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Langevin Equations for $\mathscr{S}'$-Valued Gaussian Processes and Fluctuation Limits of Infinite Particle Systems

Tomasz Bojdecki[1] and Luis G. Gorostiza[2,*]

[1] Centro de Investigación y de Estudios Avanzados, México 07000 D.F., México, and Institute of Mathematics, University of Warsaw, Poland

[2] Centro de Investigación y de Estudios Avanzados, México 07000 D.F., and SEGICyT-UPIICSA-IPN, México

Summary. $\mathscr{S}'$-valued Gaussian processes of a certain class are shown to satisfy generalized Langevin equations. Examples are fluctuation limits of several infinite particle systems, in particular infinite particle branching Brownian motions with immigration under various scalings and the voter model with hydrodynamic scaling.

1. Introduction

Let $\mathscr{S}(R^d)$ denote the space of C^∞ rapidly decreasing functions on R^d, and $\mathscr{S}'(R^d)$ its topological dual, the Schwartz space of tempered distributions. Let $A: \mathscr{S}(R^d) \to \mathscr{S}(R^d)$ be a continuous linear operator. We will study when a Gaussian $\mathscr{S}'(R^d)$-valued process $X \equiv \{X_t, t \in [0, T]\}$ satisfies a generalized Langevin equation of the form

$$dX_t = A^* X_t\, dt + dW_t, \tag{1.1}$$

where A^* is the adjoint of A and $W \equiv \{W_t, t \in [0, T]\}$ is an $\mathscr{S}'(R^d)$-valued Wiener process, in general time-inhomogeneous. (Our definition of W is slightly more general than the processes that can be obtained as stochastic integrals of deterministic processes with respect to the standard Wiener $\mathscr{S}_2'$-process introduced by Itô [20].) The precise meaning of a solution of (1.1) in this paper is taken to be the so called "mild" solution (Definition 3.1). Solutions of (1.1) are called generalized Ornstein-Uhlenbeck processes.

Let $X \equiv \{X_t, t \in [0, T]\}$ be an $\mathscr{S}'(R^d)$-valued process, continuous or right-continuous with left limits, and define $\tilde{X}$ by

$$\langle \hat{X}, \Phi \rangle = \int_0^T \langle X_t, \Phi(t, \cdot) \rangle\, dt, \qquad \Phi \in \mathscr{S}(R^{d+1}), \tag{1.2}$$

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where $\langle\cdot,\cdot\rangle$ denotes the duality on the appropriate spaces ($\mathscr{S}'(R^{d+1})$ and $\mathscr{S}(R^{d+1})$ on the left-hand side, $\mathscr{S}'(R^d)$ and $\mathscr{S}(R^d)$ on the right-hand side). We know that $\tilde{X}$ is a random element of $\mathscr{S}'(R^{d+1})$ and the distribution of $\tilde{X}$ determines the distribution of X [2]. Moreover, X is Gaussian if and only if $\tilde{X}$ is Gaussian. (The case $T=\infty$ requires a special treatment (see [2]); however, for (1.1) in this case we may assume T finite but arbitrary.)

We will show that for a continuous Gaussian $\mathscr{S}'(R^d)$-valued process X, being a solution of (1.1) is equivalent to the covariance functional of $\tilde{X}$ defined by (1.2) satisfying a certain condition (Theorem 3.5). If the operator A generates a strongly continuous semigroup of continuous linear operators on $\mathscr{S}(R^d)$, and if the covariance functional of X (which can be obtained from that of $\tilde{X}$) satisfies a certain condition, then X is Markovian and satisfies (1.1) (Theorem 3.6). In any case the distribution of the $\mathscr{S}'$-Wiener process W is determined explicitly by the covariance of X or $\tilde{X}$. The processes W that can appear in (1.1) belong to a general class of $\mathscr{S}'$-Wiener processes (Definition 2.1).

The relevance of the formulation in terms of $\mathscr{S}'(R^{d+1})$-valued random variables $\tilde{X}$ is that not only the distribution of $\tilde{X}$ suffices to obtain a Langevin equation for the process X in the Gaussian case, but in general weak convergence of a tight sequence of $\mathscr{S}'(R^d)$-valued processes $(X_n)_n$ follows if the corresponding sequence $(\tilde{X}_n)_n$ converges weakly in $\mathscr{S}'(R^{d+1})$ [2]. Thus the whole study of convergence of $\mathscr{S}'(R^d)$-valued processes to generalized Ornstein-Uhlenbeck processes may be carried out in a space of (space-time) distributions. This approach is useful for the asymptotic analysis of various types of infinite particle systems, where one wishes to show that the fluctuation process with respect to the mean converges to a generalized Ornstein-Uhlenbeck process, as will be illustrated below.

We shall give several illustrations of fluctuation limits of infinite particle systems and their corresponding Langevin equations, including Itô's example of an infinite system of Brownian motions [20] and Presutti and Spohn's hydrodynamic limits of the voter model and the simple exclusion process [26]. The others form a collection of examples concerning a certain infinite system of Brownian motions, including branching and immigration phenomena, that have been studied by one of us; most of these results are new and contain as special cases results of Dawson and Ivanoff [6, 7, 9, 21, 22], Holley and Stroock [18], Martin-Löf [23], and Gorostiza [13]; such results involve several different scalings and they were treated before using different approaches; one of our objectives here was developing a unified treatment. We remark that the Langevin equations (1.1) in [23] and [13], as well as in other papers ([12, 14]), were obtained in the form $\left\langle \tilde{X}, \left(A+\frac{\partial}{\partial t}\right)\Phi \right\rangle \sim -\left\langle \tilde{W}, \frac{\partial}{\partial t}\Phi \right\rangle$, $\Phi\in\mathscr{S}(R^{d+1})$, in the sense of equality of probability distributions; (in other of the papers cited above the meaning of the Langevin equations is not specified). Our results imply that solutions in the latter sense, which is apparently weaker than the mild sense, are in fact mild solutions of (1.1) (this follows from Proposition 3.4 and Theorem 3.5).

The results presented in [2] and here grew out of attempts to adapt the approach of Martin-Löf [23] to the analysis of more complicated infinite particle systems; while this was feasible in the cases of [8, 13] and [14], for other models it proved to be very difficult; the main problem was that one had to guess the operator A in the Langevin equation and then perform intricate computations to determine W, which became very unwieldy except in relatively simple cases. Thus it seemed desirable to extend the ideas of [23] in a general fashion that could apply to a broad class of models. This is what is achieved in [2] and in this paper.

In Sect. 2 we define the $\mathscr{S}'$-Wiener processes we need and prove their existence. Section 3 concerns the generalized Langevin equations. The examples are contained in Sect. 4.

2. $\mathscr{S}'$-Valued Wiener process

We recall that an $\mathscr{S}'(R^d)$-valued process $\{W_t, t\in R_+\}$ is said to be (centered) Gaussian if $\{\langle W_t, \phi\rangle : t\in R_+, \phi\in\mathscr{S}(R^d)\}$ is a (centered) Gaussian system.

