect. 4 and some consequences are derived. It is shown in Sect. 4 that every square integrable process $(u_t(w), t \in T)$ possesses an orthogonal decomposition $u = DF + u^0$ where, very roughly, DF is the gradient of a Wiener functional $F(w)$ and u^0 is orthogonal to all processes which are representable as gradients of Wiener functionals and the Skorohod integral of u^0, $\int_T u^0 \delta w$, vanishes.

The Malliavin calculus is a powerful method for proving the existence of a density for the probability laws of functionals of the Wiener process and, more generally, of functionals of a Gaussian measure with orthogonal increments. The ideas of Malliavin have been developed by several authors (Stroock [17], Shigekawa [14], Ikeda-Watanabe [5]). An alternative approach to the problem of the existence of a density has been proposed by Bismut [1], cf. [19] for a general survey of the Malliavin calculus and a comparison of the approaches. The method of Bismut is based on the Girsanov theorem which allows to deduce an integration by parts formula and is, roughly speaking, based on directional derivatives in adapted directions i.e. directions which are admissible by Girsanov's theorem. While directional derivatives in adapted directions suffice for the case of solutions of stochastic differential equations this is not the case in general. In Sect. 5 we use the integration by parts formula of Sect. 4 to extend the Bismut approach, obtaining sufficient conditions for the absolute continuity, which are strictly weaker than in Malliavin approach. We give a particular example where the Malliavin method is not applicable because the functional does not belong to the domain of the Ornstein-Uhlenbeck operator L, but the generalized Bismut approach can be applied using a derivative in a suitable non-adapted direction.

In some recent papers ([12, 13]) Ogawa has introduced a non-causal stochastic integral with respect to the Brownian motion, which may depend on the particular orthonormal basis that we choose in the Hilbert space $L^2([0,1])$. In Sect. 6 we give some conditions for this integral to have an intrinsic meaning in the general set-up of a Gaussian orthogonal measure, and we discuss its relation with the Skorohod integral. It turns out that the hypotheses for the existence of an "intrinsic" Ogawa integral are stronger than those for the construction of the Skorohod integral. In some particular cases we introduce a symmetric integral, which is similar to the Ogawa integral, and is an extension of the Stratonovich integral.

Finally, in Sect. 7 we study the transformation of the probability law of a Gaussian orthogonal measure under changes of the form $w(B) \to w(B) + \int_B u(t,w)\,\mu(dt)$, where μ is the intensity of the Gaussian measure w, and u is some square-integrable process without the assumption on this transformation to be one-to-one. Some conditions for the absolute continuity are given, the density is computed and a Girsanov type result is presented. In the framework of an abstract Wiener space this problem has been considered by Ramer [14], and, under more general assumptions, by Kusuoka [9], following earlier work of Cameron and Martin, Gross, Shepp and Kuo.

2. The Malliavin Operators

Let $(T, \mathbf{B})$ be a measurable space with a finite atomless measure μ. Consider a zero mean Gaussian process $\{w(B), B \in \mathbf{B}\}$ with covariance function given by $E(w(B_1)\,w(B_2)) = \mu(B_1 \cap B_2)$, defined in some probability space $(\Omega, \mathbf{F}, P)$. This process will be called a Gaussian orthogonal measure on the space $(T, \mathbf{B}, \mu)$. We assume that the sigma field $\mathbf{F}$ is generated by the random variables $w(B)$, $B \in \mathbf{B}$. We will denote by H the Hilbert space $L^2(T, \mathbf{B}, \mu)$ and we will suppose that H is separable. For any $h \in H$, we denote by $w(h)$ the Wiener integral of h with respect to w.

Recall that any square integrable functional $F \in L^2(\Omega, \mathbf{F}, P)$ can be expanded into the series of multiple Ito-Wiener integrals ([6])

$$F = E(F) + \sum_{m=1}^{\infty} I_m(f_m) \tag{2.1}$$

which converges in quadratic mean. For all $m \geqq 1$, $I_m(f_m)$ denotes the multiple Ito-integral of the deterministic function $f_m \in L^2(T^m, \mathbf{B}^m, \mu^m)$. The main properties of these integrals are the following:

(a) I_m is linear,

(b) $I_m(f) = I_m(\tilde{f})$, where $\tilde{f}(t_1, \ldots, t_m) = \frac{1}{m!} \sum_\sigma f(t_{\sigma(1)}, \ldots, t_{\sigma(m)})$, σ running over all permutations of $\{1, \ldots, m\}$,

(c) $$E(I_m(f)\,I_p(g)) = \begin{cases} 0, & \text{if } m \neq p, \\ m!\,\langle \tilde{f}, \tilde{g} \rangle_{L^2(T^m)}, & \text{if } m = p. \end{cases}$$

Note that, in the decomposition of F we may assume that the kernels f_m are symmetric and, in this case, they are uniquely determined by F as elements of $L^2(T^m, \mathbf{B}^m, \mu^m)$. In the following we will always suppose that the f_m are symmetric.

The *Malliavin operator* L is defined as

$$LF = \sum_{m=1}^{\infty} m I_m(f_m), \tag{2.2}$$

provided the series converges in quadratic mean. Than means, LF exists if and only if the sum

$$\|LF\|_2^2 = \sum_{m=1}^{\infty} m^2 m! \, \|f_m\|_2^2$$

is finite. The operator L has the following properties:

(a) L is a closed, non-negative, self-adjoint operator with a domain dense in $L^2(\Omega, \mathbf{F}, P)$ and $E(LF) = 0$.

(b) L possesses a self-adjoint square root given by

$$L^{1/2} F = \sum_{m=1}^{\infty} \sqrt{m} \, I_m(f_m).$$

(c) As pointed out in [19, 20] LF can be interpreted as the derivative of F with respect to a scale parameter: $LF(w) = (\partial F(\lambda w)/\partial \lambda)_{\lambda=1}$. In fact, let λ be a real number such that $|\lambda| < 1$, and define the functional

$$F_\lambda = E(F) + \sum_{m=1}^{\infty} \lambda^m I_m(f_m). \tag{2.3}$$

Then LF exists if and only if $\frac{1}{\varepsilon}(F - F_{1-\varepsilon})$ converges in L^2 as ε tends to zero, and in this case $LF = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(F - F_{1-\varepsilon})$.

(d) L is the infinitesimal generator of the semigroup $\{T_\tau F = F_{e^{-\tau}};\ \tau > 0\}$.

The *Malliavin Derivative* $D_h F$ can be defined as follows. Let $F \in L^2(\Omega, \mathbf{F}, P)$ be given by (2.1), for $h \in H$ set

$$\begin{aligned} D_h F &= \sum_{m=1}^{\infty} m \langle h, I_{m-1}(f_m(t_1, \dots, t_{m-1}, \cdot)) \rangle \\ &= \sum_{m=1}^{\infty} m I_{m-1}(\langle h, f_m(t_1, \dots, t_{m-1}, \cdot) \rangle), \end{aligned} \tag{2.4}$$

provided the series converges in $L^2(\Omega, \mathbf{F}, P)$ where $\langle f, g \rangle$ denotes the scalar product of two functions f, g in H. Let $\{e_i, i \geq 1\}$ be a complete orthonormal system in H. Then, DF exists as a square integrable H-valued random variable if and only if the following quantity is finite

$$E(\|DF\|_H^2) = \sum_{i=1}^{\infty} E((D_{e_i} F)^2)$$
$$= \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} m^2 E[(\int_T e_i(t) I_{m-1}(f_m(t_1, \dots, t_{m-1}, t)) \mu(dt))^2]$$
$$= \sum_{m=1}^{\infty} m^2 (m-1)! \|f_m\|_2^2 = E((L^{1/2} F)^2). \tag{2.5}$$

The set of functionals $F \in L^2(\Omega, \mathbf{F}, P)$ such that $E(\|DF\|_H^2) < \infty$ will be denoted by $H^{2,1}$. Note that $H^{2,1} = \operatorname{Dom} L^{1/2}$. The set $H^{2,1}$ is a Hilbert space with the scalar product $E(FG) + E(\langle DF, DG \rangle_H)$.

The next proposition relates the Malliavin operators D_h and L.

Proposition 2.1. *Let $h \in H$. Then*

$$D_h LF - LD_h F = D_h F,$$

if all terms exist.

Proof. It suffices to suppose $F = I_m(f_m)$, and in this case $D_h LF = m D_h F$ and $LD_h F = (m-1) D_h F$.

Fix an element $h \in H$. As in the case of the operator L, we want to interpret the random variable $D_h F$ as a directional derivative of the functional F. Without loss of generality, we may assume that $(\Omega, \mathbf{F}, P)$ is the canonical probability space associated to the process $\{w(B), B \in \mathbf{B}\}$, that means, $\Omega = \mathbb{R}^{\mathbf{B}}$ and $\mathbf{F}$ is the product sigma field completed with respect to the law of w. It is easy to see that the probability measure induced on $(\Omega, \mathbf{F})$ by the mapping $\{w(B)\} \to \{w(B) + \int_B h \, d\mu\}$ is absolutely continuous with respect to the law of w, with a density equal to $\exp(w(h) - \frac{1}{2} \int_T h^2 \, d\mu)$. Indeed, let $B_1, \dots, B_M$ be measurable disjoint subsets of T. We have

$$E\left[\exp\left(i \sum_{j=1}^{M} t_j w(B_j) + w(h) - \frac{1}{2} \int_T h^2 \, d\mu\right)\right]$$
$$= \exp\left(-\frac{1}{2} \sum_{j=1}^{M} t_j^2 \mu(B_j) + i \sum_{j=1}^{M} t_j \int_{B_j} h \, d\mu\right)$$
$$= E\left[\exp\left(i \sum_{j=1}^{M} t_j [w(B_j) + \int_{B_j} h \, d\mu]\right)\right],$$

for all $t_1, \dots, t_M$ in $\mathbb{R}$. Thus, for any real number, ε, the functional $F(w + \varepsilon \int h \, d\mu)$ is well defined.

