Probab. Th. Rel. Fields 73, 211–226 (1986)

Probability Theory and Related Fields

Mesure du Voisinage and Occupation Density

M. Csörgő [1]* and P. Révész [2]**

[1] Carleton University, Department of Mathematic and Statistics, Ottawa K1S 5B6
[2] Hungarian Academy of Sciences, Mathematical Institute, Reáltanoda u. 13–15, 1053 Budapest, Hungary

Summary. Let $W(t)$ be a standard Wiener process with occupation density (local time) $\eta(x,t)$. Paul Lévy showed that for each x, $\eta(x,t)$ is a.s. equal to the "mesure du voisinage" of W, i.e., to the limit as h approaches zero of $h^{\frac{1}{2}}$ times $N(h,x,t)$, the number of excursions from x, exceeding h in length, that are completed by W up to time t. Recently, Edwin Perkins showed that the exceptional null sets, which may depend on x, can be combined into a single null set off which the above convergence is uniform in x. The main aim of the present paper is to estimate the rate of convergence in Perkins' theorem as h goes to zero. We also investigate the connection between N and η in the case when we observe a Wiener process through a long time t and consider the number of long (but much shorter than t) excursions.

1. Introduction and Statement of Results

Let $\{W(t), t \geqq 0\}$ be a Wiener process. For any Borel set A of the real line let

$$H(A,t)=\lambda\{s\colon s\leqq t,\ W(s)\in A\}$$

be the occupation time of W where λ is the Lebesgue measure. It is well-known that $H(A,t)$ is a random measure, absolutely continuous with respect to λ. The Radon-Nikodym derivative of H is called the occupation density (local time) of W and it will be denoted by η, i.e., $\eta(x,t)$ is defined by

$$H(A,t)=\int_A \eta(x,t)\,dx.$$

The concept of mesure du voisinage of W is strongly related to the concept of its occupation density. It can be defined as follows. Let $N(h,x,t)$ be the number of excursions of W away from x that are greater than h in length and

* Research partially supported by a NSERC Canada grant

** *Present address:* Institut für Statistik und Wahrscheinlichkeitstheorie, Technische Universität Wien, Wiedner Hauptstraße 8–10, A-1040 Wien, Austria

are completed by time t. Then the "mesure du voisinage" of W at time t is $\lim_{h\searrow 0} h^{\frac{1}{2}} N(h, x, t)$, and the connection between η and N is given by the following result of P. Lévy (cf. Itô and McKean 1965, p. 43).

Theorem A. *For all real x and for all positive t we have*

$$\lim_{h\searrow 0} h^{\frac{1}{2}} N(h, x, t) = \sqrt{\frac{2}{\pi}}\, \eta(x, t) \qquad a.s.$$

Recently, Perkins (1981) proved that Theorem A holds uniformly in x and t. In fact his result says

Theorem B. *For any fixed $t' > 0$ we have*

$$\lim_{h\searrow 0} \sup_{(x,t)\in R\times[0,t']} \left| h^{\frac{1}{2}} N(h, x, t) - \sqrt{\frac{2}{\pi}}\, \eta(x, t)\right| = 0 \qquad a.s.$$

where $R = (-\infty, \infty)$.

The main aim of this paper is to estimate the rate of convergence of Theorem B. Our fundamental result is

Theorem 1. *For any fixed $t' > 0$ we have*

$$\lim_{h\searrow 0} h^{-\frac{1}{4}} (\log h^{-1})^{-1} \sup_{(x,t)\in R\times[0,t']} \left| h^{\frac{1}{2}} N(h, x, t) - \sqrt{\frac{2}{\pi}}\, \eta(x, t)\right| = 0 \qquad a.s.$$

We also investigate the connection between N and η in the case when we observe a Wiener process through a long time t and consider the number of long (but much shorter than t) excursions. We obtain

Theorem 2. *For some $0 < \alpha < 1$ let $0 < a_t < t^\alpha$ $(t > 0)$ be a non-decreasing function of t so that a_t/t is non-increasing. Then*

$$\lim_{t\to\infty} \left(\frac{a_t}{t}\right)^{\frac{1}{4}} \left(\log \frac{t}{a_t}\right)^{-1} \sup_{x\in R} \left| N(a_t, x, t) - \sqrt{\frac{2}{\pi a_t}}\, \eta(x, t)\right| = 0 \qquad a.s.$$

The proofs of Theorems 1 and 2 are based on two large deviation type inequalities (respectively) which are of interest on their own.

Theorem 3. *For any $K > 0$ and $t' > 0$ there exist a $C = C(K, t') > 0$ and a $D = D(K, t') > 0$ such that*

$$P\left\{ h^{-\frac{1}{4}} (\log h^{-1})^{-\frac{3}{4}} \sup_{(x,t)\in R\times[h,t']} \left| h^{\frac{1}{2}} N(h, x, t) - \sqrt{\frac{2}{\pi}}\, \eta(x, t)\right| \geqq C \right\} \leqq D h^K,$$

where $h < t'$.

Theorem 4. *For any $K > 0$ there exist a $C = C(K) > 0$ and a $D = D(K) > 0$ such that*

$$P\left\{\left(\frac{a_t}{t}\right)^{\frac{1}{4}}\left(\log\frac{t}{a_t}\right)^{-\frac{3}{4}}\sup_{x\in R}\left|N(a_t,x,t)-\sqrt{\frac{2}{\pi a_t}}\,\eta(x,t)\right|\geqq C\right\}\leqq D\left(\frac{a_t}{t}\right)^K,$$

where $0<a_t<t$.

The authors are indebted to E. Perkins for pointing out a serious mistake in the original manuscript.