Definition 2.1. A centered Gaussian $\mathscr{S}'(R^d)$-valued process $W\equiv\{W_t, t\in R_+\}$ is called a (generalized) Wiener process if it has continuous trajectories and its covariance functional $K(s,\phi;t,\psi)=E(\langle W_s,\phi\rangle\langle W_t,\psi\rangle)$ has the form

$$K(s,\phi;t,\psi)=\int_0^{s\wedge t}\langle Q_u\phi,\psi\rangle\,du, \qquad s,t\in R_+,\ \phi,\psi\in\mathscr{S}(R^d), \tag{2.1}$$

where the operators $Q_u\colon \mathscr{S}(R^d)\to\mathscr{S}'(R^d)$ have the properties:

(i) Q_u is linear, continuous, symmetric and positive for each $u\in R_+$, and

(ii) the function $u\to\langle Q_u\phi,\psi\rangle$ is right-continuous with left limits for each $\phi,\psi\in\mathscr{S}(R^d)$.

We say that W is associated to $Q\equiv\{Q_u, u\in R_+\}$.

Remarks. (a) From the definition it follows immediately that if W is an $\mathscr{S}'(R^d)$-valued Wiener process, then $W_0=0$ and it has independent increments in the sense that for any $0\leqq s<t$ and $\phi\in\mathscr{S}(R^d)$ the real random variable $\langle W_t,\phi\rangle-\langle W_s,\phi\rangle$ is independent of the σ-algebra generated by $\{\langle W_u,\psi\rangle : 0\leqq u\leqq s,\ \psi\in\mathscr{S}(R^d)\}$.

(b) The standard one-dimensional Wiener process $w=\{w_t, t\in R_+\}$ can be considered as an $\mathscr{S}'(R^d)$-valued Wiener process. Indeed, for arbitrary fixed $x_0\in R^d$ the process $W=w\delta_{x_0}$ satisfies the conditions of Definition 2.1 with $Q_u\phi=\phi(x_0)\,\delta_{x_0}$, $u\in R_+$. An analogous identification can be made for the standard n-dimensional Wiener process.

(c) The standard Wiener $\mathscr{S}'_2$-process b defined by Itô [19] is also an $\mathscr{S}'$-valued Wiener process in our sense, with $\langle Q_u\phi,\psi\rangle=\int_R\phi(x)\psi(x)\,dx$. Moreover, the stochastic integral $\int_0^t\alpha_s\,db_s$ defined in [20] for deterministic α is also a generalized Wiener process under the additional, but not restrictive, condition (ii) of Definition 2.1; however, not all generalized Wiener processes are of that form; the process defined in (b) is a counterexample.

In order to prove the existence theorem for generalized Wiener processes we need the following lemma.

Lemma 2.2. *Let* $Q_u\colon \mathscr{S}(R^d)\to\mathscr{S}'(R^d)$ *be a linear, continuous, symmetric operator for* $u\in R_+$, *such that the function* $u\to\langle Q_u\phi,\psi\rangle$ *is right-continuous with left limits for each* $\phi,\psi\in\mathscr{S}(R^d)$. *Then the function* $K\colon(R_+\times\mathscr{S}(R^d))\times(R_+\times\mathscr{S}(R^d))\to R$ *defined by* $K(s,\phi;t,\psi)=\int_0^{s\wedge t}\langle Q_u\phi,\psi\rangle\,du$ *is positive-definite if and only if* Q_u *is positive for each* $u\in R_+$.

Proof. Using the symmetry of $K(s,\phi;t,\psi)$ in the pairs $(s,\phi),(t,\psi)$, one obtains for any $0=t_0\leqq t_1\leqq\ldots\leqq t_n$, $\phi_1,\ldots,\phi_n\in\mathscr{S}(R^d)$,

$$\sum_{j,k=1}^{n}K(t_j,\phi_j;t_k,\phi_k)=\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left\langle Q_u\sum_{k=j}^{n}\phi_k,\sum_{k=j}^{n}\phi_k\right\rangle du. \tag{2.2}$$

If Q_u is positive, $u\in R_+$, the left-hand side of (2.2) is positive and this implies the positive-definiteness of K. Conversely, suppose that K is positive-definite but $\langle Q_t\phi,\phi\rangle<0$ for some $t\in R_+,\phi\in\mathscr{S}(R^d)$. Then by right-continuity there exists $t'>t$ such that $\langle Q_u\phi,\phi\rangle<0$ for $u\in[t,t']$. Define $t_1=t$, $t_2=t'$, $\phi_1=-\phi$, $\phi_2=\phi$; then (2.2) yields

$$0\leqq\sum_{j,k=1}^{2}K(t_j,\phi_j,t_k,\phi_k)=\int_{t_1}^{t_2}\langle Q_u\phi_2,\phi_2\rangle\,du<0.$$

This contradiction ends the proof.

The main result of this section is the following existence theorem.

Theorem 2.3. *For each family of operators* $Q=\{Q_u,u\in R_+\}$ *satisfying conditions* (i) *and* (ii) *of Definition* 2.1 *there exists an* $\mathscr{S}'(R^d)$*-valued Wiener process associated to* Q.

Proof. It suffices to verify that $K(t,\phi;t,\phi)$ satisfies the conditions of Theorem 4.1 of [19], i.e. that it is continuous in t for every ϕ, and that it is a continuous positive-definite quadratic functional of ϕ for every t; the continuity in ϕ follows from Proposition 2 of [27].

3. Langevin Equations for $\mathscr{S}'$-Gaussian Processes

In what follows we consider processes on a fixed time interval $[0,T]$, where $0<T\leqq\infty$, and for $T=\infty$ $[0,T]$ should be read as $[0,\infty)$.

For unity within this section let us write equation (1.1) here:

$$dX_t=A^*X_t\,dt+dW_t. \tag{3.1}$$

We will investigate when a given Gaussian $\mathscr{S}'(R^d)$-valued process $X\equiv\{X_t,t\in[0,T]\}$ satisfies this equation, where A^* is the adjoint of a linear, continuous operator $A\colon\mathscr{S}(R^d)\to\mathscr{S}(R^d)$, and W is an $\mathscr{S}'(R^d)$-valued Wiener process, defined in the previous section. Equation (3.1) is understood in the following, so called "mild" sense:

Definition 3.1. We say that a process X is a solution of (3.1) if for each $\phi \in \mathscr{S}(R^d)$,

$$\langle X_t, \phi\rangle = \langle X_0, \phi\rangle + \int_0^t \langle X_u, A\phi\rangle\, du + \langle W_t, \phi\rangle \quad \text{for } t \in [0, T]. \tag{3.2}$$

For any continuous $\mathscr{S}'(R^d)$-valued process $X \equiv \{X_t, t \in [0, T]\}$ we consider $\tilde{X}$, defined by (1.2), which is a random element of $\mathscr{S}'(R^{d+1})$ if $T < \infty$. If $T = \infty$ a larger space of distributions is needed (see [2]), but for the sake of brevity we prefer not to distinguish between the cases $T < \infty$ and $T = \infty$ in this paper. We know from Proposition 4.1 of [2] that the distribution of $\tilde{X}$ determines that of X, and moreover, from its proof we obtain immediately:

Lemma 3.2. *The process X is Gaussian if and only if $\tilde{X}$ is Gaussian.*

In the first part of this section the fact that X is a solution of (3.1) will be expressed in terms of $\tilde{X}$. To this end, for any $\mathscr{S}'(R^d)$-valued Wiener process W we define ∂W by

$$\langle \partial W, \Phi\rangle = -\int_0^T \left\langle W_t, \frac{\partial}{\partial t}\Phi(t, \cdot)\right\rangle dt, \qquad \Phi \in \mathscr{S}(R^{d+1}). \tag{3.3}$$

Henceforth we will restrict Φ to $\mathscr{D}([0, T]) \hat{\otimes} \mathscr{S}(R^d)$, where $\mathscr{D}([0, T])$ is the subspace of $\mathscr{S}(R)$ of functions having supports contained in the open interval $(0, T)$.