Proposition 2.2. *Let $F \in L^2(\Omega, \mathbf{F}, P)$. Assume that $\frac{1}{\varepsilon}[F(w + \varepsilon \int h \, d\mu) - F(w)]$ converges in L^2 as $\varepsilon \to 0$. Then $D_h F$ exists and coincides with the limit of this expression.*

Proof. Let $F = E(F) + \sum_{m=1}^{\infty} I_m(f_m)$. We will denote by F_n, $n = 1, 2, \dots$, the sum $E(F) + \sum_{m=1}^{n} I_m(f_m)$. We remark first that

$$I_m(f_m)(w+\varepsilon \int h\,d\mu)=\sum_{i=0}^{m}\binom{m}{i}\varepsilon^{m-i}I_i(\int_{T^{m-i}} f_m(t_1,\dots,t_i,t_{i+1},\dots,t_m)\cdot h(t_{i+1})\dots h(t_m)\,\mu(dt_{i+1})\dots\mu(dt_m)).$$

In fact, this formula is obviously true when f_m is a simple function. In the general case it is proved by a usual convergence argument. As a consequence, we obtain

$$F_n(w+\varepsilon \int h\,d\mu)=F_n(w)+\varepsilon D_h F_n+\varepsilon^2\sum_{i=0}^{n-2}\sum_{m=i+2}^{n}\binom{m}{i}\varepsilon^{m-i-2}\cdot I_i(\int_{T^{m-i}} f_m(t_1,\dots,t_i,t_{i+1},\dots,t_m)\cdot h(t_{i+1})\dots h(t_m)\,\mu(dt_{i+1})\dots\mu(dt_m)),$$

and, therefore,

$$\frac{1}{\varepsilon}\,[F_n(w+\varepsilon \int h\,d\mu)-F_n(w)]\xrightarrow[e\to 0]{L^2} D_h F_n,$$

for all $n\geqq 1$. That means, for functionals belonging to the Wiener homogeneous chaos of order n the operator D_h can be expressed as a directional derivative. Denote by $\tilde{F}_h$ the limit $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[F(w+\varepsilon \int h\,d\mu)-F(w)]$ in the L^2 sense. For any $G\in L^2(\Omega, \mathbf{F}, P)$ we compute

$$\frac{1}{\varepsilon}\,E[(F(w+\varepsilon \int h\,d\mu)-F(w))\,G_n(w)]$$
$$=\frac{1}{\varepsilon}\,E[F(w)(G_n(w-\varepsilon \int h\,d\mu)-G_n(w))+F(w)\,G_n(w-\varepsilon \int h\,d\mu)\cdot(e^{\varepsilon w(h)-\frac{\varepsilon^2}{2}\int_T h^2\,d\mu}-1)].$$

When $\varepsilon\to 0$, this expression converges to

$$E[-FD_h G_n+FG_n w(h)],$$

which will now be shown to be equal to $E(G_n D_h F_{n+1})$. In fact, by the properties of the multiple Ito integrals, we have

$$E[F(G_n w(h)-D_h G_n)]$$
$$=E\left[F\sum_{m=0}^{n} I_{m+1}\left(\frac{1}{m+1}\left\{g_m(t_1,\dots,t_m)\,h(t_{m+1})+\sum_{i=1}^{m} g_m(t_1,\dots,t_{m+1}^{i},\dots,t_m)\,h(t_i)\right\}\right)\right]$$
$$=\sum_{m=0}^{n}(m+1)!\int_{T^{m+1}} f_{m+1}(t_1,\dots,t_{m+1})\,g_m(t_1,\dots,t_m)\,h(t_{m+1})\,d\mu^{m+1}$$
$$=E(G_n D_h F_{n+1}),$$

where $G=E(G)+\sum_{m=1}^{\infty} I_m(g_m)$.

Therefore $E(G_n D_h F_{n+1}) = E(G_n \tilde{F}_n)$, consequently $D_h F_{n+1}$ is the projection of $\tilde{F}_h$ on the Wiener homogeneous chaos of order n and, therefore, $\tilde{F}_h = D_h F$.

Let $\psi: \mathbb{R}^n \to \mathbb{R}$ be a three times differentiable function on $\mathbb{R}^n$ such that ψ together with its fist three derivatives are of polynomial growth. The functionals $F(w) = \psi(w(h_1), \ldots, w(h_n))$, $h_i \in H$, will be called simple functionals. From Proposition 2.2 it is clear that for any $h \in H$

$$D_h F = \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i} \langle h, h_i \rangle_H. \tag{2.6}$$

Moreover, it can also be proved (cf. [18, 19]) that

$$LF = \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i} w(h_i) - \sum_{i,j=1}^{n} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \langle h_i, h_j \rangle_H. \tag{2.7}$$

The main rules of the calculus associated with the Malliavin operators D and L can be summarized as follows:

Proposition 2.3. *Let $\varphi: \mathbb{R}^d \to \mathbb{R}$ be a real valued twice continuously differentiable function such that the first and second derivatives of φ are bounded. We consider square-integrable functionals $F_1, \ldots, F_d$, and set $F = (F_1, \ldots, F_d)$. We have*

(*i*) *$\varphi(F) \in H^{2,1}$ if $F_i \in H^{2,1}$ for all $i = 1, \ldots, d$, and $D\varphi(F) = \sum_{i=1}^{d} \frac{\partial \varphi}{\partial x_i}(F) DF_i$.*

(ii) *If $F_i \in \operatorname{Dom} L$ for all $i = 1, \ldots, d$, then $\varphi(F)$ belongs to the closed L^1 extension of L and*

$$L\varphi(F) = \sum_{i=1}^{d} \frac{\partial \varphi}{\partial x_i}(F) LF_i - \sum_{i,j=1}^{d} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(F) \langle DF_i, DF_j \rangle_H.$$

Proof. Consider first the case where the F_i are simple functionals, and then pass to the limit.

3. The Skorohod Integral

Let $\{w(B), B \in \mathbf{B}\}$ be a Gaussian measure with orthogonal increments on the underlying space $(T, \mathbf{B}, \mu)$. In this section we introduce a stochastic integral with respect to the measure w, defined in [17] by Skorohod, and discuss its basic properties.

Suppose that $u = \{u(t, w), (t, w) \in T \times \Omega\}$ is a $\mathbf{B} \otimes \mathbf{F}$-measurable process such that $E(u_t^2) < \infty$ for all $t \in T$. Then for each t the random variable u_t has a representation in a series of multiple Wiener-Ito integrals

$$u_t = E(u_t) + \sum_{m=1}^{\infty} I_m(f_m(t \mid t_1, \ldots, t_m)), \tag{3.1}$$

where the kernels f_m can be chosen to possess the following properties:

(a) for every t in $T f_m(t \mid \cdot)$ is symmetric in the coordinates $t_1, \ldots, t_m$ and belongs to $L^2(T^m, \mathbf{B}^m, \mu^m)$.

(b) f_m is a measurable function of all its variables.

The proof of (b) is as follows. Let $\{u^k, k \geq 1\}$ be a sequence of simple processes such that $\lim_k u_t^k(w) = u_t(w)$ for all (t, w) and $u_t^k \xrightarrow[k\to\infty]{L^2(\Omega)} u_t$ for all t.

We will have a representation of the form $u_t^k = E(u_t^k) + \sum_{m=1}^{\infty} I_m(f_m^k(t|t_1, \ldots, t_m))$, for any $k \geq 1$, where the functions f_m^k verify the desired properties. Then it is easy to check that $f_m^k(t|\cdot) \xrightarrow[k\to\infty]{L^2(T^m)} f_m(t|\cdot)$ for all t, and this implies the existence of a measurable version of f_m.

We will denote by

$$\tilde{f}_m(t_1, \ldots, t_m, t) = \frac{1}{m+1}\left[f_m(t|t_1, \ldots, t_m) + \sum_{i=1}^{m} f_m(t_i|t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_m)\right],$$

i.e. the symmetrization of f_m as a function of $m+1$ variables. Then, the *Skorohod integral* of the process u is defined as

$$\int_T u\,\delta w = w(E(u_t)) + \sum_{m=1}^{\infty} I_{m+1}(\tilde{f}_m), \tag{3.2}$$

provided that the multiple Ito integrals exist and the series converges in quadratic mean. This means that the Skorohod integral can be defined for all measurable processes u such that $E(u_t^2) < \infty$ for all t and such that the following seminorm is finite

$$\|u\| = \left\{\int_T (Eu_t)^2\,\mu(dt) + \sum_{m=1}^{\infty} (m+1)!\,\|\tilde{f}_m\|^2\right\}^{1/2}. \tag{3.3}$$

In this case $E\int_T u\,\delta w = 0$ and $E[(\int_T u\,\delta w)^2] = \|u\|^2$. The Skorohod integral of the process u will be also denoted by δu. Let $L^2(T \times \Omega)$ be the space of measurable processes u such that $\int_T E(u_t^2)\,\mu(dt) < \infty$. We have the following result.

Proposition 3.1. *Let $u \in L^2(T \times \Omega)$ such that $\int_T E(\|Du_t\|_H^2)\,\mu(dt) < \infty$. Then, the Skorohod integral of the process u exists, and we have*

$$\begin{aligned} E[(\delta u)^2] &= \int_T E(u_t^2)\,\mu(dt) + \int_{T^2} E[Du_t(s)\,Du_s(t)]\,\mu(dt)\,\mu(ds) \\ &\leq \int_T E(u_t^2)\,\mu(dt) + \int_T E(\|Du_t\|_H^2)\,\mu(dt) \end{aligned} \tag{3.4}$$

$$= \int_T E(u_t^2)\,\mu(dt) + \int_T E[(L^{1/2} u_t)^2]\,\mu(dt). \tag{3.4a}$$

Proof. Note that Du is a square integrable random variable valued on $H \otimes H = L^2(T^2, \mathbf{B}^2, \mu^2)$, and we can choose a version of $Du_t(s)$ which belongs to $L^2(T^2 \times \Omega, \mathbf{B}^2 \otimes \mathbf{F}, \mu^2 \otimes P)$. For almost all (s, t, w) we will have

$$Du_t(s) = \sum_{m=1}^{\infty} m I_{m-1}(f_m(t|s, t_1, \ldots, t_{m-1})).$$

Then, using (3.3), a simple computation shows that

$$E[(\delta u)^2]=\int_T (Eu_t)^2\,\mu(dt)+\sum_{m=1}^{\infty}\frac{m!}{(m+1)}\,((m+1)\int_{T^{m+1}} f_m^2(t\,|\,t_1,\ldots,t_m),$$

$$\mu(dt)\,\mu(dt_1)\ldots\mu(dt_m)+m(m+1)\int_{T^{m+1}} f_m(t\,|\,s,t_1,\ldots,t_{m-1})\,f_m(s\,|\,t,t_1,\ldots,t_{m-1}),$$

$$\mu(dt)\,\mu(ds)\,\mu(dt_1)\ldots\mu(dt_{m-1}))=\int_T E(u_t^2)\,\mu(dt)+\int_{T^2} E[Du_t(s)\,Du_s(t)]\,\mu(dt)\,\mu(ds).$$

Finally we apply the Schwarz inequality and equation (2.5).