2. Proof of Theorem 4 in the Case of $a_t=t^{1/4}$

Introduce the following notations:

$$\begin{aligned}
\tau_0&=\tau_0(x)=\inf\{t\colon t\geqq 0,\ W(t)=x\},\\
\tau_1&=\tau_1(x)=\inf\{t\colon t>\tau_0,\ |W(t)-x|=1\},\\
\tau_2&=\tau_2(x)=\inf\{t\colon t>\tau_1,\ W(t)=x\},\\
&\dots\dots\dots\dots\dots\dots\\
\tau_{2i+1}&=\tau_{2i+1}(x)=\inf\{t\colon t>\tau_{2i},\ |W(t)-x|=1\},\\
\tau_{2i+2}&=\tau_{2i+2}(x)=\inf\{t\colon t>\tau_{2i+1},\ W(t)=x\},\\
&\dots\dots\dots\dots\dots\dots
\end{aligned}$$

$$\psi_{2i}=\sup\{t\colon \tau_{2i}<t<\tau_{2i+1},\ W(t)=x\}\qquad (i=0,1,2,\dots),$$

$$\tau_2-\tau_1=\beta_1(x)=\beta_1,\ \tau_4-\tau_3=\beta_2(x)=\beta_2,\ \dots,\ \tau_{2i}-\tau_{2i-1}=\beta_i(x)=\beta_i,\ \dots,$$
$$\tau_1-\tau_0=\alpha_1(x)=\alpha_1,\ \tau_3-\tau_2=\alpha_2(x)=\alpha_2,\ \dots,\ \tau_{2i+1}-\tau_{2i}=\alpha_{i+1}(x)=\alpha_{i+1},\ \dots$$

$$M(a,x,n)=\mathcal{N}\{i\colon i\leqq n,\ \beta_i>a\},$$

where $\mathcal{N}\{\dots\}$ is the cardinality of the set in brackets. Finally let $n(t)=n(x,t)$ be the largest integer for which $\tau_{2n(t)}\leqq t$.

The following lemma is well-known (cf. Knight 1981, Lemma 2.11).

Lemma 1. *For any fixed* x $\{\beta_i\}$ *is a sequence of i.i.d. rv's with*

$$P(\beta_i>a)=\frac{1}{\sqrt{2\pi}}\int_a^\infty v^{-\frac{3}{2}}e^{-\frac{1}{2v}}\,dv=P(a)=\sqrt{\frac{2}{\pi a}}\left(1+O\left(\frac{1}{a}\right)\right).$$

Lemma 2. *For any positive integer* n, $a\geqq 1$ *and* $C>0$ *we have*

$$P\left\{\left|\frac{M(a,x,n)-nP(a)}{\sqrt{nP(a)(1-P(a))\log(nP(a))}}\right|\geqq\sqrt{C}\right\}\leqq 2\left(\frac{1}{P(a)\cdot n}\right)^{\frac{2C}{9}}$$

provided that

$$C\log(nP(a))<nP(a)(1-P(a)).$$

Proof. A simple consequence of the Bernstein-inequality (cf. e.g., Rényi 1970, p. 387).

Lemma 3. *There exists a universal constant $L>0$ such that*

$$P\left\{\left|\frac{\eta(x,\tau_{2n})-n}{\sqrt{n}}\right|\geqq\sqrt{C\log nP(a)}\right\}\leqq L\left(\frac{1}{P(a)n}\right)^{C/2}$$

holds for any positive integer n, $C>0$ and $1\leqq a\leqq n^{\rho}$ $(\rho<2)$.

Proof. It follows from the well-known fact that

$$\eta(x,\tau_1)-\eta(x,\tau_0),\ \eta(x,\tau_3)-\eta(x,\tau_2),\ \ldots$$

are independent, exponentially distributed rv's with parameter 1.

Lemmas 2 and 3 together imply

Lemma 4. *For any $K>0$ there exist a $C=C(K)>0$ and a $D=D(K)>0$ such that*

$$P\left\{\left|\frac{M(a,x,n)-\eta(x,\tau_{2n})P(a)}{\sqrt{nP(a)(1-P(a))\log(nP(a))}}\right|\geqq C\right\}\leqq D\left(\frac{1}{P(a)n}\right)^{K}$$

holds for any integer n and $1\leqq a\leqq n^{\rho}$ $(\rho<2)$.

A simple consequence of this lemma is

Lemma 5. *For any $K>0$ there exist a $C=C(K)>0$ and a $D=D(K)>0$ such that*

$$P\left\{\left|\frac{M(a,x,n)-\eta(x,\tau_{2n})\sqrt{\frac{2}{\pi a}}}{n^{\frac{1}{2}}\left(\frac{2}{\pi a}\right)^{\frac{1}{4}}(\log n/\sqrt{a})^{\frac{1}{2}}}\right|\geqq C\right\}\leqq D\left(\frac{\sqrt{a}}{n}\right)^{K}$$

holds for any integer n and $n^{\psi}\leqq a\leqq n^{\rho}$ $(2/5<\psi<\rho<2)$.

This lemma in turn implies

Lemma 6. *For any $K>0$ and $2/5<\psi<\rho<2$ there exist a $C=C(\psi,\rho,K)>0$ and a $D=D(\psi,\rho,K)>0$ such that*

$$P\left\{\sup_{n^{\psi}<j<n^{\rho}}\left|\frac{M(j,x,n)-\eta(x,\tau_{2n})\sqrt{\frac{2}{\pi j}}}{n^{\frac{1}{2}}\left(\frac{2}{\pi j}\right)^{\frac{1}{4}}(\log n/\sqrt{j})^{\frac{1}{2}}}\right|>C\right\}\leqq D\left(\frac{n^{\psi/2}}{n}\right)^{K}$$

holds for any integer n, where j runs over the integers.