Lemma 3.3. *If W is an $\mathscr{S}'(R^d)$-valued Wiener process associated to Q, then ∂W is a Gaussian random element of $(\mathscr{D}([0, T]) \hat{\otimes} \mathscr{S}(R^d))'$ with mean 0 and covariance functional determined by*

$$E\langle \partial W, \phi f\rangle \langle \partial W, \psi g\rangle = \int_0^T \langle Q_u \phi, \psi\rangle f(u) g(u)\, du, \tag{3.4}$$

for $\phi, \psi \in \mathscr{S}(R^d)$, $f, g \in \mathscr{D}([0, T])$.

Proof. Only (3.4) needs verification, all the other assertions being trivial. The derivation of (3.4) is based upon straightforward calculations, using the integration by parts formula several times and the fact that the elements of $\mathscr{D}([0, T])$ vanish at 0 and T.

Remark. ∂W can be viewed as a "generalized $\mathscr{S}'(R^d)$-valued process" $\{\partial W_s, s \in [0, T]\}$ such that, formally, $W_t = \int_0^t \partial W_s\, ds$ and

$$E\langle \partial W_s, \phi\rangle \langle \partial W_t, \psi\rangle = \delta(s - t)\langle Q_t \phi, \psi\rangle, \qquad \phi, \psi \in \mathscr{S}(R^d).$$

The so called "space-time Gaussian white noise" corresponds to $Q_{\cdot} \equiv$ identity on $\mathscr{S}(R^d)$.

It will be convenient to introduce one more notation. To each continuous linear operator $A: \mathscr{S}(R^d) \to \mathscr{S}(R^d)$ we associate a continuous linear operator $A + \partial: \mathscr{D}([0, T]) \hat{\otimes} \mathscr{S}(R^d) \to \mathscr{D}([0, T]) \hat{\otimes} \mathscr{S}(R^d)$ determined by

$$(A + \partial)\phi f = f A\phi + \phi f', \qquad \phi \in \mathscr{S}(R^d), \quad f \in \mathscr{D}([0, T]), \tag{3.5}$$

where $f' \equiv df/dt$. Its adjoint is of course $A^* - \partial: (\mathscr{D}([0, T]) \hat{\otimes} \mathscr{S}(R^d))' \to (\mathscr{D}([0, T]) \hat{\otimes} \mathscr{S}(R^d))'$ such that

$$\langle (A^* - \partial)\xi, \phi f \rangle = \langle \xi, (A + \partial)\phi f \rangle, \quad \phi \in \mathscr{S}(R^d), \quad f \in \mathscr{D}([0, T]), \quad \xi \in (\mathscr{D}([0, T])) \hat{\otimes} \mathscr{S}(R^d))'. \tag{3.6}$$

Proposition 3.4. *A continuous $\mathscr{S}'(R^d)$-valued process X is a solution of* (3.1) *if and only if*

$$-(A^* - \partial)\tilde{X} = \partial W, \tag{3.7}$$

where ∂W is defined by (3.3).

Proof. By (3.5), (3.6) and (1.2) we have for $\phi \in \mathscr{S}(R^d)$, $f \in \mathscr{D}([0, T])$,

$$\begin{aligned}\langle -(A^* - \partial)\tilde{X}, \phi f \rangle &= \langle \tilde{X}, -fA\phi - \phi f' \rangle = -\int_0^T \langle X_t, A\phi \rangle f(t)\, dt - \int_0^T \langle X_t, \phi \rangle f'(t)\, dt \\ &= \int_0^T \left(\langle X_0, \phi \rangle + \int_0^t \langle X_u, A\phi \rangle\, du - \langle X_t, \phi \rangle \right) f'(t)\, dt, \end{aligned} \tag{3.8}$$

where the last equality was obtained by integration by parts using the fact that $f(0) = f(T) = 0$.

Let us define an $\mathscr{S}'(R^d)$-valued continuous process Y by

$$\langle Y_t, \phi \rangle = \langle X_0, \phi \rangle + \int_0^t \langle X_u, A\phi \rangle\, du - \langle X_t, \phi \rangle, \quad t \in [0, T],\ \phi \in \mathscr{S}(R^d). \tag{3.9}$$

By (3.3) and (3.8) equality (3.7) is clearly equivalent to

$$\int_0^T \langle Y_t, \phi \rangle f'(t)\, dt = -\int_0^T \langle W_t, \phi \rangle f'(t)\, dt, \quad \phi \in \mathscr{S}(R^d),\ f \in \mathscr{D}([0, T]),$$

or, for any fixed $\phi \in \mathscr{S}(R^d)$,

$$\frac{d}{dt}(\langle Y_t, \phi \rangle + \langle W_t, \phi \rangle) = 0$$

in the $\mathscr{D}'([0, T])$-sense. Since the function $t \to \langle Y_t, \phi \rangle + \langle W_t, \phi \rangle$ is continuous and equal to 0 at $t = 0$, the last expression is equivalent to $\langle Y_t, \phi \rangle + \langle W_t, \phi \rangle = 0$, $t \in [0, T]$, and this, by (3.9), is precisely (3.2). The proof is complete.

We will formulate now a condition for X to be a solution of (3.1) in terms of the covariance functional $\tilde{K}$ of $\tilde{X}$, i.e.

$$\tilde{K}(\Phi, \Psi) = \operatorname{Cov}(\langle \tilde{X}, \Phi \rangle, \langle \tilde{X}, \Psi \rangle), \quad \Phi, \Psi \in \mathscr{D}([0, T]) \hat{\otimes} \mathscr{S}(R^d). \tag{3.10}$$

Theorem 3.5. *Let $X \equiv \{X_t, t \in [0, T]\}$ be a continuous, centered Gaussian $\mathscr{S}'(R^d)$-valued process, and let $\tilde{K}$ be given by* (3.10). *Assume that there exist a continuous linear operator $A: \mathscr{S}(R^d) \to \mathscr{S}(R^d)$ and a family $Q \equiv \{Q_u, u \in [0, T]\}$ of operators satisfying conditions* (i) *and* (ii) *of Definition* 2.1, *such that*

$$\tilde{K}((A+\partial)\phi f,(A+\partial)\psi g)=\int_0^T\langle Q_u\phi,\psi\rangle f(u)g(u)\,du, \tag{3.11}$$

$$\phi,\psi\in\mathscr{S}(R^d),\quad f,g\in\mathscr{D}([0,T]).$$

Then there exists an $\mathscr{S}'$-valued Wiener process W associated to Q such that X is a solution of (3.1). *Conversely, if* (3.2) *holds for some A, W (associated to Q), then* (3.11) *is satisfied.*

Proof. (3.11) is clearly the same as

$$E(\langle(A^*-\partial)\tilde{X},\phi f\rangle\langle(A^*-\partial)\tilde{X},\psi g\rangle)=\int_0^T\langle Q_u\phi,\psi\rangle f(u)g(u)\,du. \tag{3.12}$$

The "converse" part of the theorem then follows immediately from Proposition 3.4 and Lemma 3.3.