Remark. A result of this type appears in the article of Skorohod, in a more general set-up, where w is a generalized Gaussian element on an arbitrary separable Hilbert space H. Furthermore, a similar stochastic integral has been considered by Ramer [14] and Kusuoka [9] in the context of square-integrable functionals defined in an abstract Wiener space.

Now we are going to present the main properties of this integral.

Proposition 3.2. *Let $F\in L^2(\Omega,\mathbf{F},P)$. Then $LF=\delta DF$ in the sense that F belongs to the domain of L if and only if $F\in H^{2,1}$ and DF is Skorohod integrable.*

Proof. Note that here we identify the H-valued random variable $DF\in L^2_H(\Omega,\mathbf{F},P)$ with a real random process of $L^2(T\times\Omega)$. To prove the proposition, it suffices to consider the case $F=I_m(f_m)$. Then $LF=mF$, and $\delta DF=\delta(mI_{m-1}(f_m(t,t_1,\ldots,t_{m-1})))=mF$.

Proposition 3.3. *$L\delta u-\delta Lu=\delta u$, for any measurable process u such that $E(u_t^2)<\infty$ for all t and such that all terms in this expression exist.*

Proof. We may assume that $u_t=I_m(f_m(t\,|\,\cdot))$. Then,

$$L\delta u-\delta Lu=(m+1)\,I_{m+1}(\tilde f_m)-mI_{m+1}(\tilde f_m)=I_{m+1}(\tilde f_m)=\delta u.$$

Proposition 3.4. *$D_h\delta u-\delta D_h u=\langle u,h\rangle$ if $h\in H$ and u is a process of $L^2(T\times\Omega)$ such that the variables $D_h\delta u$ and $\delta D_h u$ exist.*

Proof. As before, we may assume that $u_t=I_m(f_m(t\,|\,\cdot))$. In this case,

$$\begin{aligned}D_h\delta u&=D_h I_{m+1}\left(\frac{1}{m+1}\left[f_m(t\,|\,t_1,\ldots,t_m)+\sum_{i=1}^{m} f_m(t_i\,|\,t_1,\ldots,t_{i-1},t,t_{i+1},\ldots,t_m)\right]\right)\\&=I_m\left(\left\langle h(\cdot),f_m(\cdot\,|\,t_1,\ldots,t_m)+\sum_{i=1}^{m} f_m(t_i\,|\,t_1,\ldots,t_{i-1},\cdot,t_{i+1},\ldots,t_m)\right\rangle\right)\\&=\langle h,u\rangle+\delta D_h u.\end{aligned}$$

Proposition 3.5. *Let $F\in H^{2,1}$ and $f\in H$. Then the process $f(t)F(w)$ is Skorohod integrable and*

$$\begin{aligned}\delta(fF)&=w(f)\,F-D_fF\\&=F\delta f-\langle DF,f\rangle_H.\end{aligned}$$

Proof. It suffices to assume that $F=I_m(f_m)$. Then, the above equality follows from the properties of the multiple Ito integrals:

$$\delta(f \cdot F) = I_{m+1}(\tilde{f} \cdot f_m) = I_1(f)\, I_m(f_m) - m I_{m-1}(\langle f, f_m(t_1, \ldots, t_{m-1}, \cdot)\rangle)$$
$$= w(f)\, F - D_f F.$$

Remark. The result of Proposition 3.5 can be generalized to yield

$$\delta(F \cdot u_t(w)) = F \cdot \delta u - \langle u, DF \rangle_H$$

under suitable conditions.

As a consequence of Proposition 3.5 for a simple process $u(t, w) = \sum_{k=1}^{d} F_k(w)\, 1_{B_k}(t)$, where B_k are measurable subsets of T, and $F_k \in H^{2,1}$, we have

$$\delta u = \sum_{k=1}^{d} w(B_k)\, F_k - \sum_{k=1}^{d} \langle DF_k, 1_{B_k} \rangle_H.$$

We can also consider the behavior of the Skorohod integral under a change of scale. Let λ be a real parameter such that $|\lambda| < 1$. We have already introduced in (2.3) the «natural» extension $F(\lambda w)$ of the functional $F \in L^2(\Omega, \mathbf{F}, P)$, which has been denoted by $F_\lambda(w)$. With this definition we obtain the following expressions

$$LF_\lambda = (LF)_\lambda, \quad D_h F_\lambda = \lambda (D_h F)_\lambda, \quad \text{and} \quad \delta u_\lambda = \frac{1}{\lambda} (\delta u)_\lambda,$$

which are compatible with the different relations among the operators L, D and δ.

Example 3.6. Consider the particular case $T = [0, 1]$, μ the Lebesgue measure and $w[0, t] = w_t$ an ordinary Brownian motion. Here we may take Ω equal to the space of continuous functions $C(T)$ and P the Wiener measure on it. Let $u = \{u_t, t \in T\}$ be a measurable and adapted process such that $\int_0^1 E(u_t^2)\, dt < \infty$. Then as we show now, u is Skorohod integrable, and the Ito and Skorohod integrals coincide. Consequently the Skorohod integral for square-integrable processes is an extension of the Ito integral. In fact, the kernels $f_m(t \mid t_1, \ldots, t_m)$ can be chosen with the property $f_m(t \mid t_1, \ldots, t_m) = 0$ unless $\max\{t_1, \ldots, t_m\} \leq t$, and then, the seminorm $\|u\|$ defined by the expression (3.3) coincides with $\left(\int_0^1 E(u_t^2)\, dt\right)^{1/2}$, because using Proposition 3.1 we have

$$\|u\|^2 = \int_0^1 E(u_t^2)\, dt + \sum_{m=1}^{\infty} m m! \int_0^1 \int_0^1 f_m(t \mid s, t_1, \ldots, t_{m-1})$$
$$\cdot f_m(s \mid t, t_1, \ldots, t_{m-1})\, ds\, dt\, dt_1 \ldots dt_{m-1} = \int_0^1 E(u_t^2)\, dt.$$

Therefore $\int_0^1 u\, \delta w$ exists. Furthermore, $I_{m+1}(\tilde{f}_m) = \int_0^1 I_m(f_m(t \mid t_1, \ldots, t_m))\, dw_t$, and the Skorohod integral δu is equal to the Ito integral $\int_0^1 u_t\, dw_t$.

In a similar way, if the process u is measurable, adapted to the filtration $\mathbf{G}_t = \sigma\{w([s,1]),\ t \leqq s \leqq 1\}$, and satisfies $\int_0^1 E(u_t^2)\,dt < \infty$, then the Skorohod integral of u exists and coincides with the backward Ito integral [7].

Two particular examples of Skorohod integrals are as follows

$$\int_0^T w(T)\,\delta w = \int_0^T \int_0^T dw_{t_1}\,dw_{t_2} = w^2(T) - T$$

hence for $T_1 < T_2$

$$\int_0^{T_2} w(T_1)\,\delta w = w(T_1)\,w(T_2) - T_1. \tag{3.5}$$

The second example:

$$\begin{aligned} \int_0^T w_t(w_T - w_t)\,\delta w &= \int_0^T \left(\int_0^T \int_0^T 1_{t_1 < t < t_2}\,dw_{t_1}\,dw_{t_2} \right) \delta w \\ &= \int_0^T \int_0^{t_2} \int_0^t dw_{t_1}\,dw_t\,dw_{t_2} \\ &= \int_0^T \left(\int_0^{t_2} w_t\,dw_t \right) dw_{t_2} \\ &= \int_0^T \frac{w^2(t_2) - t_2}{2}\,dw_{t_2}. \end{aligned} \tag{3.6}$$

Remark (a). Let $u = \{u_t,\ t \in T\}$ be a measurable process such that $\int_0^1 E(u_t^2)\,dt < \infty$, with the integral representation $u_t = E(u_t) + \sum_{m=1}^{\infty} I_m(f_m(t \mid t_1, \ldots, t_m))$. We can consider the predictable projection u^P of u, which will be equal to

$$u_t^P = E(u_t) + \sum_{m=1}^{\infty} I_m(\hat{f}_m(t \mid t_1, \ldots, t_m)),$$

where

$$\hat{f}_m(t \mid t_1, \ldots, t_m) = \begin{cases} f_m(t \mid t_1, \ldots, t_m) & \text{if } \max\{t_1, \ldots, t_m\} \leqq t \\ 0 & \text{otherwise.} \end{cases}$$

Then, if u is adapted we know that $u = u^P$. However, it follows from (3.5) or (3.6) that for a non-adapted process u, the Skorohod integral $\int_T u_t\,\delta w_t$ does not coincide, in general, with the Ito integral $\int_T u_t^P\,dw_t$.

Remark (b). Unlike the Ito integral we may well have $\int_0^1 u\,\delta w = 0$ a.s. while $E \int_0^1 u_s^2\,ds > 0$. An example of such a case is

$$u_t = \int_0^1 (t - t_1)\,dw_{t_1}.$$

In this case $f_1(t\,|\,t_1)=t-t_1$; $\tilde{f}_1(t_1,t)=0$ and $\int_0^1 u\,\delta w=0$ by 3.2. Another example is given in the next remark.

Remark (c). Consider the functional $F(w)=\int_0^T u_s\,\delta w_s$ where u is a nonadapted Skorohod integrable integrand. Then F has the Ito integral representation $F(w) = \int_0^T u_s^I\,dw_s$ for some adapted process u^I therefore

$$F(w)=\int_0^T u_s\,\delta w_s$$

and also

$$F(w)=\int_0^T u_s^I\,dw_s=\int_0^T u_s^I\,\delta w_s$$

and

$$\int_0^T (u_s-u_s^I)\,\delta w=0$$

but $\int_0^T (u_s-u_s^I)^2\,ds>0$, cf. Proposition 4.4.

4. An Integration by Parts Formula

Let $u=\{u_t, t\in T\}$ be a measurable process of the space $L^2(T\times\Omega)$. For any $F\in H^{2,1}$, we define $D_uF=\langle DF, u\rangle_H$. Then D_uF is an integrable random variable such that $(E\,|D_uF|)^2\leqq E(\|DF\|_H^2)\int_T E(u_t^2)\,\mu(dt)$. In [4], Gaveau and Trauber have shown that the Skorohod integral of the process u is equivalent to the adjoint of the differential operator D introduced by Malliavin. More precisely, we have the following result.