Using the same method as above an applying the trivial formula

$$P\{j<\beta_1<j+1\}=\frac{1}{\sqrt{2\pi}}\int_j^{j+1}v^{-\frac{3}{2}}e^{-\frac{1}{2v}}dv=O(j^{-\frac{3}{2}}),$$

one gets

Lemma 7. *For any $K>0$ and $2/5<\psi<\rho<2$ there exist a $C=C(\psi,\rho,K)>0$ and a $D=D(\psi,\rho,K)>0$ such that*

$$P\left\{\sup_{n^\psi<j<n^\rho}\frac{M(j,x,n)-M(j+1,x,n)}{n^{\frac{1}{2}}\left(\frac{2}{\pi j}\right)^{\frac{1}{4}}(\log n/\sqrt{j})^{\frac{1}{2}}}>C\right\}\leqq D\left(\frac{n^{\psi/2}}{n}\right)^K$$

holds for any integer n, where j runs over the integers.

Lemmas 6 and 7 together imply

Lemma 8. *For any $K>0$ and $2/5<\psi<\rho<2$ there exist a $C=C(\psi,\rho,K)>0$ and $D=D(\psi,\rho,K)>0$ such that*

$$P\left\{\sup_{n^\psi<a<n^\rho}\frac{\left|M(a,x,n)-\eta(x,\tau_{2n})\sqrt{\frac{2}{\pi a}}\right|}{n^{\frac{1}{2}}\left(\frac{2}{\pi a}\right)^{\frac{1}{4}}(\log n/\sqrt{a})^{\frac{1}{2}}}>C\right\}\leqq D\left(\frac{n^{\psi/2}}{n}\right)^K$$

holds for any integer n, where a runs over the reals.

As a simple consequence of this lemma one gets

Lemma 9. *For any $K>0$, $2/5<\psi<\rho<2$, and $0<\gamma<\delta<\infty$ there exist a $C=C(\gamma,\delta,\psi,\rho,K)>0$ and a $D=D(\gamma,\delta,\psi,\rho,K)>0$ such that*

$$P\left\{\sup_{t^\gamma<n<t^\delta}\ \sup_{n^\psi<a<n^\rho}\frac{\left|M(a,x,n)-\eta(x,\tau_{2n})\sqrt{\frac{2}{\pi a}}\right|}{n^{\frac{1}{2}}\left(\frac{2}{\pi a}\right)^{\frac{1}{4}}(\log n/\sqrt{a})^{\frac{1}{2}}}<C\right\}<D\left(\frac{t^{\psi/2}}{t}\right)^K$$

holds for any $t>1$, where n runs over the integers and a runs over the reals.

By Lemma 1 one gets

Lemma 10. *For any $K>0$ there exists a $C=C(K)>0$ and a $D=D(K)>0$ such that*

$$P\left(\frac{\tau_{2n}}{n^2}<\frac{1}{C\log n}\right)\leqq D n^{-K}$$

for any positive integer n and

$$P(n(t)>(C t\log t)^{\frac{1}{2}})\leqq D t^{-K}$$

for any $t>1$.

Lemma 11. *For any $0<\gamma<1/2$, $\varepsilon>0$, $C>0$ there exists a $D=D(\gamma,\varepsilon,C)>0$ such that*

$$P(\eta(x,t)>t^{\gamma+\varepsilon},\ n(t)<t^\gamma)<D t^{-C}.$$

Proof of Lemma 11. For any $0<u<v<\infty$ let $\eta(x,(u,v))=\eta(x,v)-\eta(x,u)$. Then

$$\eta(x,t)\leqq\sum_{i=0}^{n(t)}\eta(x,[\tau_{2i},\tau_{2i+1})).$$

Hence

$$\begin{aligned} &P\{\eta(x,t)>t^{\gamma+\varepsilon}, n(t)<t^{\gamma}\}\\ &\leqq \sum_{k\leqq t^{\gamma}} P\left\{\sum_{i=0}^{n(t)} \eta(x,[\tau_{2i},\tau_{2i+1}))>t^{\gamma+\varepsilon}\,|\,n(t)=k\right\} P\{n(t)=k\}\\ &\leqq P\left\{\max_{k\leqq t^{\gamma}} \sum_{i=0}^{k} \eta(x,[\tau_{2i},\tau_{2i+1}))>t^{\gamma+\varepsilon}\right\}\\ &=P\left\{\sum_{i=0}^{t^{\gamma}} \eta(x,[\tau_{2i},\tau_{2i+1}))>t^{\gamma+\varepsilon}\right\}. \end{aligned}$$

Since (cf. Knight 1969)

$$P\{\eta(x,[\tau_{2i},\tau_{2i+1})<y\}=1-e^{-y} \qquad (y\geqq 0,\ i=0,1,\ldots),$$

the statement of Lemma 11 follows.

Introduce the following notations

$$\begin{aligned} &\mathscr{A}=\mathscr{A}(t)=\{n(t)<t^{\gamma}\}, \qquad \gamma=3/16,\\ &\mathscr{B}=\mathscr{B}(t)=\{t^{\gamma}\leqq n(t)\leqq (Ct\log t)^{\frac{1}{2}}\},\\ &\mathscr{C}=\mathscr{C}(t)=\{n(t)>(Ct\log t)^{\frac{1}{2}}\},\\ &\mathscr{D}=\mathscr{D}(t)=\left\{\frac{\left|M(a_t,x,n(t))-\eta(x,t)\sqrt{\dfrac{2}{\pi a_t}}\right|}{(t/a_t)^{\frac{1}{4}}(\log t/a_t)^{\frac{3}{4}}}>C\right\}, \qquad a_t=t^{\frac{1}{4}}. \end{aligned}$$

Lemma 12. *For any $K>0$ there exist a $C=C(K)>0$ and a $D=D(K)>0$ such that*

$$P(\mathscr{D})\leqq D\left(\frac{a_t}{t}\right)^{K}.$$

Proof. Clearly

$$P(\mathscr{D})=P(\mathscr{A}\mathscr{D})+P(\mathscr{B}\mathscr{D})+P(\mathscr{C}\mathscr{D}).$$