Assume that (3.11) holds. Let $\hat{W}$ be an $\mathscr{S}'$-Wiener process associated to Q, existence of which is guaranteed by Theorem 2.3 (that theorem of course remains valid for a finite time interval). It is clear, by continuity of A, that $-(A^*-\partial)\tilde{X}$ is a Gaussian random element of $(\mathscr{D}([0,T])\hat{\otimes}\mathscr{S}(R^d))'$ with mean 0, so (3.12) and Lemma 3.3 imply that $-(A^*-\partial)\tilde{X}$ and $\partial\hat{W}$ have the same finite-dimensional distributions. By (3.8) we have

$$\langle-(A^*-\partial)\tilde{X},\phi f\rangle=\int_0^T\langle Y_t,\phi\rangle f'(t)\,dt,$$

where Y, defined by (3.9), is a continuous centered Gaussian process. Since a function $h\in\mathscr{D}([0,T])$ has the form $h=f'$ for some $f\in\mathscr{D}([0,T])$ if and only if $\int_0^T h(t)\,dt=0$, we see that the families

$$\left\{\int_0^T\langle Y_t,\phi\rangle h(t)\,dt:\ \phi\in\mathscr{S}(R^d),\ h\in\mathscr{D}([0,T]),\ \int_0^T h(t)\,dt=0\right\}$$

and (by (3.3))

$$\left\{-\int_0^T\langle\hat{W}_t,\phi\rangle h(t)\,dt:\ \phi\in\mathscr{S}(R^d),\ h\in\mathscr{D}[0,T]),\ \int_0^T h(t)\,dt=0\right\}$$

are Gaussian with the same finite-dimensional distributions. For any fixed $s\in[0,T]$ let $f_n\in\mathscr{D}([0,T])$, $n=1,2,\ldots$, be such that f_n converges to δ_s in $\mathscr{D}'(R)$ as $n\to\infty$, and let $\bar{f}_n\in\mathscr{D}([0,T])$, $n=1,2,\ldots$ be such that $\bar{f}_n$ converges to δ_0 in $\mathscr{D}'(R)$ as $n\to\infty$. We assume additionally that $\int_0^T f_n(t)\,dt=\int_0^T\bar{f}_n(t)\,dt=1$ for all n. Then $h_n=f_n-\bar{f}_n$ is in $\mathscr{D}([0,T])$ and satisfies $\int_0^T h_n(t)\,dt=0$, and $\lim_{n\to\infty}\int_0^T\langle Y_t,\phi\rangle h_n(t)\,dt=\langle Y_s,\phi\rangle$ a.s. and in L^2, since $Y_0=0$. Analogously, $\lim_{n\to\infty}\int_0^T\langle\hat{W}_t,\phi\rangle h_n(t)\,dt=\langle\hat{W}_s,\phi\rangle$ for each $\phi\in\mathscr{S}(R^d)$. Hence we conclude that the Gaussian systems $\{\langle Y_s,\phi\rangle,\ s\in[0,T],\ \phi\in\mathscr{S}(R^d)\}$ and $\{-\langle\hat{W}_s,\phi\rangle,\ s\in[0,T],\ \phi\in\mathscr{S}(R^d)\}$ have the same distributions, and this means (see Definition 2.1) that Y has the form $Y = -W$ for some $\mathscr{S}'$-Wiener process associated to Q. The theorem is proved.

In the second part of this section we will give a sufficient condition for X to satisfy (3.1). The usefulness of this criterion will be illustrated by several examples in the next section.

Theorem 3.6. *Assume that*

(a) $X \equiv \{X_t, t \in [0,T]\}$ *is a continuous, centered Gaussian* $\mathscr{S}'(R^d)$*-valued process with covariance functional*

$$K(s,\phi;t,\psi) = \mathrm{Cov}(\langle X_s,\phi\rangle, \langle X_t,\psi\rangle), \qquad s,t \in [0,T],\ \phi,\psi \in \mathscr{S}(R^d);$$

(b) *for each* $\phi \in \mathscr{S}(R^d)$ *the function* $s \to K(s,\phi;s,\phi)$ *is continuously differentiable*;

(c) $\{T_t, t \in [0,T]\}$ *is a strongly continuous semigroup of continuous linear operators on* $\mathscr{S}(R^d)$ *such that its infinitesimal generator* A (*in the sense that*

$$T_t\phi - \phi = \int_0^t T_s A\phi\, ds,\ t \in [0,T],\ \phi \in \mathscr{S}(R^d))$$

is continuous from $\mathscr{S}(R^d)$ *into itself*;

(d) *for each* $0 \leqq s \leqq t \leqq T$, $\phi,\psi \in \mathscr{S}(R^d)$, K *satisfies*

$$K(s,\phi;t,\psi) = K(s,\phi;s,T_{t-s}\psi). \tag{3.13}$$

Then X *is a Markov process and there exists an* $\mathscr{S}'(R^d)$*-valued Wiener process* W *such that* X is a solution of (3.1). W *is associated to the family* $Q \equiv \{Q_u, u \in [0,T]\}$ *defined by*

$$\langle Q_u\phi,\psi\rangle = \frac{d}{du}K(u,\phi;u,\psi) - K(u,A\phi;u,\psi) - K(u,\phi;u,A\psi), \quad \phi,\psi \in \mathscr{S}(R^d) \tag{3.14}$$

(fluctuation-dissipation relation).