Theorem 4.1. ([4]). *The mapping $F\to E(D_uF)$ defined on the space $H^{2,1}\subset L^2(\Omega,\mathbf{F},P)$ is continuous in the norm of $L^2(\Omega,\mathbf{F},P)$ if and only if the Skorohod integral of u, exists, and, in this case, $E(D_uF)=E(F\int_T u\,\delta w)$.*

Let $F\in H^{2,1}$, for the case where u is deterministic we have already seen that D_uF can be interpreted as a directional derivative. In general, in order to identify D_uF as a directional derivative it is necessary to impose some restrictions to the process u, but we will not treat this problem here (cf. [19]). The following property, which has been proved in [19] for directional derivatives, is an immediate consequence of the above definitions.

Proposition 4.2. *Let $F\in H^{2,1}$. Then*

$$\|DF\|_H^2=\sup\{(D_uF)^2 : u\in L_H^2(\Omega,\mathbf{F},P) \text{ and such that } \|u\|_H=1\}.$$

Proof. Clearly $(D_uF)^2\leqq\|DF\|_H^2$ if $\|u\|_H=1$. Conversely $\|DF\|_H^2=D_{DF}F=(D_uF)^2$, where $u=\dfrac{DF}{\|DF\|_H}$.

Let $u \in L^2(T \times \Omega)$. We may ask the following question: When is u equal to DF for some functional F in $H^{2,1}$?

Proposition 4.3. *Suppose that $u \in L^2(T \times \Omega)$. There exists a functional $F \in H^{2,1}$ such that $DF = u$ if and only if the kernels $f_m(t \mid t_1, \ldots, t_m)$ which appear in the integral decomposition of u are symmetric functions of all the variables.*

Proof. The condition is obviously necessary. To show the sufficiency, define $F = w(Eu_t) + \sum_{m=1}^{\infty} \frac{1}{m+1} I_{m+1}(f_m)$. This series converges in quadratic mean and $F \in \operatorname{Dom} L^{1/2}$ because $E[(L^{1/2} F)^2] = \sum_{m=0}^{\infty} m! \, \|f_m\|_2^2 = \int_T E(u_t^2)\, \mu(dt)$. Also, it is clear that $DF = u$.

Remark. Recall that under the hypotheses of Proposition 4.3, it follows by Proposition 3.2 that the Skorohod integral of the process u exists if and only if $F \in \operatorname{Dom} L$, and then $\delta u = LF$.

Theorem 4.4. *Every process $u \in L^2_H(\Omega, \mathbf{F}, P)$ has a unique orthogonal decomposition $u = DF + u^0$ where $F \in H^{2,1}$ and $E\langle u^0, DG \rangle_H = 0$ for all G in $H^{2,1}$. Furthermore, u^0 is Skorohod integrable and $\delta u^0 = 0$.*

Proof. Note that the Hilbert spaces $L^2_H(\Omega, \mathbf{F}, P)$ and $L^2(T \times \Omega, \mathbf{B} \otimes \mathbf{F}, \mu \otimes P)$ can be identified by the natural isometry. The elements of the form DF, $F \in H^{2,1}$, constitute a closed subspace of $L^2_H(\Omega, \mathbf{F}, P)$ by Proposition 4.3. Therefore, any process $u \in L^2_H(\Omega, \mathbf{F}, P)$ has a unique orthogonal decomposition $u = DF + u_0$, where $F \in H^{2,1}$ and $u_0 \perp DG$ for all $G \in H^{2,1}$. Note that, by Theorem 4.1, u^0 is always Skorohod integrable and $\delta u^0 = 0$.

Remark. I. Shigekawa has recently introduced differential n-forms in abstract Wiener space and derived a de Rham-Hodge-Kodaira decomposition for such forms [16]. The decomposition of Theorem 4.4 above corresponds to the case of $n=1$ of [16], i.e. the decomposition of 1-forms as the sum of the exterior derivative of a zero form, the Hodge-star operation on a 2-form and a harmonic component which vanishes in this case. We wish to thank an anonymous reviewer for calling our attention to [16].

5. The Generalized Bismut Approach

Let F be a square-integrable functional of the Gaussian orthogonal measure w. The question arises whether the probability distribution of F possesses a density with respect to the Lebesgue measure. The following lemma, introduced by Malliavin, is a basic tool in establishing this property.

Lemma 5.1. ([10]). *Let X be a random variable and assume that for all $\varphi \in C_b^\infty(\mathbb{R})$ (the class of all real valued functions on $\mathbb{R}$ which are bounded and possess bounded derivatives of all orders) the following inequality holds $|E(\varphi'(x))| \leq c \, \|\varphi\|_\infty$. Then the law of x has a density.*

Following the approach of Bismut [1] and applying the general integration by parts formula established in the last section, we can state the following result.

Theorem 5.2. *Let* $F \in L^2(\Omega, \mathbf{F}, P)$. *Let* $u = \{u_t,\ t \in T\}$ *be a Skorohod integrable process of* $L^2(T \times \Omega)$. *Assume that* $F, D_u F \in H^{2,1}$, *and* $D_u F \neq 0$ *a.s., then the probability law of* F *possesses a density.*

Proof. Set $G = D_u F/(\varepsilon + (D_u F)^2)$ for any $\varepsilon > 0$. Then, by the rules of Malliavin calculus, $D_u G = (\varepsilon - (D_u F)^2)/(\varepsilon + (D_u F)^2)^2$. Applying Theorem 4.1 we have for any $\varphi \in C_b^\infty(\mathbb{R})$

$$E\Big(G\varphi(F) \int_T u\,\delta w\Big) = E(G\varphi'(F)\, D_u F + \varphi(F)\, D_u G).$$

Hence

$$E\left|\varphi'(F)\,\frac{(D_u F)^2}{\varepsilon + (D_u F)^2}\right| \leqq \|\varphi\|_\infty\, E\left\{\left|\frac{(D_u F)^2}{\varepsilon + (D_u F)^2} \cdot \int_T \delta w\right| + \left|\frac{\varepsilon - (D_u F)^2}{(\varepsilon + (D_u F)^2)^2} \cdot D_u D_u F\right|\right\}.$$

For any fixed $\varepsilon > 0$, the right-hand side is bounded by hypothesis. Therefore, if P^ε is a new measure on the original probability space, defined by $\frac{dP^\varepsilon}{dP} = \frac{(D_u F)^2}{\varepsilon + (D_u F)^2}$, the measure $P_0^\varepsilon F^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Let B be a Borel set of $\mathbb{R}$ with Lebesgue measure zero. Then $P^\varepsilon(F \in B) = 0$ and, by monotone convergence, $P(F \in B) = 0$, which completes the proof.

In the Malliavin approach, the main condition to assure the existence of a density is $\|DF\|_H > 0$ a.s. More precisely, we have ([10, 15], cf. the proof of Proposition 2.2.1 of [19]):

Theorem 5.3. *Let* $F \in \operatorname{Dom} L$ *such that* F *and* $\|DF\|_H^2 \in H^{2,1}$. *Assume* $\|DF\|_H > 0$ *a.s. Then, the probability law of* F *is absolutely continuous with respect to the Lebesgue measure.*

For the particular case where we choose $u = DF$ in Proposition 5.2, the smoothness requirements in Propositions 5.2 and 5.3 are the same (in fact, $F \in \operatorname{Dom} L$ iff $F \in H^{2,1}$ and DF is Skorohod integrable, cf. Proposition 3.2). Comparing, now, the generalized Bismut approach (Proposition 5.2) with the Malliavin result (Proposition 5.3) yields that Proposition 5.2 is more general. In fact, from Proposition 4.2 we see that hypotheses of Proposition 5.2 imply $\|DF\|_H > 0$ a.s., but it may happen that u is such that $D_u F > 0$ a.s., u is Skorohod integrable and $D_u F \in H^{2,1}$ although DF does *not* satisfy the conditions of Proposition 5.3.

Example 5.4. Let $F(w) = \varphi(w(T))$, where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is such that: $\varphi(x) = 2x - x\log x^2$, for all $|x| \leqq \varepsilon < 1$, and extend $\varphi(x)$ for all x for which $|x|$ is in (ε, ∞) in such a way that φ is C_b^∞ and $\varphi' > 0$ for $|x| > \varepsilon/2$.

Then $D_h F = \varphi'(w(T)) \int_T h\,d\mu$ and, therefore, $F \in H^{2,1}$, because $\varphi'(x) = -\log x^2$ if $|x| \leqq \varepsilon$. If we choose $u_t = w_T^2$, then $D_u F = \varphi'(w(T))\, w_T^2\, \mu(T)$ belongs to $H^{2,1}$ and

$D_u F > 0$ a.s. Therefore the generalized Bismut method (Proposition 5.2) is applicable. However, $F \notin \operatorname{Dom} L$, and $\|DF\|_H^2 \notin H^{2,1}$.

6. The Ogawa Integral and its Relation with the Skorohod Integral

In [12, 13] Ogawa has introduced a stochastic integral of non-adapted processes with respect to the Brownian motion using a method which differs from that of Skorohod. In the context of a Gaussian orthogonal measure w, the idea for constructing this integral is the following. Let $\{e_i, i \geqq 1\}$ be a complete orthonormal system in $H = L^2(T)$ and consider a measurable process u satisfying $P\{\int_T u_t^2 \, \mu(dt) < \infty\} = 1$. Then the Ogawa integral of u is defined as the sum in the sense of the convergence in probability of the series $\sum_i \langle u, e_i \rangle \, w(e_i)$ if it exists. In general, this integral may depend on the particular orthonormal system. Ogawa has proved the existence of such integral when u is a continuous quasi-martingale, w is the Brownian motion on $[0, 1]$ and e_i is the system of trigonometric functions.

Consider a process $u \in L^2(T \times \Omega)$ with the integral representation (3.1) and put

$$u_t^n = E(u_t) + \sum_{m=1}^{n} I_m(f_m(t \mid t_1, \ldots, t_m))$$

for any $n \geqq 1$. From Proposition 3.5 we have

$$\delta(e_i(t)\langle u^n, e_i \rangle) = w(e_i)\langle u^n, e_i \rangle - \langle D(\langle u^n, e_i \rangle), e_i \rangle, \tag{6.1}$$

for all $n \geqq 1$ and $i \geqq 1$. Assume that $\int_T E(\|Du_t\|_H^2) \, \mu(dt) < \infty$. Then the sequence of processes $\{e_i(t)\langle u^n, e_i \rangle, n \geqq 1\}$ converges to the process $e_i(t)\langle u, e_i \rangle$ in the norm (3.3). Therefore we may take the limit in the expression (6.1) for the L^2 convergence, when $n \to \infty$, obtaining

$$\delta(e_i(t)\langle u, e_i \rangle) = w(e_i)\langle u, e_i \rangle - \int_{T^2} Du_t(s) \, e_i(t) \, e_i(s) \, \mu(dt) \, \mu(ds). \tag{6.2}$$

Thus, in order to give an intrinsic meaning to the sum of the series $\sum_i w(e_i)\langle u, e_i \rangle$ we have to impose some conditions to the kernel $Du_t(s)$.