By Lemma 10

$$P(\mathscr{C}\mathscr{D})\leqq D\left(\frac{a_t}{t}\right)^{K}.$$

Since $M(a_t,x,n(t))<n(t)$, by Lemma 11 we have

$$P(\mathscr{A}\mathscr{D})\leqq D\left(\frac{a_t}{t}\right)^{K}.$$

Now let $2/5<\psi<1/2$ and $4/3<\rho<2$. Since $\mathscr{B}$ implies

$$(n(t))^{\psi}<a_t=t^{\frac{1}{4}}<(n(t))^{\rho}$$

and

$$(C\log t/a)^{\frac{1}{4}}\geqq (n(t))^{\frac{1}{2}},$$

we obtain

$$\mathscr{B}\mathscr{D} \subset \left\{ \sup_{(n(t))^{\psi} < a < (n(t))^{\rho}} \frac{\left| M(a, x, n(t)) - \eta(x, t) \sqrt{\frac{2}{\pi a}} \right|}{t^{\frac{1}{4}} a^{-\frac{1}{4}} (\log t/a)^{\frac{3}{4}}} > C, \mathscr{B} \right\}$$

$$\subset \left\{ \sup_{t^{\gamma} \leqq n \leqq (Ct \log t)^{\frac{1}{2}}} \sup_{n^{\psi} < a < n^{\rho}} \frac{\left| M(a, x, n) - \eta(x, \tau_{2n}) \sqrt{\frac{2}{\pi a}} \right|}{n^{\frac{1}{2}} a^{-\frac{1}{4}} (\log n/a)^{\frac{1}{2}}} > C \right\}.$$

Hence Lemma 9 implies Lemma 12.

The following lemma is well-known (cf. Csörgő and Révész 1981, Lemma 1.6.1).

Lemma 13.

$$P\{ \sup_{0 \leqq t \leqq T} T^{-\frac{1}{2}} |W(t)| \leqq x \} \leqq (4/\pi) \exp(-\pi^2/8x^2).$$

Lemma 14. *For any* $K>0$ *there exist a* $C=C(K)>0$ *and a* $D=D(K)>0$ *such that*

$$P\{ \inf_{0 \leqq t \leqq T-b_T} \sup_{0 \leqq s \leqq b_T} |W(t+s)-W(t)| \leqq 1 \} \leqq D T^{-K},$$

where $b_T = C \log T$.

The proof of this lemma is essentially the same as that of Step 1 of Theorem 1.6.1 of Csörgő and Révész (1981). We present the details for convenience.

Proof of Lemma 12. By Lemma 13

$$P\{ \min_{0 \leqq i \leqq T-b_T} \sup_{0 \leqq s \leqq b_T} |W(i+s)-W(i)| \leqq 2 \}$$

$$\leqq (T-b_T) \exp\left(-\frac{\pi^2}{8} \cdot \frac{C \log T}{4} \right) \leqq D T^{-K}, \qquad \text{if } \pi^2 C/32 > K+1.$$

Since

$$\{ \inf_{0 \leqq t \leqq T-b_T} \sup_{0 \leqq s \leqq b_T} |W(t+s)-W(t)| \leqq 1 \}$$

$$\subset \{ \min_{0 \leqq i \leqq T-b_T} \sup_{0 \leqq s \leqq b_T} |W(i+s)-W(i)| \leqq 2 \},$$

we have Lemma 14.

Let $L(a_t, x, t)$ be the number of excursions of W away from x that are greater than a_t in length and not higher than one. Then by Lemma 14 we have the following two lemmas.

Lemma 15. *For any* $K>0$ *there exists a* $D=D(K)>0$ *such that*

$$P\{L(a_t, x, t) \geqq 1\} \leqq D t^{-K}.$$

Lemma 16. *For any* $K>0$ *there exists a* $C=C(K)>0$ *and a* $D=D(K)>0$ *such that*

$$P\{ \max_{1 \leqq k \leqq n(t)} \alpha_k > C \log t \} \leqq D t^{-K}.$$

Next we prove

Lemma 17. *For any $K>0$ there exists a $C=C(K)>0$ and a $D=D(K)>0$ such that*

$$P\{N(a_t, x, t) \notin (M(a_t, x, n(t)), M(a_t - C\log t, x, n(t)))\} \leqq D t^{-K}.$$

Proof. Let

$$N^*(a_t, x, t) = N(a_t, x, t) - L(a_t, x, t).$$

By Lemma 15 it suffices to prove that

$$P\{N^*(a_t, x, t) \notin (M(a_t, x, n(t)), M(a_t - C\log t, x, n(t)))\} \leqq D t^{-K}.$$

For any fixed x the end points of the excursions away from x that are higher than one are (ψ_{2i}, τ_{2i+2}) $(i=0, 1, 2, \ldots)$. The lengths of these excursions are

$$\beta_{i+1} < \tau_{2i+2} - \psi_{2i} < \alpha_{i+1} + \beta_{i+1}.$$

Hence by Lemma 16 we have our Lemma 17.