Proof. The Markov property for a Gaussian $\mathscr{S}'$-process is implied directly by (3.13) (see e.g. [23]). To prove the other assertions we again define Y by formula (3.9). It will clearly suffice to show that Y is an $\mathscr{S}'$-Wiener process associated to the Q given by (3.14). Y is continuous, centered Gaussian; let us compute its covariance functional. For $0 \leqq s \leqq t \leqq T$, $\phi,\psi \in \mathscr{S}(R^d)$, we have

$$\begin{aligned}
E(\langle Y_s,\phi\rangle\langle Y_t,\psi\rangle) = {} & K(0,\phi;0,\psi) + \int_0^t K(0,\phi;u,A\psi)\,du \\
& + \int_0^s K(0,\psi;u,A\phi)\,du - K(0,\phi;t,\psi) - K(0,\psi;s,\phi) \\
& + \int_0^s\int_0^t K(u,A\phi;r,A\psi)\,dr\,du - \int_0^s K(u,A\phi;t,\psi)\,du \\
& - \int_0^t K(u,A\psi;s,\phi)\,du + K(s,\phi;t,\psi).
\end{aligned}$$

The double integral term is transformed as follows:

$$\begin{aligned}
&\int_0^s \int_0^t K(u, A\phi; r, A\psi)\, dr\, du \\
&\quad = \int_0^s \int_u^t K(u, A\phi; r, A\psi)\, dr\, du + \int_0^s \int_r^s K(u, A\phi; r, A\psi)\, du\, dr \\
&\quad = \int_0^s K\left(u, A\phi; u, \int_u^t T_{r-u} A\psi\, dr\right) du + \int_0^s K\left(r, A\psi; r, \int_r^s T_{u-r} A\phi\, du\right) dr \\
&\quad = \int_0^s K(u, A\phi; u, T_{t-u}\psi)\, du - \int_0^s K(u, A\phi; u, \psi)\, du \\
&\quad + \int_0^s K(r, A\psi; r, T_{s-r}\phi)\, dr - \int_0^s K(r, A\psi; r, \phi)\, dr.
\end{aligned}$$

We have used successively (3.13), the fact that K is a continuous linear functional with respect to each of the $\mathscr{S}(R^d)$-variables, and assumption (c). Transforming similarly the remaining terms we obtain

$$\begin{aligned}
E(\langle Y_s, \phi\rangle \langle Y_t, \psi\rangle) &= K(s, \phi; s, \psi) - K(0, \phi; 0, \psi) \\
&\quad - \int_0^s (K(u, A\phi; u, \psi) + K(u, \phi; u, A\psi))\, du.
\end{aligned}$$

Assumption (b) and bilinearity of K imply that the function $s \to K(s, \phi; s, \psi)$ is continuously differentiable for every $\phi, \psi \in \mathscr{S}(R^d)$; hence for each $s, t \in [0, T]$, $\phi, \psi \in \mathscr{S}(R^d)$ we have

$$\begin{aligned}
\operatorname{Cov}(\langle Y_s, \phi\rangle, \langle Y_t, \psi\rangle) = \int_0^{s \wedge t} \Big(\frac{d}{du} K(u, \phi; u, \psi) &- K(u, A\phi; u, \psi) \\
&- K(u, \phi; u, A\psi)\Big)\, du. \qquad (3.15)
\end{aligned}$$

To complete the proof we must show that (3.14) defines a continuous operator Q_u from $\mathscr{S}(R^d)$ into $\mathscr{S}'(R^d)$ which is symmetric and positive. Observe that by the Banach-Steinhaus theorem, $dK(u, \phi; u, \psi)/du$ is linear and continuous in ϕ, ψ, as the pointwise limit of the continuous bilinear forms $h^{-1}(K(u+h, \phi; u+h, \psi) - K(u, \phi; u, \psi))$ as $h \to 0$. Therefore the whole expression at the right-hand side of (3.14) is a continuous bilinear form on $\mathscr{S}(R^d) \times \mathscr{S}(R^d)$, hence it must be equal to $\langle Q_u \phi, \psi\rangle$ for some continuous linear Q_u: $\mathscr{S}(R^d) \to \mathscr{S}'(R^d)$. The symmetry of Q_u is obvious, and Lemma 2.2 together with (3.15) imply that Q_u is positive.

Remarks. (a) If the assumption of differentiability of K is dropped, then we still have (3.2), where W is a continuous $\mathscr{S}'(R^d)$-valued Gaussian process with covariance functional

$$K(s \wedge t, \phi; s \wedge t, \psi) - K(0, \phi; 0, \psi) - \int_0^{s \wedge t} (K(u, A\phi; u, \psi) + K(u, \phi; u, A\psi))\, du;$$

therefore it has independent increments (see Remark (a) after Definition 2.1).

(b) From the construction in Theorems 3.5 and 3.6 it follows that W is adapted to the filtration generated by X. Hence the solutions of (3.1) obtained by these theorems are "weak" (see e.g. [30]). There exist some results concerning strong solutions of (3.1) in special cases ([1], [28]); they are expressed in the form $X'_t = T_t X_0 + \int_0^t T_{t-u}\, dW_u$. We will treat elsewhere the problem of existence and uniqueness of strong solutions of equations more general than (3.1) (with A depending on t). See [5] for the case of Hilbert space, where it holds that $X = X'$.

(c) We do not know what can be said on necessity of the assumptions of Theorem 3.6.

(d) Wittig [29] considers Langevin equations for continuous centered Gaussian $\mathscr{S}'(R^d)$-valued processes X having covariance functionals of the form

$$K(s, \phi; t, \psi) = \int_0^s Q(T_{s-r}\phi, T_{t-r}\psi)\, dr, \qquad s \leqq t,$$

where Q is a continuous covariance and $\{T_t\}$ is a contraction semigroup with generator A. It is shown that X satisfies the Langevin equation $dX_t = A^* X_t\, dt + dW_t$, $X_0 = 0$, where W is a time-homogeneous $\mathscr{S}'$-Wiener process associated to (in our notation) $\langle Q_s \phi, \psi \rangle = Q(\phi, \psi)$ for all s. This is a special case of Theorem 3.6 in the sense that we allow time-inhomogeneous $\mathscr{S}'$-Wiener processes and we do not require T_t to be a contraction (e.g. in example 3 below with $\alpha > 0$ it is not a contraction). On the other hand, the main emphasis in [29] is on the Sobolev subspaces where the solutions live. The methods are different from ours.

(e) Results of related interest have been obtained recently by Chari [4].

4. Fluctuation Limits of Infinite Particle Systems

The procedure we use for the asymptotic analysis of most of the particle systems considered here is the following. We have a sequence $(X_n)_{n=1,2,\ldots}$ (or a continuous sequence $(X_K)_{K \geqq 1}$) of $\mathscr{S}'(R^d)$-valued processes, continuous or right-continuous with left limits. Tightness of the sequence is determined combining results of [17, 18] and [25]. Weak convergence of X_n to a continuous Gaussian $\mathscr{S}'(R^d)$-valued process X is obtained from convergence of finite-dimensional distributions [25], or from convergence of $\tilde{X}_n$ to $\tilde{X}$ in $\mathscr{S}'(R^{d+1})$ (or other nuclear spaces of distributions) [2]; the latter is somewhat simpler because it involves only weak convergence of random variables, and since the spaces are nuclear, a Lévy-type continuity theorem can be used [3], [24] (see [2]). (Convergence can also be proved by martingale methods [18] when the processes are Markovian.) If the covariance of $\tilde{X}$ satisfies condition (3.11), then we can write down the Langevin equation for X according to Theorem 3.5. The covariance of X can be obtained directly from that of $\tilde{X}$, and if A generates a semigroup $\{T_t\}$ and the covariance of X satisfies condition (3.13), then X is Markovian and we have its Langevin equation given by Theorem 3.6. The following examples fall into the latter setup. The simplest one is Itô's model [20]; convergence using the $\tilde{X}$ approach needs only a direct application of the central limit theorem (see [2]). The examples involving branching and immigration are all based upon results described later on in this section. Since the tightness and convergence proofs are analogous to those in [13, 14], here we generally restrict ourselves to the matter of interest, namely the Langevin equations. Other aspects of this study not included here are laws of large numbers for the systems (see the survey [15]) and Hilbert subspaces of $\mathscr{S}'(R^d)$ supporting the values of the fluctuation limit processes. A different type of fluctuation limits of supercritical infinite particle branching motions is studied in [8], [16]. The last examples are related to results of Presutti and Spohn [26] on the voter model.