Proposition 6.1. *Let $u \in L^2(T \times \Omega)$ such that $\int_T E(\|Du_t\|_H^2) \, \mu(dt) < \infty$. Assume that the kernel $Du_t(s)$ has a finite trace, for all w, except in a set of probability zero. Then, the Ogawa integral exists for any complete orthonormal system and it is equal to $\delta u + \operatorname{trace} Du$.*

Proof. The sum $\sum_{i=1}^{M} e_i(t)\langle u, e_i \rangle$ converges to the process u when M tends to infinity in the norm (3.3). Indeed, it is easy to check that

$$\sum_{i=1}^{M} e_i(t)\langle u, e_i \rangle \xrightarrow[M \to \infty]{L_H^2(\Omega, \mathbf{F}, P)} u,$$

and

$$\sum_{i=1}^{M} e_i(t) \int_T Du_v(s)\, e_i(v)\, \mu(dv) \xrightarrow[M\to\infty]{L^2_{H^2}(\Omega, \mathbf{F}, P)} Du. \tag{6.3}$$

As a consequence, $\delta\left(\sum_{i=1}^{M} e_i(t)\langle u, e_i\rangle\right)$ converges in L^2 to δu when $M\to\infty$. Moreover, for any complete orthonormal system $\{e_i, i\geqq 1\}$ the series $\sum_{i=1}^{\infty} \int_{T^2} Du_t(s)\, e_i(t)\, e_i(s)\, \mu(dt)\, \mu(ds)$ converges for almost all w to the finite random variable trace Du. This achieves the proof of the proposition.

If the Ogawa integral of some process u exists and has the same value for any complete orthonormal system, it will be denoted by $\mathring{\delta} u$.

Examples. (i) Let $F\in H^{2,1}$ and $f\in H$. Consider the process $u(t, w)=f(t)\, F(w)$. Then, $Du_t(s)=f(t)\, DF(s)$ has a finite trace and, therefore, the Ogawa integral of u is well defined and is given by $\mathring{\delta}(fF)=\delta(fF)+D_f F=w(f)\, F$, by Proposition 3.4. More generally, for simple processes $u(t, w)=\sum_{k=1}^{d} F_k(w)\, 1_{B_k}(t)$, such that $F_k\in H^{2,1}$, we have

$$\mathring{\delta}\left(\sum_{k=1}^{d} F_k(w)\, 1_{B_k}(t)\right)=\sum_{k=1}^{d} w(B_k)\, F_k.$$

(ii) Consider a measurable function $\varphi\colon \mathbb{R}^n\times T\to\mathbb{R}$ such that for any $t\in T$, $\varphi(\cdot, t)$ is a continuously differentiable function, and such that the function φ together with its derivatives $\frac{\partial\varphi}{\partial x_i}$ have an absolute value bounded by $\psi(t)(1+|x|^\nu)$, where $\psi\in L^2(T)$ and $\nu\geqq 0$ is some integer. Let $h_1, \ldots, h_n\in H$. Then, the process $u(t, w)=\varphi(w(h_1), \ldots, w(h_n), t)$ satisfies the hypotheses of Proposition 6.1. In fact, we have

$$Du_t(s)=\sum_{i=1}^{n} \frac{\partial\varphi}{\partial x_i}(w(h_1), \ldots, w(h_n), t)\, h_i(s).$$

So, we know that the Ogawa integral $\mathring{\delta}$ exists. In order to compute its value, denote by $\{\varphi(x, w),\ x\in\mathbb{R}^n\}$ a continuous version of the process $w(\varphi(x, \cdot)) =\int_T \varphi(x, t)\, dw_t$. The existence of this version follows from the Kolmogorov continuity criterion. In fact, for any $p\geqq 2$ and $|x|, |y|\leqq k$, we have

$$E(|w(\varphi(x, \cdot))-w(\varphi(y, \cdot))|^p)\leqq C_p\Big(\int_T |\varphi(x, t)-\varphi(y, t)|^2\, \mu(dt)\Big)^{p/2}\leqq \text{const}\, |x-y|^p.$$

We will show now that

$$\mathring{\delta} u=\varphi(w(h_1), \ldots, w(h_n), w), \tag{6.4}$$

and, by Proposition 6.1,

$$\delta u=\varphi(w(h_1), \ldots, w(h_n), w)-\sum_{i=1}^{n} \int_T \frac{\partial\varphi}{\partial x_i}(w(h_1), \ldots, w(h_n), t)\, h_i(t)\, \mu(dt).$$

To show Eq. (6.4), define

$\varphi_M(x, w) = \sum_{i=1}^{M} \langle \varphi(x, \cdot), e_i \rangle w(e_i)$. We know that $\varphi_M(w(h_1), \ldots, w(h_n), w)$ converges in probability to $\overset{\circ}{\delta} u$ when M tends to infinity. We are going to see that this sequence of random variables converges in L^2 to the right-hand side of (6.4). To do this, assume that the h_i belong to the linear span of $\{e_i, \ldots, e_n\}$. Then, for $M \geqq n$, we have

$$\begin{aligned} &E(|\varphi_M(w(h_1), \ldots, w(h_n), w) - \varphi(w(h_1), \ldots, w(h_n), w)|^2) \\ &= E\left(\left(\sum_{i=M+1}^{\infty} w(e_i) \int_T \varphi(\cdot, t)\, e_i(t)\, \mu(dt)\right)(w(h_1), \ldots, w(h_n))^2\right) \\ &= E \sum_{i=M+1}^{\infty} \left(\int_T \varphi(w(h_1), \ldots, w(h_n), t)\, e_i(t)\, \mu(dt)\right)^2 \xrightarrow[M \to \infty]{} 0. \end{aligned}$$

(iii) Let $F \in \operatorname{Dom} L$. Assume that $D^2 F$ has a finite trace for almost all w. Then, the process $u = DF$ is Ogawa integrable, and we have

$$\overset{\circ}{\delta}(DF) = \delta(DF) + \operatorname{trace} D^2 F = LF + \operatorname{trace} D^2 F.$$

We recover here a familiar expression for the operator L. In particular, in the context of Example 3.6, if F is extendable to all continuous functions vanishing at zero and this extension, $F_c(\cdot)$, is twice Frechet differentiable then $\overset{\circ}{\delta} DF(w) = (\partial F_c(\lambda w)/\delta \lambda)_{\lambda=1}$ (cf. Eq. 14 of [19]).

We remark that the conclusion of Proposition 6.1 is still true if, instead of assuming that $Du_t(s)$ is a trace class kernel, we suppose that the series

$$\sum_i \int_{T^2} Du_t(s)\, e_i(t)\, e_i(s)\, \mu(dt)\, \mu(ds)$$

converges in probability and the sum does not depend on the orthonormal system $\{e_i, i \geqq 1\}$.

Consider now the situation of Example 3.6. That means, $T = [0, 1]$, μ is the Lebesgue measure and w is the Wiener process. Let $u \in L^2(T \times \Omega)$ be a process such that $\int_T E(\|Du_t\|_H^2)\, \mu(dt) < \infty$, and assume that a.s.,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon^2} \int_{([x-\varepsilon, x+\varepsilon] \cap T)^2} Du_t(s)\, ds\, dt = \varphi(x), \quad \text{for all } x \in T, \text{ a.e.,} \tag{6.5}$$

where $\varphi(t)$ is an integrable function, which will be denoted by $Du_t(t)$. Under these assumptions we can define the *symmetric integral* of the process u by the expression

$$\int_T u \overset{s}{\delta} w = \int_T u \delta w + \int_T Du_t(t)\, dt.$$

This symmetric stochastic integral is similar to the Ogawa integral in the sense that they coincide if, for instance, the kernel $Du_t(s)$ is continuous. Finally we have the following result.

Proposition 6.2. *Let x and y be adapted processes of $L^2(T \times \Omega)$ verifying*

$$\int_T E(\|Dx_t\|_H^2 + \|Dy_t\|_H^2)\, dt < \infty.$$

Consider the continuous semimartingale, with respect to the Wiener filtration, given by $u_t = \int_0^t x_s\,dw_s + \int_0^t y_s\,ds$. Then the process u verifies condition (6.5) and the symmetric integral of u is equal to the Stratonovich integral, that means,

$$\int_T u \overset{s}{\delta} w = \int_T u\,dw + \tfrac{1}{2}\langle u, w\rangle_1.$$

Proof. It suffices to see that (6.5) holds and $Du_t(t) = \frac{1}{2} x_t$ for all $t \in T$, a.e. In fact, if this is true we will have

$$\int_T Du_t(t)\,dt = \tfrac{1}{2}\int_T x_t \cdot dt = \tfrac{1}{2}\langle u, w\rangle_1.$$

It is clear that $u_t \in L^2(T \times \Omega)$. Consider the integral representations

$$x_t = E(x_t) + \sum_{m=1}^{\infty} I_m(f_m(t\,|\,t_1, \ldots, t_m)),$$

and

$$y_t = E(y_t) + \sum_{m=1}^{\infty} I_m(g_m(t\,|\,t_1, \ldots, t_m)),$$

where the kernels f_m and g_m satisfy properties (a) and (b) of Sect. 3, and, also they vanish unless $\max\{t_1, \ldots, t_m\} \leqq t$. Assume first that $x_t = I_m(f_m(t\,|\,\cdot))$, and $y_t = I_m(g_m(t\,|\,\cdot))$. In this case we obtain, from Proposition 3.4, that

$$Du_t(s) = \int_s^t Dx_s(\sigma)\,dw_\sigma + \int_s^t Dy_s(\sigma)\,d\sigma + x_s, \tag{6.6}$$

for all $s \leqq t$.