Lemmas 12 and 17 together imply

Lemma 18. *For any $K>0$ there exists a $C=C(K)>0$ and a $D=D(K)>0$ such that*

$$P\left\{\frac{\left|N(a_t, x, t) - \sqrt{\frac{2}{\pi a_t}}\,\eta(x, t)\right|}{(t/a_t)^{\frac{1}{4}}(\log t/a_t)^{\frac{3}{4}}} > C\right\} \leqq D\left(\frac{a_t}{t}\right)^K.$$

Now let

$$x_i = x_i(t) = i t^{-c} \qquad (i=0, \pm 1, \pm 2, \ldots, \pm [t^{c+1}]).$$

Then by Lemma 18 we have

Lemma 19. *For any $K>0$ and $c>0$ there exist a $C=C(K,c)$ and a $D=D(K,c)>0$ such that*

$$P\left\{\max_{|i| \leqq [t^{c+1}]} \frac{\left|N(a_t, x_i, t) - \sqrt{\frac{2}{\pi a_t}}\,\eta(x_i, t)\right|}{(t/a_t)^{\frac{1}{4}}(\log t/a_t)^{\frac{3}{4}}} > C\right\} \leqq D\left(\frac{a_t}{t}\right)^K.$$

Let $\{E(t), 0 \leqq t \leqq 1\}$ be a positive Brownian excursion and for any $0<\varepsilon<1/2$ put

$$M(\varepsilon) = \min_{\varepsilon<t<1-\varepsilon} E(t).$$

Lemma 20. *There exists a constant $C>0$ such that*

$$P\{M(\varepsilon^{\frac{1}{8}}) < \varepsilon\} < C\varepsilon^{\frac{3}{4}}$$

for any $0<\varepsilon<2^{-9}$.

Proof. By Theorem 5.2.7 of Knight (1981) we have

$$P(E(t)<2\varepsilon)=\sqrt{\frac{2}{\pi}}(t(1-t))^{-\frac{3}{2}}\int_0^{2\varepsilon} y^2 \exp\left(-\frac{y^2}{2t(1-t)}\right)dy$$
$$\leqq\frac{8}{3}\sqrt{\frac{2}{\pi}}(t(1-t))^{-\frac{3}{2}}\varepsilon^3.$$

In case $\varepsilon^{\frac{1}{8}}\leqq t\leqq 1-\varepsilon^{\frac{1}{8}}$ we obtain

$$P(E(t)<2\varepsilon)\leqq 4\varepsilon^{45/16}.$$

Let

$$\varepsilon^{\frac{1}{8}}=t_0<t_1<\ldots<t_n$$

be a partition of the interval $(\varepsilon^{\frac{1}{8}}, 1-\varepsilon^{\frac{1}{8}})$ with

$$t_{i+1}-t_i=\varepsilon^{33/16}, \qquad n=[\varepsilon^{-33/16}]+1.$$

Then

$$P\{\min_{0\leqq i\leqq n} E(t_i)<2\varepsilon\}\leqq 8\varepsilon^{\frac{3}{4}}.$$

Knowing the transition function of the inhomogeneous Markov-process $E(t)$ (Theorem 5.2.7 of Knight (1981)) it is easy to prove that

$$P\{\max_{0\leqq i\leqq n-1}\sup_{t_i\leqq t\leqq t_{i+1}}|E(t)-E(t_i)|\geqq\varepsilon\}\leqq C\varepsilon^{\frac{3}{4}}.$$

Hence we have our Lemma.

Put

$$E_a(\tau)=E_a(a t)=a^{\frac{1}{2}}E(t) \qquad (a>0,\ 0\leqq t\leqq 1)$$

where $0\leqq\tau=a t\leqq a$. Further let

$$M_a(\varepsilon)=\min_{a\varepsilon\leqq\tau\leqq a(1-\varepsilon)} E_a(\tau).$$

Lemma 21. *There exists a constant $C>0$ such that*

$$P(M_a(\varepsilon^{\frac{1}{8}})<\varepsilon a^{\frac{1}{2}})<C\varepsilon^{\frac{3}{4}}$$

for any $0<\varepsilon<2^{-9}$, $a>1$. Similarly

$$P(M_a(t^{-\frac{1}{8}(c+\frac{1}{8})})<t^{-c})<Ct^{-\frac{3}{4}(c+\frac{1}{8})}$$

for any $c>0$, $a\geqq a_t$ and t big enough.

Proof. Our first statement is a trivial analogue of Lemma 20. Choosing

$$a\geqq a_t=t^{\frac{1}{4}}, \qquad \varepsilon=t^{-c}a^{\frac{1}{2}}$$

where c is the constant of x_i (cf. definition between Lemmas 18 and 19) we have the second statement as well.

Let $N^+(h, x, t)$ resp. $N^-(h, x, t)$ be the number of positive resp. negative excursions of W away from x that are greater than h in length and completed by time t.

Lemma 22. *There exists a constant* $C>0$ *such that*

$$P\{\inf_{x_i \leqq x \leqq x_{i+1}} N^+(a_t - 2\varepsilon_t, x, t) \leqq N^+(a_t, x_i, t) - 2\}$$
$$\leqq C(t^{\frac{3}{4}})^2 (t^{-\frac{3}{4}(c+\frac{1}{8})})^2 = C t^{-(\frac{3}{2})c + 21/16},$$

consequently

$$P\{\sup_{0 \leqq |i| \leqq [t^{c+1}]} (N^+(a_t, x_i, t) - \inf_{x_i \leqq x \leqq x_{i+1}} N^+(a_t - 2\varepsilon_t, x, t)) \geqq 2\}$$
$$\leqq C t^{-(\frac{1}{2})c + 37/15}$$

where $\varepsilon_t = t^{-\frac{1}{8}(c+\frac{1}{8})}$.

Proof. Let $x_i = x_i(t) < x < x_{i+1}(t) = x_{i+1}$ and consider a positive excursion away from x_i that is greater than a_t in length and is completed by time t. Say the end points of this excursion are a and b. Such an excursion is called bad if in the interval $(a+\varepsilon_t, b-\varepsilon_t)$ there is a point u where $W(u) \leqq x_{i+1}$. By Lemma 21 the probability that an excursion (away from x_i, greater than a_t) is a bad one is less than $C t^{-\frac{3}{4}(c+\frac{1}{8})}$. Considering two such excursions (away from x_i, greater than a_t) the probability that both are bad ones is less than $C t^{-\frac{3}{2}(c+\frac{1}{8})}$. Since the number of such excursions is less than $t^{\frac{3}{4}}$ and the number of such pairs is less than $t^{\frac{3}{2}}$, we obtain our Lemma.