For our examples we have chosen Brownian particle motion for simplicity, but other Markov particle motions can be taken (see e.g. [7, 23]. In all the following $\{T_t\}$ stands for the Brownian semigroup. We stress that the Langevin equations are understood in the "mild" sense (Definition 3.1); furthermore, in the Langevin equations in all the examples the $\mathscr{S}'$-Wiener process and the initial condition are independent.

(1) Let $B_k \equiv \{B_k(t), t \in R_+\}$, $k=1,2,\dots$ be independent d-dimensional Brownian motions with the same initial distribution μ. Let

$$X_n(t, A) = n^{-\frac{1}{2}}(N_n(t, A) - EN_n(t, A)),$$

where $N_n(t, A)$ is the number of $k \leqq n$ such that $B_k(t) \in A$ and A is a Borel set of R^d. Then $X_n \equiv \{X_n(t), t \in R_+\}$ is a continuous $\mathscr{S}'(R^d)$-valued process, and X_n converges weakly as $n \to \infty$ to a continuous, centered Gaussian $\mathscr{S}'(R^d)$-valued process X with covariance functional

$$K(s, \phi; t, \psi) = \operatorname{Cov}(\phi(B_1(s)), \psi(B_1(t))).$$

(The usual central limit theorem gives convergence of $\tilde{X}_n$ to $\tilde{X}$ [2].) This covariance can be written

$$K(s, \phi; t, \psi) = \int_{R^d} \phi(x) T_{t-s} \psi(x) \mu_s(x)\, dx - \int_{R^d} \phi(x) \mu_s(s)\, dx \int_{R^d} T_{t-s} \psi(x) \mu_s(x)\, dx, \quad s \leqq t,$$

where $\mu_s = T_s \mu$. Therefore K satisfies condition (3.13) for the semigroup $\{T_t\}$. Hence X is a Markov process, and application of Theorem 3.6 yields the Langevin equation

$$dX_t = \tfrac{1}{2} \Delta X_t\, dt + dW_t,$$

where the initial value X_0 is Gaussian with characteristic functional

$$E e^{i\langle X_0, \phi\rangle} = e^{-\frac{1}{2} \operatorname{Var} \phi(Z)},$$

Z having distribution μ, and the $\mathscr{S}'$-Wiener process W is determined by

$$\langle Q_s \phi, \psi \rangle = \int_{R^d} \mu_s(x) \nabla\phi(x) \cdot \nabla\psi(x)\, dx,$$

where $\cdot$ stands for the usual scalar product in R^d. This result was obtained by Itô [20], interpreting the noise term in the Langevin equation in a different way.

Examples (2) to (5) can be treated using the same approach, which is explained after example (5).

(2) Consider an infinite system of independent d-dimensional Brownian motions whose initial positions (particles) are of two types: (a) a collection of particles at time 0 distributed in R^d according to a homogeneous Poisson random field with intensity γK ($\gamma \geqq 0$), and (b) (immigration) a collection of particles appearing in $R^d \times [0, \infty)$ according to a homogeneous space-time Poisson random field with intensity βK ($\beta \geqq 0$). Let

$$X_K(t, A) = K^{-\frac{1}{2}}(N_K(t, A) - EN_K(t, A)),$$

where $N_K(t, A)$ is the number of particles in the Borel set A of R^d at time t. Then X_K is an $\mathscr{S}'(R^d)$-valued right-continuous with left limits process, and X_K converges weakly as $K \to \infty$ to a continuous centered Gaussian $\mathscr{S}'(R^d)$-valued process X with covariance functional

$$K(s, \phi; t, \psi) = (\gamma + \beta s) \int_{R^d} \phi(x) T_{t-s} \psi(x) \, dx, \qquad s \leqq t.$$

Hence condition (3.13) is satisfied for the semigroup $\{T_t\}$ and therefore X is a Markov process obeying the Langevin equation

$$dX_t = \tfrac{1}{2} \Delta X_t \, dt + dW_t,$$

with $X_0 = \gamma^{\frac{1}{2}} G$, where G is a spatial standard Gaussian white noise, and the $\mathscr{S}'$-Wiener process W is specified by

$$\langle Q_s \phi, \psi \rangle = \beta \int_{R^d} \phi(x) \psi(x) \, dx + (\gamma + \beta) \int_{R^d} \nabla \phi(x) \cdot \nabla \psi(x) \, dx.$$

Setting $\beta = 0$ we obtain the result of Martin-Löf [23].

(3) Let us add branching to the model in the previous example, i.e. assume that each particle independently branches at an exponentially distributed time (after its birth) with parameter V, the branching law $\{p_n\}_{n=0,1,\dots}$ having mean m_1 and finite second factorial moment m_2, and the new particles appearing at the location where their parent branched. $\alpha = V(m_1 - 1)$ is the Malthusian parameter of the underlying branching process. Let X_K be defined as in the previous example. Then X_K converges weakly as $K \to \infty$ in the Skorohod space $D([0, \infty), \mathscr{S}'(R^d))$ to a continuous centered Gaussian $\mathscr{S}'(R^d)$-valued process X with covariance functional

$$\begin{aligned} K(s, \phi; t, \psi) = \gamma &\left[e^{\alpha s} \int_{R^d} \phi(x) T^\alpha_{t-s} \psi(x) \, dx + m_2 V \int_{R^d} \phi(x) \int_0^s e^{\alpha r} T^\alpha_{t+s-2r} \psi(x) \, dr \, dx \right] \\ &+ \beta \left[(e^{\alpha s} - 1)/\alpha \int_{R^d} \phi(x) T^\alpha_{t-s} \psi(x) \, dx \right. \\ &\left. + m_2 V \int_{R^d} \phi(x) \int_0^s (e^{\alpha r} - 1)/\alpha T^\alpha_{t+s-2r} \psi(x) \, dr \, dx \right], \qquad s \leqq t, \end{aligned}$$

where $\{T_t^\alpha\}$ is the semigroup $T_t^\alpha = e^{\alpha t} T_t$. Hence condition (3.13) is satisfied for this semigroup, and the Langevin equation for the Markov process X is

$$dX_t = (\tfrac{1}{2}\Delta + \alpha) X_t \, dt + dW_t,$$

with $X_0 = \gamma^{\frac{1}{2}} G$ (as in the previous example), and the $\mathscr{S}'$-Wiener process W is given by

$$\begin{aligned}\langle Q_s \phi, \psi \rangle = {} & \gamma e^{\alpha s} [(m_2 V - \alpha) \int_{R^d} \phi(x) \psi(x) \, dx + \int_{R^d} \nabla\phi(x) \cdot \nabla\psi(x) \, dx] \\ & + \beta \{ [m_2 V(e^{\alpha s} - 1)/\alpha + 2 - e^{\alpha s}] \int_{R^d} \phi(x) \psi(x) \, dx \\ & + (e^{\alpha s} - 1)/\alpha \int_{R^d} \nabla\phi(x) \cdot \nabla\psi(x) \, dx \}.\end{aligned}$$

Setting $\beta = 0$ we have the result of [13].