We remark that the kernels $Du_t(s)$, $Dx_t(s)$ and $Dy_t(s)$ are zero if $s > t$. In the expression (6.6) we sum the index m from 1 to infinity obtaining that $\int_T E(\|Du_t\|_H^2)\,dt < \infty$ and that the equality (6.6) holds in general. Set

$$\alpha(s, t) = \int_s^t Dx_s(\sigma)\,dw_\sigma + \int_s^t Dy_s(\sigma)\,d\sigma.$$

Then, we have $\lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon^2} \int_{([x-\varepsilon, x+\varepsilon] \cap T)^2} \alpha(s, t)\,ds\,dt = 0$ for almost all x and w. In fact, using the Lebesgue differentiation theorem, we have

$$\frac{1}{4\varepsilon^2} \int_{x-\varepsilon}^{x+\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |\alpha(s, t)|\,ds\,dt \leqq \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} \Big(\sup_{s \leqq t \leqq s+(1/n)} |\alpha(s, t)| \Big)\,ds \underset{\varepsilon \downarrow 0}{\longrightarrow} \sup_{x \leqq t \leqq x+(1/n)} |\alpha(x, t)|,$$

for almost all x, and finally, the expression $\sup_{x \leqq t \leqq x+(1/n)} |\alpha(x, t)|$ converges to zero when n tends to infinity, by continuity. Moreover, for almost all t and w, we have

$$\begin{aligned} Du_t(t) &= \lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon^2} \int_{([t-\varepsilon, t+\varepsilon] \cap T)^2} x_\sigma 1_{\{\sigma \leqq \tau\}}\,d\sigma\,d\tau \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon^2} \int_{t-\varepsilon}^{t+\varepsilon} (t + \varepsilon - \sigma)\,x_\sigma\,d\sigma = x_t. \end{aligned}$$

7. Transformation of Measure

Let $w(B)$ be a Gaussian measure with orthogonal increments defined in the canonical probability space $(\Omega, \mathbf{F}, P)$. Assume that u is a stochastic process of $L^2(T\times\Omega)$ such that $\int_T E(\|Du_t\|_H^2)\,\mu(dt)<\infty$. We know that with these assumptions the process u is Skorohod integrable. We consider a new stochastic measure defined by $\{w(B)-\int_B u\,d\mu,\ B\in\mathbf{B}\}$, and we want to know whether the law of this stochastic process indexed by $\mathbf{B}$ is absolutely continuous with respect to the law of $\{w(B),\ B\in\mathbf{B}\}$. Note that we can also formulate this problem in terms of stochastic processes indexed by the Hilbert space $H=L^2(T)$.

Suppose that $A(s,t)$ is a square-integrable kernel on the measure space $(T,\mathbf{B},\mu)$. The Carleman-Fredholm determinant of A is defined by the product expansion

$$d_c(A)=\Pi_j(1-\lambda_j)\exp\lambda_j,$$

where the λ_j's are the nonzero eigenvalues of A counted with their multiplicities. The following properties of this determinant are well-known:

(i) $d_c(A)$ is a continuous function of A with respect to the Hilbert-Schmidt norm

$$\|A\|=\{\int_{T^2} A(s,t)^2\,\mu(ds)\,\mu(dt)\}^{1/2}.$$

(ii) If $Af(t)=\int_T A(s,t)f(s)\,\mu(ds)$ is a nuclear operator of $H=L^2(T)$, and I denotes the identity operator then $d_c(A)=\det(I-A)\exp(\operatorname{trace}A)$.

(iii) Let A and B be two square-integrable kernels and denote by AB the composition of these kernels: $(AB)(s,t)=\int_T A(s,u)\,B(u,t)\,\mu(du)$. Then, we have $d_c(A)\cdot d_c(B)=d_c(A+B-AB)\exp(\operatorname{trace}(AB))$. In particular, if $A+B=AB$, then $d_c(A)\,d_c(B)=\exp(\operatorname{trace}(AB))$.

An alternative expression for $d_c(A)$ is the following

$$d_c(A)=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int_{T^n}\det(\hat{A}(s_i,s_j))\,\mu(ds_1)\ldots\mu(ds_n),$$

where $\hat{A}(s_i,s_j)=A(s_i,s_j)$ if $i\neq j$, and $\hat{A}(s_i,s_i)=0$.

Denote by $\{e_i, i\geq 1\}$ a complete orthonormal system in H. A process $v\in L^2(T\times\Omega)$ will be called an *elementary process* if for some integer N, v can be expressed like

$$v_t=\sum_{j=1}^{N}\psi_j(w(e_1),\ldots,w(e_N))\,e_j(t), \tag{7.1}$$

where $\psi\colon\mathbb{R}^N\to\mathbb{R}$ is a continuously differentiable function such that ψ together with its first derivative has polynomial growth.

Let $u\in L^2(T\times\Omega)$ such that $\int_T E(\|Du_t\|_H^2)\,\mu(dt)<\infty$. It is easy to see that the process u can be approximated in the norm $\{\int_T E(u_t^2)\,\mu(dt)$

$+\int_T E(\|Du_t\|_H^2)\,\mu(dt)\}^{1/2}$ by elementary processes. Indeed, first we approximate the process u by a finite sum of the form $E(u_t)+\sum_{m=1}^{n} I_m(f_m(t\,|\,\cdot))$. Then, we may assume that the kernels $f_m(t\,|\,t_1,\ldots,t_m)$ are finite linear combinations of products like

$$h(t)\,e_{j_1}(t_1)\ldots e_{j_1}(t_{i_1})\,e_{j_2}(t_{i_1+1})\ldots e_{j_2}(t_{i_2})\ldots e_{j_k}(t_{i_1+\ldots+i_{k-1}+1})\ldots e_{j_k}(t_m),$$

where $h\in L^2(T)$, $i_1+\ldots+i_k=m$, and the indices $j_1,\ldots,j_k$ are different. That means, the process u is a finite linear combination of products of the form $h(t)\,h_{i_1}(w(e_{j_1}))\ldots h_{i_k}(w(e_{j_k}))$ where $h_n(x)$ are Hermite polynomials, normalized in such a way that the coefficients of x^n are equal to 1.

An elementary process v will be called *one-to-one* if the mapping $(I-\psi)(x)=(x_1-\psi_1(x),\ldots,x_N-\psi_N(x))$ is one-to-one. A process $u\in L^2(T\times\Omega)$ such that $\int_T E(\|Du_t\|_H^2)\,\mu(dt)<\infty$ will be called *weakly one-to-one* if u can be approximated in the norm $\{E(\|u\|_H^2)+E(\|Du\|_{H.S.}^2)\}^{1/2}$ by one-to-one elementary processes. Consider, for instance, the situation of Example 3.6, that means, $T=[0,1]$ and w_t is the Wiener process. In this case, any adapted process $u\in L^2(T\times\Omega)$ such that $E(\|Du\|_{H.S.}^2)<\infty$ is weakly one-to-one.

Let v be an elementary process, given w, then x and $y=(I-\psi)(x)$ are well defined hence $y=y(w)$. The cardinality of the x's such that $(I-\psi)(x)=y(w)$ is the multiplicity function of v and will be denoted $M_v(w)$; obviously $M_v(w)\geqq 1$ and may be infinite. For example, let $v(t)=(w,e_1)^3\cdot e_1$ (where $\|e_1\|=1$) then $M_v(w)=3$ for $|(w,e)|<2\cdot 3^{-2/3}$ and $M_v(w)=1$ otherwise. Let u be a process $u\in L^2(\Omega\times T)$ and $E\int_T \|Du_t\|_H^2\,\mu(dt)<\infty$, an integer valued finite or infinite random variable $M_u(w)$ will be said to be a weak multiplicity function associated with u if u can be approximated in the norm $\{E(\|u\|_H^2)+E(\|Du\|_{H.S.}^2)\}^{1/2}$ by elementary processes u_n such that $M_{u_n}(w)\xrightarrow{\text{a.s.}} M_u(w)$. In particular, if u is weakly one-to-one then $M_u(w)=1$ a.s.

Theorem 7.1. *Let $u\in L^2(T\times\Omega)$ such that $\int_T E(\|Du_t\|_H^2)\,\mu(dt)<\infty$, and assume that u possesses a multiplicity function $M_u(w)$ such that $M_u(w)<\infty$ a.s. Let $g\colon \mathbb{R}^d\to\mathbb{R}$ be a positive, continuous and bounded function and consider the non-negative functional $F(w)=g(w(B_1),\ldots,w(B_d))$, where $B_1,\ldots,B_d\in\mathbf{B}$. Then*

$$E(F(w))\geqq E\left((M_u(w))^{-1}\,|d_c(Du)|\exp\Big(\int_T u\,\delta w-\tfrac{1}{2}\|u\|_H^2\Big)\,F\Big(w-\int u\,d\mu\Big)\right),\qquad (7.2)$$

where $d_c(Du)$ represents the Carleman-Fredholm determinant of the kernel Du. Furthermore, if $d_c(Du)\neq 0$ a.s. then, the probability distribution of the process $\{w(B)-\int_B u\,d\mu,\ B\in\mathbf{B}\}$ is absolutely continuous with respect to the law of $\{w(B),\ B\in\mathbf{B}\}$.

Proof. Denote by $\{e_i, i\geqq 1\}$ a complete orthonormal system in H. Assume that v is an elementary process of the form (7.1) and denote by $\Delta(w)$ the $N\times N$ matrix given by

$$\Delta_{ij}(w)=\delta_{ij}-\langle D_{e_i}v(w),e_j\rangle=\delta_{ij}-\partial_i\psi_j(w(e_1),\ldots,w(e_n)).$$

Then we have

$$E[M_v^{-1}(w)\,|d_c(Dv)|\,\exp(\int_T v\,\delta w-\tfrac{1}{2}\|v\|_H^2)\,F(w-\int v\,d\mu)]$$
$$=E[M_v^{-1}(w)\,|\det\Delta(w)|\,\exp(\operatorname{trace}Dv+\int_T v\,\delta w-\tfrac{1}{2}\|v\|_H^2)\,F(w-\int v\,d\mu)]$$
$$=E\left[M_v^{-1}(w)\,|\det\Delta(w)|\,\exp\left(\sum_{j=1}^N\langle v,e_j\rangle\,w(e_j)-\frac{1}{2}\sum_{j=1}^N\langle v,e_j\rangle^2\right)F(w-\int v\,d\mu)\right]. \tag{7.3}$$