Lemma 23. *For any* $c>0$ *there exists a* $K = K(c) > 0$ *such that*

$$P\{N^* > 1\} \leqq K t^{-2c},$$

where

$$N^* = \sup_{-t^{-c} \leqq x \leqq 0} N^+(a_t, x, \tau),$$
$$\tau = \inf\{s \colon W(s) = -t^{-c}\}.$$

Proof. Let

$$A = \{\omega \colon -t^{-c} \leqq W(s) \leqq 1 \text{ for all } 0 \leqq s \leqq \log^2 t\},$$
$$B = \{\omega \colon \mu < \tau\},$$

where

$$\mu = \tau_0(1) = \inf\{s \colon W(s) = 1\}.$$

Let

$$\psi = \inf\{s \colon s > \mu,\ W(s) = 0\},$$
$$A^\psi = \{\omega \colon -t^{-c} \leqq W(s+\psi) \leqq 1 \text{ for all } 0 \leqq s \leqq \log^2 t\},$$
$$\mu_\psi = \inf\{s \colon s > \psi,\ W(s) = 1\},$$
$$B^\psi = \{\omega \colon \mu^\psi < \tau\}.$$

We note that by ψ being a stopping time, we have

$$P\{A\} = P\{A^\psi\} \leqq P\{\sup_{0 \leqq s \leqq \log^2 t} |W(s)| \leqq 1\} \leqq K t^{-2c}, \tag{1}$$

and
$$P\{B\}=P\{B^{\psi}|B\}\leqq Kt^{-c}. \tag{2}$$
Consider
$$P\{N^*\geqq 2\}=P\{N^*\geqq 2, A\}+P\{N^*\geqq 2, \bar{A}\}$$
$$\leqq Kt^{-2c}+P\{N^*\geqq 2, \bar{A}\}.$$
We have also
$$\{\bar{A}, N^*\geqq 2\}\subseteq\{\mu<\log^2 t\}$$
$$\{N^*\geqq 2\}\subseteq\{\tau>\log^2 t\}$$
$$\{\mu<\log^2 t,\ \tau>\log^2 t\}\subseteq B$$
$$\{N^*\geqq 2,\ \mu<\log^2 t\}\subseteq\{\psi<\tau\},$$
which together imply
$$\{\bar{A}, N^*\geqq 2\}\subseteq\{B, \psi<\tau\}. \tag{3}$$
Hence
$$P\{N^*\geqq 2, \bar{A}\}=P\{N^*\geqq 2, \bar{A}, B, \psi<\tau\}$$
$$=P\{N^*\geqq 2, \bar{A}, B, \psi<\tau, A^{\psi}\}+P\{N^*\geqq 2, \bar{A}, B, \psi<\tau, \overline{A^{\psi}}\}$$
$$\leqq Kt^{-2c}+P\{N^*\geqq 2, \bar{A}, B, \psi<\tau, \overline{A^{\psi}}\}, \quad \text{by (1)}.$$
The way we have seen (3) we see also that
$$\{\overline{A^{\psi}}, N^*\geqq 2\}\subseteq B^{\psi}.$$
Consequently,
$$P\{N^*\geqq 2, \bar{A}, B, \psi<\tau, \overline{A^{\psi}}\}=P\{N^*\geqq 2, \bar{A}, B, \psi<\tau, \overline{A^{\psi}}, B^{\psi}\}$$
$$\leqq P\{B, B^{\psi}\}\leqq Kt^{-2c}, \quad \text{by (2)}.$$

The above inequalities together imply Lemma 23.

This lemma implies

Lemma 24. *For all $i=\pm 1, \pm 2, \ldots$ and $x_i=it^{-c}$, we have*
$$P\{\sup_{x_i\leqq x\leqq x_{i+1}} N^+(a_t, x, \tau_0(x_i))>1\}\leqq Kt^{-2c}.$$

Lemma 25. *Let*
$$\alpha=\inf\{s\colon N^+(a_t, 0, s)=1\}.$$
Then for any $l\geqq 2$ and $C>0$ we have
$$P\{\sup_{0\leqq x\leqq 1/t^c} N^+(a_t, x, \alpha)>l-1\}\leqq Kt^{-(l-1)c}.$$

Proof. Let
$$\beta=\inf\{s\colon \sup_{0\leqq x\leqq t^{-c}} N^+(a_t, x, s)=1\},$$
$$A^{\beta}=\{\omega\colon 0<W(s+\beta)<1 \text{ for all } 0\leqq s\leqq\log^2 t\},$$
$$\mu_{\beta}=\inf\{s\colon s>\beta,\ W(s)=1\},$$
$$B^{\beta}=\{\omega\colon \mu_{\beta}<\alpha\}.$$

Just like in (1) and (2) of the above proof

$$P\{A^{\beta}\} \leqq K t^{-c} \quad \text{and} \quad P\{B^{\beta}\} \leqq K t^{-c}. \tag{4}$$

Consider the case of $l=2$.

$$\begin{aligned} P\{&\sup_{0 \leqq x \leqq t^{-c}} N^{+}(a_t, x, \alpha) \geqq 2\} \\ &= P\{\sup_{0 \leqq x \leqq t^{-c}} N^{+}(a_t, x, \alpha) \geqq 2, A^{\beta}\} + P\{\sup_{0 \leqq x \leqq t^{-c}} N^{+}(a_t, x, \alpha) \geqq 2, A^{\beta}\} \\ &\leqq K t^{-c} + P\{\sup_{0 \leqq x \leqq t^{-c}} N^{+}(a_t, x, \alpha) \geqq 2, \overline{A^{\beta}}\}. \end{aligned}$$

We have also

$$\{\sup_{0 \leqq x \leqq t^{-c}} N^{+}(a_t, x, \alpha) \geqq 2, \overline{A^{\beta}}\} \subseteq \{\mu_{\beta} \leqq \beta + \log^2 t\},$$

$$\{\sup_{0 \leqq x \leqq t^{-c}} N^{+}(a_t, x, \alpha) \geqq 2\} \subseteq \{\alpha \geqq \beta + \log^2 t\},$$

which together imply

$$\{\sup_{0 \leqq x \leqq t^{-c}} N^{+}(a_t, x, \alpha) \geqq 2, \overline{A^{\beta}}\} \subseteq B^{\beta}.$$

Hence by (4) we have Lemma 25 with $l=2$. For $l>2$ a similar argument completes the proof.