In this example it is easy to see from the covariance functional of X_t that for $\alpha = 0$ (the critical case) and $\beta > 0$, X_t does not have a limit as $t \to \infty$, and for $\alpha < 0$ (the subcritical case) and $\beta > 0$, X_t converges weakly as $t \to \infty$ to the centered Gaussian random element X_∞ of $\mathscr{S}'(R^d)$ with covariance functional

$$\begin{aligned}\operatorname{Cov}(\langle X_\infty, \phi \rangle, \langle X_\infty, \psi \rangle) = {} & -(\beta/\alpha) \{ \int_{R^d} \phi(x) \psi(x) \, dx \\ & + m_2 V \int_{R^d} \int_{R^d} \phi(x) \psi(y) k(x, y) \, dx \, dy \},\end{aligned}$$

with

$$k(x, y) = \begin{cases} e^{-(-2\alpha)^{\frac{1}{2}} \|x - y\|} / 2(-2\alpha)^{\frac{1}{2}}, & d = 1, \\ (-2\alpha)^{d/4 - 1/2} K_{d/2 - 1}((-2\alpha)^{\frac{1}{2}} \|x - y\|) / (2\pi)^{d/2} \|x - y\|^{d/2 - 1}, & d \geqq 2, \end{cases}$$

where $K_{d/2-1}$ is the usual modified Bessel function [10]. $k(x, y)$ is one half of the potential kernel of Brownian motion killed at an independent exponentially distributed time with paramer $-\alpha$. Observe that X_∞ depends only on the immigration.

A different scaling that yields the same results of this example is the following. The Poisson intensities are still γK and βK; replace V by VK and assume that the branching law has mean $m_{K,1}$ and second and third factorial moments $m_{K,2}$ and $m_{K,3}$ such that $m_{K,1} = 1 + \mu/K$ for some $\mu \in R$, $m_{K,2} = \upsilon/K$ with $\upsilon > \max\{\mu, 0\}$, and $\sup_{K \geqq 1} m_{K,3} < \infty$. Then N^K has the same asymptotic behavior above, with α replaced by μV and m_2 by υ in all the expressions. This scaling is related to Dawson's diffusion approximation [7] (see [15]).

(4) For the model considered in the previous example we will give now a result for the critical case ($\alpha = 0$) and dimension $d \geqq 3$, assuming that the branching law has finite third moment. Let the intensity of the initial Poisson field be γ and the intensity of the immigration Poisson field be β/K^2. Introduce the space-time scaling $(x, t) \to (Kx, K^2 t)$, i.e. $\langle N_K(t), \phi \rangle = \langle N_K(K^2 t), \phi(\cdot/K) \rangle$ where N_K is as before. Let

$$X_K(t, A) = K^{-d/2 - 1} (N_K(t, A) - E N_K(t, A)).$$

Then X_K converges weakly as $K \to \infty$ in $D([0,\infty), \mathscr{S}'(R^d))$ to the continuous, centered Gaussian $\mathscr{S}'(R^d)$-valued process X with covariance functional

$$K(s,\phi;t,\psi)=m_2 V\left\{\gamma \int_{R^d} \phi(x) \int_0^s T_{t+s-2r}\psi(x)\,dr\,dx + \beta \int_{R^d} \phi(x) \int_0^s r T_{t+s-2r}\psi(x)\,dr\,dx\right\}, \quad s \leqq t.$$

Condition (3.13) is satisfied for the semigroup $\{T_t\}$, and X is a Markov process such that

$$dX_t = \tfrac{1}{2}\Delta X_t\,dt + dW_t,$$

with $X_0=0$, the $\mathscr{S}'$-Wiener process W being given by

$$\langle Q_s \phi, \psi \rangle = m_2 V(\gamma+\beta s) \int_{R^d} \phi(x)\psi(x)\,dx.$$

Letting $\beta=0$ yields the results of Dawson [6, 7] and Holley and Stroock [18]. (In the case $\alpha \neq 0$, the space-time scaling with central limit theorem normalization does not give results of the type we are considering).

(5) In the previous example the particle motion (Brownian motion) is preserved by the space-time scaling. If we only introduce the space scaling $x \to Kx$, with Poisson intensities γ and β, then the motion is annihilated in the limit $K \to \infty$ while the branching is preserved. Let

$$X_K(t,A) = K^{-d/2}(N_K(t,A) - EN_K(t,A)),$$

with N_K as before. Then X_K converges weakly in $D([0,\infty), \mathscr{S}'(R^d))$ to the continuous, centered Gaussian $\mathscr{S}'(R^d)$-valued process X with covariance functional

$$K(s,\phi;t,\psi) = q(s,t) \int_{R^d} \phi(x)\psi(x)\,dx,$$

where

$$q(s,t)=\begin{cases} e^{\alpha t}\{\gamma[1+m_2 V(e^{\alpha s}-1)/\alpha] + \beta[(1-e^{-\alpha s})/\alpha + m_2 V(e^{\alpha s}+e^{-\alpha s}-2)/2\alpha^2]\} & \text{if } \alpha \neq 0 \\ \gamma(1+m_2 Vs) + \beta(s+m_2 Vs^2/2) & \text{if } \alpha = 0, \quad s \leqq t. \end{cases}$$

Convergence of $X_K(t)$ to $X(t)$ for fixed t was established by Dawson and Ivanoff [9, 21, 22]. Condition (3.13) is satisfied for the (multiplicative) semigroup $\{e^{\alpha t}\}$, and X is a Markov process with Langevin equation

$$dX_t = \alpha X_t\,dt + dW_t,$$

with $X_0 = \gamma^{\frac{1}{2}} G$ (as in examples (2) and (3)), and the $\mathscr{S}'$-Wiener process W is defined by

$$\langle Q_s \phi, \psi \rangle = \{\gamma e^{\alpha s}(m_2 V - \alpha) + \beta[2 - e^{\alpha s} + m_2 V(e^{\alpha s}-1)/\alpha]\} \int_{R^d} \phi(x)\psi(x)\,dx.$$

In this scaling it turns out that the results are exactly the same as for the model where the particles do not migrate, hence the migrations have no effect in the limit. This example was studied in [14] before we had the present method.