For any $h\in H$ we set $\tilde{w}(h)=w(h)-\sum_{j=1}^N\langle h,e_j\rangle\,w(e_j)$. The random processes $\tilde{w}(h)$ and $\sum_{j=1}^N\langle h,e_j\rangle\,w(e_j)$ are independent, and, therefore, the expectation (7.3) will be equal to

$$E_{\tilde{w}}\int_{\mathbb{R}^N}M_v^{-1}\left(\tilde{w}+\sum_{k=1}^N x_k e_k\right)\left|\det\Delta\left(\tilde{w}+\sum_{k=1}^N x_k e_k\right)\right|$$
$$\cdot\exp\left(\sum_{j=1}^N\psi_j(x_1,\ldots,x_N)\,x_j-\tfrac{1}{2}\sum_{j=1}^N\psi_j(x_1,\ldots,x_N)^2\right)$$
$$\cdot F\left(\tilde{w}+\sum_{j=1}^N x_j e_j-\sum_{j=1}^N\psi_j(x_1,\ldots,x_N)\,e_j\right)$$
$$\cdot\exp\left(-\tfrac{1}{2}\sum_{j=1}^N x_j^2\right)(2\pi)^{-N/2}\,dx_1\ldots dx_N$$
$$=E_{\tilde{w}}\int_{\mathbb{R}^N}M_v^{-1}\left(\tilde{w}+\sum_{k=1}^N x_k e_k\right)|\det(I-j\psi)|$$
$$\cdot\exp\left(-\tfrac{1}{2}\sum_{j=1}^N(x_j-\psi_j(x_1,\ldots,x_N))^2\right)$$
$$\cdot F\left(\tilde{w}+\sum_{j=1}^N(x_j-\psi_j(x_1,\ldots,x_N))\,e_j\right)(2\pi)^{-N/2}\,dx_1\ldots dx_N,$$

where J_ψ denotes the Jacobian matrix of ψ. Then we make the change of variables $x_j-\psi_j(x_1,\ldots,x_N)=y$, in the above integral. The Jacobian matrix of this transformation $\left[\frac{\partial y_i}{\partial x_j}\right]$ coincides with $I-J_\psi$. In applying the change of variable formula we have to multiply by $M_v(w)$ ([3], Theorem 3.2.5, p. 244) and, consequently the expression (7.3) is upper bounded by

$$E_{\tilde{w}}\int_{\mathbb{R}^N}\exp\left(-\tfrac{1}{2}\sum_{j=1}^N y_j^2\right)F\left(\tilde{w}+\sum_{j=1}^N y_j e_j\right)(2\pi)^{-N/2}\,dy_1\ldots dy_N=E(F(w)).$$

The reason for it's being an upper bound for (7.3) rather than an equality is that in the left hand side of the last equality we also integrate over regions for which $\det(I-J\psi)=0$.

Finally, let $\{v_n, n\geq 1\}$ be a sequence of one-to-one elementary processes converging to the process u in the norm $\{E(\|u\|_H^2)+E(\|Du\|_{H.S.}^2)\}^{1/2}$. Then, there exists a subsequence $\{v_{n_i}, i\geq 1\}$ such that

$$M_{u_{n_i}}^{-1} |d_c(D v_{n_i})| \exp(\int_T v_{n_i} \delta w - \tfrac{1}{2} \|v_{n_i}\|_H^2) F(w - \int v_{n_i} d\mu)$$

converges almost surely to

$$M_u^{-1} |d_c(Du)| \exp(\int_T u \delta w - \tfrac{1}{2} \|u\|_H^2) \hat{F}(w - \int u d\mu).$$

Therefore, by Fatou's lemma we obtain the inequality (7.2). The absolute continuity of $\{w(B) - \int_B u d\mu\}$ follows directly from (7.2) since $|d_c(Du)| > 0$ a.s.

Lemma 7.2. *Let v be an elementary process of the form* (7.1) *and assume that $d_c(Dv) \neq 0$ a.s. Then, the probability distribution of the process $\{w(B) - \int_B v d\mu, B \in \mathbf{B}\}$ is absolutely continuous with respect to the law of $\{w(B), B \in \mathbf{B}\}$.*

Proof. Suppose that F is a bounded functional such that $E(F(w)) = 0$. Using the same notation as in the demonstration of Theorem 7.1, for all $\tilde{w}$, almost surely, we will have

$$\int_{\mathbb{R}^N} F\left(\tilde{w} + \sum_{j=1}^N y_j e_j\right) \exp\left(-\tfrac{1}{2} \sum_{j=1}^N y_j^2\right) dy_1 \ldots dy_N = 0.$$

For these $\tilde{w}$ it is clear that

$$\int_{\mathbb{R}^N} |\det(I - J\psi)| \exp\left(-\tfrac{1}{2} \sum_{j=1}^N (x_j - \psi_j(x_1, \ldots, x_N))^2\right),$$

$$F\left(\tilde{w} + \sum_{j=1}^N (x_j - \psi_j(x_1, \ldots, x_N)) e_j\right) dx_1 \ldots dx_N = 0.$$

Integrating with respect to $\tilde{w}$ we obtain

$$E[|d_c(Dv)| \exp(\int_T v \delta w - \tfrac{1}{2} \|v\|_H^2) F(w - \int v d\mu)] = 0.$$

Therefore, $E(F(w - \int v d\mu)) = 0$, which implies the desired result.

We remark that, in the last proposition, the condition $d_c(Dv) \neq 0$ a.s. is equivalent to saying that $I - J\psi \neq 0$ a.e. with respect to the Lebesgue measure on $\mathbb{R}^N$.

Proposition 7.3. *Assume that $u_t = \sum_{i=1}^N F_i(w) e_i(t)$, where the F_i are random variables such that the second Malliavin derivatives $D^2 F_i$ exist and $E(\|D^2 F_i\|_{H.S.}^2) < \infty$. Suppose $d_c(Du) \neq 0$ a.s. Then, the probability distribution of the process $\{w(B) - \int_B u d\mu, B \in \mathbf{B}\}$ is absolutely continuous with respect to the law of $\{w(B), B \in \mathbf{B}\}$.*

Proof. As before we will write $\tilde{w}(h) = w(h) - \sum_{j=1}^N \langle h, e_j \rangle w(e_j)$. We denote by $\mathbf{G}$ the σ-field generated by the random variables $\{\tilde{w}(h), h \in H\}$. An arbitrary bounded and measurable functional can be expressed as $F(w) = \psi(w(e_1), \ldots, w(e_N), G)$, where ψ is a bounded Borel function and G is $\mathbf{G}$-measurable.

Assume that $E(F(w))=0$. Then for all $\tilde{w}$, almost surely, we have

$$\int_{\mathbb{R}^N} \psi(x_1, \dots, x_N, G(\tilde{w})) \exp\left(-\tfrac{1}{2}\sum_{j=1}^{N} x_j^2\right) dx_1 \dots dx_N = 0. \tag{7.4}$$

The translated functional will be $F^u(w)=\psi(w(e_1)-F_1(w), \dots, w(e_N)-F_N(w), G(\tilde{w}))$, and our aim is to show that $E(F^u(w))=0$.

Denote by $P^u(dx|\tilde{w})$ a regular version of the probability distribution of the random vector $\xi=(w(e_1)-F_1(w), \dots, w(e_N)-F_N(w))$ conditioned by $\mathbf{G}$. Then,

$$E(F^u(w)) = E_{\tilde{w}} \int_{\mathbb{R}^N} \psi(x_1, \dots, x_N, G(\tilde{w}))\, P^u(dx/\tilde{w}).$$

Taking into account (7.4), in order to show that this expectation is zero, it suffices with proving that for all $\tilde{w}$, almost surely, the probability $P^u(dx|\tilde{w})$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^N$. This will be achieved using the Malliavin technique.

We fix a function $\varphi \in C_b^\infty(\mathbb{R}^N)$. For any integer $m \geqq 1$, we consider a function $\gamma_m \in C_0^\infty(\mathbb{R}^N \otimes \mathbb{R}^N)$ such that

(a) $\gamma_m(\sigma)=1$ if $\sigma \in K_m$,

(b) $\gamma_m(\sigma)=0$ if $\sigma \notin K_{m+1}$,

where $K_m=\left\{\sigma \in \mathbb{R}^N \otimes \mathbb{R}^N : |\sigma^{ij}| \leqq m \text{ for all } i, j; \text{ and } |\det \sigma| \geqq \frac{1}{m}\right\}$. We have

$$De_j[\varphi(\xi)] = \sum_{i=1}^{N} (\partial_i \varphi)(\xi)(\delta_{ij} - De_j F_i).$$

Denote by Γ the matrix $(\delta_{ij} - De_j F_i)$ and observe that $d_c(Du) \neq 0$ a.s. implies $\det \Gamma \neq 0$ a.s. Thus,

$$\begin{aligned} &E[\gamma_m(\Gamma)(\partial_i \varphi)(\xi) H] \\ &= \sum_{j=1}^{N} E[\gamma_m(\Gamma)\, De_j[\varphi \mid \xi)](\Gamma^{-1})^{ji} H] \\ &= \sum_{j=1}^{N} E[De_j(H \cdot \varphi \mid \xi)\, \gamma_m(\Gamma)(\Gamma^{-1})^{ji}) - \varphi \mid \xi)\, De_j(\gamma_m(\Gamma)(\Gamma^{-1})^{ji}) H \\ &\quad - \varphi(\xi)\, \gamma_m(\Gamma)(\Gamma^{-1})^{ji} De_j H], \end{aligned}$$

where H is a simple functional of the form $\alpha(w(e_{N+1}), \dots, w(e_M))$. Notice that $De_j H = 0$ for $j=1, \dots, N$. Therefore, we obtain

$$\begin{aligned} &E[\gamma_m(\Gamma)(\partial_i \varphi)(\xi) H] \\ &= \sum_{j=1}^{N} E[H \cdot \varphi \mid \xi)\{w(e_j)\, \gamma_m(\Gamma)(\Gamma^{-1})^{ji} - De_j(\gamma_m(\Gamma)(\Gamma^{-1})^{ji})\}]. \end{aligned}$$

As a consequence,

$$\begin{aligned} &|E[\gamma_m(\Gamma)(\partial_i \varphi)(\xi) \mid \mathbf{G}]| \\ &\leqq \|\varphi\|_\infty E\left[\left|\sum_{j=1}^{N} \{w(e_j)\, \gamma_m(\Gamma)(\Gamma^{-1})^{ji} - De_j(\gamma_m(\Gamma)(\Gamma^{-1})^{ji})\}\right| \mid \mathbf{G}\right]. \end{aligned}$$

For any m let $P_m^u(dx/\tilde{w})$ be a regular version of the conditional distribution of ξ given $\mathbf{G}$, with respect to the measure $\gamma_m(\Gamma)\,dP$. Using the basic Lemma of Malliavin we deduce that, for almost all $\tilde{w}$, the measure $P_m^u(dx/\hat{w})$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^N$. Making m tend to infinity we obtain the desired result. □

It is interesting to remark that inequality (7.2) implies the following Girsanov-type theorem:

Theorem 7.4. *Let $u \in L^2(T \times \Omega)$ be such that $E(\|Du\|_{H.S.}^2) < \infty$ and assume that $M_u(w) < \infty$ a.s. Set*

$$\eta = |d_c(Du)| \exp\Big(\int_T u\,\delta w - \tfrac{1}{2}\|u\|_H^2\Big) M_u^{-1}(w)$$

and suppose that $E(\eta)=1$. Then the law of the process $\{w(B) - \int_B u\,d\mu,\ B \in \mathbf{B}\}$ in the probability space $(\Omega, \mathbf{F}, \eta \cdot P)$ coincides with the law of $\{w(B),\ B \in \mathbf{B}\}$ with respect to P. Furthermore if $|d_c(Du)| \neq 0$ a.s. then the measures are equivalent.