Lemma 26. *For all $i = \pm 1, \pm 2, \ldots$ we have*

$$P\{\sup_{x_i \leqq x \leqq x_{i+1}} N^{+}(a_t, x, t) > N^{+}(a_t, x_i, t) + 3\} \leqq K t^{3/2 - 2c}.$$

Proof. Let

$$\begin{aligned} &N^{+}(a_t, x, (u, v)) = N^{+}(a_t, x, v) - N^{+}(a_t, x, u), \\ &\bar{\alpha}_0 = \tau_0(x_i) \\ &\bar{\alpha}_1 = \inf\{s \colon s > \bar{\alpha}_0, N^{+}(a_t, x_i, (\bar{\alpha}_0, s)) = 1\}, \\ &\bar{\alpha}_2 = \inf\{s \colon s > \bar{\alpha}_1, N^{+}(a_t, x_i, (\bar{\alpha}_1, s)) = 1\} \\ &\quad \vdots \\ &\nu(t) = \inf\{k \colon \bar{\alpha}_k > t\}, \\ &\zeta(l) = \text{No.}\{i \colon 0 \leqq i \leqq \nu(t) - 1, \sup_{x_i \leqq x \leqq x_{i+1}} N^{+}(a_t, x, (\bar{\alpha}_i, \bar{\alpha}_{i+1})) \geqq l\}. \end{aligned}$$

We note that

$$\nu(t) \leqq t^{\frac{3}{4}}, \tag{5}$$

$$N^{+}(a_t, x_i, t) = \nu(t) - 1, \tag{6}$$

$$\begin{aligned} \sup_{x_i \leqq x \leqq x_{i+1}} N^{+}(a_t, x, t) &\leqq \sup_{x_i \leqq x \leqq x_{i+1}} N^{+}(a_t, x, \bar{\alpha}_0) + \sum_{i=0}^{\nu(t)-1} \sup_{x_i \leqq x \leqq x_{i+1}} N^{+}(a_t, x, (\bar{\alpha}_i, \bar{\alpha}_{i+1})) \\ &\leqq \sup_{x_i \leqq x \leqq x_{i+1}} N^{+}(a_t, x, \bar{\alpha}_0) + \nu(t) + \zeta(2) + 2\zeta(3) + \ldots \\ &= \sup_{x_i \leqq x \leqq x_{i+1}} N^{+}(a_t, x, \bar{\alpha}_0) + N^{+}(a_t, x_i, t) + 1 + \zeta(2) + 2\zeta(3) + \ldots \end{aligned} \tag{7}$$

by (6).

By (5) and Lemma 25 with $l=2$ we get

$$P\{\zeta(2)>1\}\leqq K t^{\frac{3}{2}-2c}. \tag{8}$$

By (5) and Lemma 25 with $l\geqq 3$ we get

$$P\{\zeta(l)>0\}\leqq K t^{\frac{3}{2}-(l-1)c}. \tag{9}$$

By Lemma 24, (8) and (9)

$$P\{\sup_{x_i\leqq x\leqq x_{i+1}} N^+(a_t, x, \bar{\alpha}_0)+\zeta(2)+2\zeta(3)+\ldots>2\}\leqq K t^{\frac{3}{2}-2c}. \tag{10}$$

Hence by (7) and (10) we get Lemma 26.

Lemma 27. *For all* $i=\pm 1, \pm 2, \ldots$ *we have*

$$P\{\sup_{x_i\leqq x\leqq x_{i+1}} N^+(a_t, x, t)>N^+(a_t-2\varepsilon_t, x_{i+1}, t)+4\}$$
$$\leqq K(t^{-(\frac{3}{2})c+21/16}+t^{\frac{3}{2}-2c}).$$

Proof. Since

$$P\{\sup_{x_i\leqq x\leqq x_{i+1}} N^+(a_t, x, t)>N^+(a_t-2\varepsilon_t, x_{i+1}, t)+4\}$$
$$\leqq P\{\sup_{x_i\leqq x\leqq x_{i+1}} N^+(a_t, x, t)>N^+(a_t, x_i, t)+3\}$$
$$+P\{N^+(a_t-2\varepsilon_t, x_{i+1}, t)<N^+(a_t, x_i, t)-1\},$$

by Lemmas 22 and 26 we get Lemma 27.

Similarly one can prove

Lemma 28. *For all* $i=\pm 1, \pm 2, \ldots$

$$P\{\inf_{x_i\leqq x\leqq x_{i+1}} N^+(a_t-2\varepsilon_t, x, t)<N^+(a_t, x_{i+1}, t)-4\}$$
$$\leqq K(t^{-(\frac{3}{2})c+21/16}+t^{\frac{3}{2}-2c}).$$

In analogy with Lemmas 22 and 26, by symmetry arguments we have also

Lemma 29. *For all* $i=\pm 1, \pm 2, \ldots$ *we have*

$$P\{\sup_{x_i\leqq x\leqq x_{i+1}} N^-(a_t, x, t)>N^-(a_t, x_{i+1}, t)+3\}\leqq K t^{\frac{3}{2}-2c} \tag{11}$$

and

$$P\{\inf_{x_i\leqq x\leqq x_{i+1}} N^-(a_t-2\varepsilon_t, x, t)<N^-(a_t, x_{i+1}, t)-1\}$$
$$\leqq K t^{-(\frac{3}{2})c+21/16}. \tag{12}$$