The general setup underlying examples (3) to (5) is the following. We have an infinite system of branching Brownian motions on R^d generated by initial particles appearing according to a homogeneous Poisson random field with intensity γ and immigrant particles appearing according to a homogeneous space-time Poisson random field with intensity β. Suppose the particle lifetime distribution is exponential with parameter V, and the branching law has mean m_1 and second factorial moment $m_2<\infty$; let $\alpha=V(m_1-1)$. Denote $N_t(A)$ the number of particles in the Borel set A of R^d at time t and denote $X_t=N_t-EN_t$; it can be shown that $\{X_t, t\geqq 0\}$ can be realized in $D([0,\infty), \mathscr{S}'(R^d))$. It can be proved using the method of [9] that the joint characteristic functional of $N_{t_1},\ldots,N_{t_n}$, $t_1<\ldots<t_m$, is given by

$$E\exp\left\{i\sum_{j=1}^m u_j\langle N_{t_j},\phi_j\rangle\right\}=\exp\left\{\gamma\int_{R^d}E\left[\exp\left(i\sum_{j=1}^m u_j\langle N_{t_j}^x,\phi_j\rangle\right)-1\right]dx\right.$$
$$\left.+\beta\int_0^{t_m}\int_{R^d}E\left[\exp\left(i\sum_{j=1}^m u_j\langle N_{t_j-s}^x,\phi_j\rangle\right)-1\right]dx\,ds\right\},$$
$$u_1,\ldots,u_m\in R,\ \phi_1,\ldots,\phi_m\in\mathscr{S}(R^d), \tag{4.1}$$

where $\{N_t^x, t\geqq 0\}$ is the system generated by a single particle located initially at $x\in R^d$ ($N_t^x=0$ for $t<0$).

From (4.1) and using the Lemma in [13] one obtains

$$E\langle N_t,\phi\rangle=\gamma\int_{R^d}E\langle N_t^x,\phi\rangle\,dx+\beta\int_0^t\int_{R^d}E\langle N_{t-s}^x,\phi\rangle\,dx\,ds$$
$$=(\gamma e^{\alpha t}+\beta(e^{\alpha t}-1)/\alpha)\int_{R^d}\phi(x)\,dx,\qquad t\geqq 0, \tag{4.2}$$

and

$$\operatorname{Cov}(\langle N_s,\phi\rangle,\langle N_t,\psi\rangle)$$
$$=\gamma\int_{R^d}E\langle N_s^x,\phi\rangle\langle N_t^x,\psi\rangle\,dx+\beta\int_0^s\int_{R^d}E\langle N_{s-r}^x,\phi\rangle\langle N_{t-r}^x,\psi\rangle\,dx\,dr$$
$$=e^{\alpha t}(\gamma+\beta(1-e^{-\alpha s})/\alpha)\int_{R^d}\phi(x)\,T_{t-s}\psi(x)\,dx$$
$$+e^{\alpha t}\gamma m_2 V\int_0^s e^{\alpha(s-r)}\int_{R^d}\phi(x)\,T_{t+s-2r}\psi(x)\,dx\,dr$$
$$+e^{\alpha t}\beta m_2 V\int_0^s e^{\alpha(s-r)}(1-e^{-\alpha r})/\alpha\int_{R^d}\phi(x)\,T_{t+s-2r}\psi(x)\,dx\,dr,\qquad s\leqq t. \tag{4.3}$$

The joint characteristic function of $\langle X_{t_1},\phi_1\rangle,\ldots,\langle X_{t_m},\phi_m\rangle$ can be written from (4.1) and (4.2). Then the calculations for examples (3) to (5) reduce essentially to bringing the corresponding scalings and normalizations into the characteristic function for X_K and into expressions (4.2) and (4.3), and applying the Lévy continuity theorem in the usual way. In some scalings, in the application of the continuity theorem the convergence of the error term to 0 requires the branching law $\{p_n\}$ to have a finite third moment (see [14]). The condition $d \geqq 3$ in example (4) is also used in proving convergence of the error term to 0. In all cases the covariance functional of the limit fluctuation process is just the expression obtained by taking the limit $K \to \infty$ in (4.3) for the corresponding scaling and normalization. A similar procedure can be carried out in terms of $\tilde{X}^K$ (e.g. [13]).

The limit in example (2) is also obtained in an analogous way by the continuity theorem and covariance calculations.

We end with two examples of a different type from those above, the voter model and the simple exclusion process on a d-dimensional discrete lattice. Presutti and Spohn [26] have obtained Gaussian fluctuation limits for these models under hydrodynamic scalings (we refer to [26] for details). However they do not mention the Markov property and the Langevin equations. We will show them now. In the following we let $\{T_t\}$ stand for the semigroup generated by Δ in order to conform with [26].

(6) For the voter model in dimension $d \geqq 3$ with symmetric nearest neighbor interactions, the fluctuation process of the magnetization field under an appropriate choice of initial measures, space-time scaling and normalization converges weakly to an $\mathscr{S}'(R^d)$-valued continuous, centered, Gaussian process X with covariance functional

$$K(s, \phi; t, \psi) = a \int_0^s \int_{R^d} [1 - (T_r m(x))^2]\, T_{s-r}\phi(x)\, T_{t-r}\psi(x)\, dx\, dr, \qquad s \leqq t,$$

where a is a positive constant and m is a continuous function from R^d into $[-1, 1]$. Condition (3.13) is satisfied for $\{T_t\}$, and therefore X is a Markov process with Langevin equation $dX_t = \Delta X_t\, dt + dW_t$, $X_0 = 0$, W being associated to

$$\langle Q_s \phi, \psi \rangle = a \int_{R^d} [1 - (T_s m(x))^2]\, \phi(x)\, \psi(x)\, dx$$

(W is space-time white noise when $m \equiv 0$). In the stationary case the Gaussian fluctuation limit X has covariance

$$K(s, \phi; t, \psi) = b \int_{R^d} (-\Delta)^{-1} \phi(x)\, T_{t-s}\psi(x)\, dx, \qquad s \leqq t,$$

where b is a positive constant. Again (3.13) holds for $\{T_t\}$, and X is a Markov process satisfying $dX_t = \Delta X_t\, dt + dW_t$, where W corresponds to

$$\langle Q_s \phi, \psi \rangle = 2b \int_{R^d} \phi(x)\, \psi(x)\, dx.$$

(7) In the simple exclusion case in any dimension the Gaussian fluctuation limit X has covariance

$$K(s, \phi; t, \psi) = K(0, T_s\phi; 0, T_t\psi) + \int_0^s \int_{R^d} [1 - (T_r m(x))^2]\, \nabla T_{s-r}\phi(x) \cdot \nabla T_{t-r}\psi(x)\, dx\, dr, \qquad s \leqq t,$$

where m is as above. Condition (3.13) holds for $\{T_t\}$, and X is a Markov process satisfying $dX_t = \Delta X_t\,dt + dW_t$, with W associated to

$$\langle Q_s\phi, \psi\rangle = \int_{R^d} [1-(T_s m(x))^2]\,\nabla\phi(x)\cdot\nabla\psi(x)\,dx.$$

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