Proof. Let F be a bounded functional and A a Borel subset of $\mathbb{R}$. By (7.2) we know that

$$E[1_A(F)] \geqq E[\eta 1_A(F(w - \textstyle\int u\,d\mu)]. \tag{7.5}$$

As functions of A both members of this inequality are nonnegative measures with total mass equal to $E(F)$. Therefore, they must coincide, that means that equality holds in (7.5) and this proves the theorem. By 7.1 and since $\eta \neq 0$ a.s., $w - \int u\,d\mu$ is absolutely continuous with respect to the Wiener measure. On the other hand if $E 1_A(w - \int u\,d\mu) = 0$ then $E\eta 1_A(w - \int u\,d\mu) = 0$ hence, by 7.5 with equality $E 1_A(w) = 0$ hence the equivalence of the measures. □

Let $u(w)$ be a process of $L^2(T \times \Omega)$ and let $v_t(w)$ be a process of $L^0(T \times \Omega)$ and such that

$$u_t(w) = -v_t(w - \textstyle\int u\,d\mu) \quad \text{a.s.}$$

Let $A \subset \Omega$ denote the range of the transformation $w \to w - \int u\,d\mu$ namely $A = \{w - \int u\,d\mu\}$. Let $y = w - \int u\,d\mu$ then a.s. for every w, the equation $y(B) = \int_B v(y)\,d\mu + w(B)$ has a unique solution y, $y \in A$ and given any $y \in A$ then there is a unique solution w which will be denoted by $w(y)$.

Theorem 7.5. *Let u and v be as specified above and assume:*

1) *$u(w)$ is weakly one-to-one, $E\|Du\|_{H.S.}^2 < \infty$.*
2) *$|d_c Du| \neq 0$ a.s. and*

$$E\{d_c(Du) \cdot \exp \int_T u\,\delta w - \tfrac{1}{2}\|u\|_H^2\} = 1$$

then $P(A)=1$, the probability law of the process $\{w(B) - \int u\,d\mu\}$, $B \in \mathbf{B}\}$ is equivalent to the law of $\{w(B),\ B \in \mathbf{B}\}$ and the density is given by

$$\frac{dP_Y}{dP_w}(y) = \frac{1}{|(d_c D u)(w(y))| \exp(\int_T v(y)\,\delta w(y) - \frac{1}{2}\|v\|_y^2)}.$$

Proof. Let $\mathbf{P}$ denote the underlying measure and $\tilde{\mathbf{P}}$ the measure induced by

$$\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}(w)=|d_c(Du)|\cdot\exp\left(\int u(w)\,\delta w-\tfrac{1}{2}\|u\|_H^2\right).$$

Now under $\tilde{\mathbf{P}}$, by Theorem 7.4, y is Wiener. Let P_Y denote the probability law of y under $\mathbf{P}$. Hence $\tilde{\mathbf{P}}$ restricted to y is Wiener namely P_w and $\mathbf{P}$ restricted to y is P_Y. Consequently

$$\frac{dP_w}{dP_Y}(y(w))=E\left[\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}\,|\mathbf{F}_Y\right]$$

where $\mathbf{F}_Y$ is the sigma field generated by $y(w)$ and the expectation is with respect to the measure $\mathbf{P}$.

Rewriting $d\tilde{\mathbf{P}}/d\mathbf{P}$:

$$\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}(w(y))=|d_c(D+u)(w(y))|\,\exp\left(-\int v(y)\,\delta w(y)-\tfrac{1}{2}\|v(y)\|_H^2\right)$$

it follows that this expression is measurable on $\mathbf{F}_Y$. Therefore

$$\frac{dP_w}{dP_Y}(y(w))=|(d_c Du)(w(y))|\cdot\exp\left(-\int v(y)\,\delta w(y)-\tfrac{1}{2}\|v\|_H^2\right)$$

since by 7.1, $P_Y \ll P_w$, and this completes the proof.

Examples. (1) Consider a process of the form $u_t(w)=f(t)\,F(w)$, where $f\in H$ and $F\in H^{2,2}$. In this case, the kernel $Du_t(s)=f(t)\,DF(s)$ is nuclear and it is easy to see that

$$d_c Du=(1-D_f F)\exp(D_f F).$$

So, as a consequence of Proposition 7.3, if $D_f F\neq 1$ a.s., then the law of the process $\{w(B)-F(w)\int_B f(t)\,\mu(dt),\ B\in\mathbf{B}\}$ is absolutely continuous with respect to the law of $\{w(B),\ B\in\mathbf{B}\}$.

(2) Consider the situation of Example 3.6, that means $T=[0,1]$ and w_t is the Wiener process. Suppose that u_t is an adapted and measurable process such that $\int_0^1 E(u_t^2)\,dt+\int_0^1\int_0^1 E(Du_t(s)^2)\,ds\,dt<\infty$. Then, $d_c Du=1$ because the kernel $Du_t(s)$ vanishes if $t<s$. Therefore, u satisfies the hypotheses of Theorem 7.1 with multiplicity $M=1$. However for this case the Girsanov theorem is at present stronger since it holds under weaker assumptions. Note that the formula for the density coincides with the result of Theorem 7.3 because $d_c Dv=1$.

(3) Consider the case $T=[0,1]$, $u(t)=(w(e))^k\cdot e_t$ and $\|e\|=1$. Then $1\leqq M_u\leqq k$; noting the shape of the graph of $x-x^k=y$, it follows that for k even, $M_u=2$ and for k odd, $k\geqq 3$ $M_u=3$ if $|y|<(k-1)\cdot k^{(1-k)/k}$ and 1 otherwise. Note that for k even there are values of y for which $x-x^k=y$ has no real solutions but $y(w)$ is a.s. not in this region. For both k even and odd, $k\geqq 2$, $d_c Du\neq 0$ a.s. and therefore by Theorem 7.1 the probability law induced by $w.-(w(e))^k\cdot e.$ is absolutely continuous with respect to the law of w. It is straightforward to verify that the assumptions of Theorem 7.4 are verified for k odd only.

Consider the case

$$u_t(w) = \pm(\tilde{w}(e))^k e_t$$
$$v_t(w) = -(w(e))^k e_t$$

where $\tilde{w}_t = w_t - \int_0^t u_s\,ds$ then $u_t(w) = -v_t(\tilde{w})$. Consider Theorem 7.5 for this case. The first relation implies that $\tilde{w}_t = w_t - (\tilde{w}(e))^k \int_0^t e_s\,ds$; hence in order that u_t be well defined we have to require that $\tilde{w}(e) = w(e) - (\tilde{w}(e))^k$ be satisfied for all values of $w(e)$. For k even this is impossible. For k odd, given $w(e)$ there is always one and only one solution for $\tilde{w}(e)$. Therefore u is one-to-one and so in v. Therefore Theorems 7.4 and 7.5 are applicable.

References

1. Bismut, J.M.: Martingales, the Malliavin calculus and hypoellipticity under general Hörmander's conditions. Z. Wahrscheinlichkeitstheor. Verw. Geb. **56**, 469–505 (1981)
2. Cameron, R.H., Martin, W.T.: The transformation of Wiener integrals by non-linear transformations. Trans. Am. Math. Soc. **66**, 253–283 (1949)
3. Federer, H.: Geometric measure theory. Berlin-Heidelberg-New York: Springer 1969
4. Gaveau, B., Trauber, P.: L'intégrale stochastique comme opérateur de divergence dans l'espace fonctionnel. J. Funct. Anal. **46**, 230–238 (1982)
5. Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes. Amsterdam-Oxford-New York: North Holland/Kodanska 1981
6. Ito, K.: Multiple Wiener integral. J. Math. Soc. Japan **3**, 157–169 (1951)
7. Kunita, H.: On backward stochastic differential equations. Stochastics **6**, 293–313 (1982)
8. Kuo, H.H.: Gaussian measures in Banach spaces. Lecture Notes Math. **463**. Berlin-Heidelberg-New York: Springer 1975
9. Kusuoka, S.: The non-linear transformation of Gaussian measure on Banach space and its absolute continuity (I). J. Fac. Sci. Univ. Tokyo Univ. Sec. **IA**, 567–597 (1982)
10. Malliavin, P.: Stochastic calculus of variations and hypoelliptic operators. In: Ito, K. (ed.). Proc. of Int. Symp. Stoch. D. Eqs. Kyoto 1976, pp. 195–263. Tokyo: Kinokuniya-Wiley (1978)
11. Malliavin, P.: Calcul des variations, intégrales stochastiques et complexes de Rham sur l'espace de Wiener. C.R. Acad. Sci. Paris, t. 299, Serie I, **8**, 347–350 (1984)
12. Ogawa, S.: Sur le produit direct du bruit blanc par luimême. C.R. Acad. Sc. Paris, t. **288** (Serie A), 359–362 (1979)
13. Ogawa, S.: Une remarque sur l'approximation de l'intégrale stochastique du type noncausal par une suite des intégrales de Stieltjes. Tohoku Math. Journ. **36**, 41–48 (1984)
14. Ramer, R.: On non-linear transformations of Gaussian measures. J. Funct. Anal. **15**, 166–187 (1974)
15. Shigekawa, I.: Derivatives of Wiener functionals and absolute continuity of induced measures. J. Math. Kyoto Univ. **20**, 263–289 (1980)
16. Shigekawa, I.: de Rham-Hodge-Kodaira's decomposition on abstract Wiener space. (preprint)
17. Skorohod, A.V.: On a generalization of a stochastic integral. Theory Probab. Appl. **XX**, 219–233 (1975)
18. Stroock, D.: The Malliavin calculus. A. functional analytical approach. J. Funct. Anal. **44**, 212–257 (1981)
19. Zakai, M.: The Malliavin calculus. Acta Applicandae Mat. **3**, 175–207 (1985)
20. Zakai, M.: Malliavin derivatives and derivatives of functionals of the Wiener process with respect to a scale parameter. Ann. Probab. **13**, 609–615 (1985)

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