Lemma 30. *For all* $i=\pm 1, \pm 2, \ldots$

$$P\{\sup_{x_i\leqq x\leqq x_{i+1}} N(a_t, x, t)<N(a_t-2\varepsilon_t, x_{i+1}, t)+7\}$$
$$\leqq K(t^{-(\frac{3}{2})c+21/16}+t^{\frac{3}{2}-2c}) \tag{13}$$

and

$$P\{\inf_{x_i \leqq x \leqq x_{i+1}} N(a_t - 2\varepsilon_t, x, t) < N(a_t, x_{i+1}, t) - 5\}$$
$$\leqq K(t^{-(\frac{3}{2})c + 21/16} + t^{\frac{3}{2} - 2c}). \tag{14}$$

Proof. By Lemma 27 and (11) we get (13), while Lemma 28 and (12) yield (14).

Lemma 31. *For any $K>0$ there exists a $C=C(K)>0$, $c=c(K)>0$ and a $D=D(K)>0$ such that*

$$P\left\{(t a_t)^{-\frac{1}{4}}\left(\log\frac{t}{a_t}\right)^{-\frac{3}{4}} \sup_{|i| \leqq t^{c+1}} \sup_{x \in [x_i, x_{i+1}]} |\eta(x, t) - \eta(x_i, t)| > C\right\} \leqq D\left(\frac{a_t}{t}\right)^K.$$

Proof. Apply inequality (3.32) of Csörgő and Révész (1984).

Choosing c big enough, Lemmas 19, 30 and 31 together imply that our Theorem 4 holds true in case of $a_t = t^{\frac{1}{4}}$, i.e., we have

Lemma 32. *For any $K>0$ there exist a $C=C(K)>0$ and a $D=D(K)>0$ such that*

$$P\left\{\left(\frac{a_t}{t}\right)^{\frac{1}{4}}\left(\log\frac{t}{a_t}\right)^{-\frac{3}{4}} \sup_{x \in R}\left|N(a_t, x, t) - \sqrt{\frac{2}{\pi a_t}}\,\eta(x, t)\right| \geqq C\right\} \leqq D\left(\frac{a_t}{t}\right)^K.$$

3. Proofs of Theorems 3 and 4

For any fixed $t>0$, let

$$W_1(s) = W_1(s, t) = \frac{W(st)}{\sqrt{t}}, \qquad s \geqq 0.$$

Then $\{W_1(s, t); s \geqq 0\}$ is a Wiener process for any $t>0$. The local time and the number of excursions of W_1 will be denoted by η_1 and N_1 respectively. Clearly we have

$$t^{\frac{1}{2}}\eta_1(x t^{-\frac{1}{2}}, 1) = \eta(x, t), \qquad N_1(h, x t^{-\frac{1}{2}}, 1) = N(a_t, x, t),$$

where $h = a_t/t$.

Applying the above transformation, Lemma 32 gives

Lemma 33. *For any $K>0$ there exist a $C=C(K)>0$ and a $D=D(K)>0$ such that*

$$P\left\{h^{-\frac{1}{4}}(\log h^{-1})^{-\frac{3}{4}} \sup_{x \in R}\left|h^{\frac{1}{2}} N(h, x, 1) - \sqrt{\frac{2}{\pi}}\,\eta(x, 1)\right| \geqq C\right\} \leqq D h^K.$$

Applying the above transformation in the opposite direction, one gets immediately Theorem 4 as a consequence of Lemma 33.

A trivial generalization of Lemma 33 is:

Lemma 34. *For any $K>0$ there exist a $C=C(K)>0$ and a $D=D(K)>0$ such that for any fixed $t>h$ we have*

$$P\left\{h^{-\frac{1}{4}}(\log h^{-1})^{-\frac{3}{4}}\sup_{x\in R}\left|h^{\frac{1}{2}}N(h,x,t)-\sqrt{\frac{2}{\pi}}\,\eta(x,t)\right|\geqq C\right\}\leqq Dh^K.$$

For $t'>h>0$, *let*

$$t_i = ih^2 \qquad (i=[h^{-1}],[h^{-1}]+1,\ldots,[t'h^{-2}]).$$

Then by Lemma 34 we obtain

Lemma 35. *For any* $K>0$ *and* $t'>h$ *there exist a* $C=C(K,t')>0$ *and a* $D=D(K,t')>0$ *such that*

$$P\left\{h^{-\frac{1}{4}}(\log h^{-1})^{-\frac{3}{4}}\max_{[h^{-1}]\leqq i\leqq[t'h^{-2}]}\sup_{x\in R}\left|h^{\frac{1}{2}}N(h,x,t_i)-\sqrt{\frac{2}{\pi}}\,\eta(x,t_i)\right|\geqq C\right\}\leqq Dh^K.$$

By Lemma 35 and a simple estimation of

$$\sup_{x\in R}|\eta(x,t_{i+1})-\eta(x,t_i)|$$

one obtains Theorem 3.

4. Proofs of Theorems 1 and 2

The proofs of Theorems 1 and 2 are based on Theorems 3 and 4, and are very similar to each other. Here we only present the proof of Theorem 1.

Let $h_n=n^{-2}$. The Borel-Cantelli lemma and Theorem 3 imply

$$\lim_{n\to\infty} h_n^{-\frac{1}{4}}(\log h_n^{-1})^{-1}\sup_{(x,t)\in R\times[0,t']}\left|h_n^{\frac{1}{2}}N(h_n,x,t)-\sqrt{\frac{2}{\pi}}\,\eta(x,t)\right|=0 \qquad \text{a.s.}$$

As a consequence we also have

$$\lim_{n\to\infty} h_{n+1}^{-\frac{1}{4}}(\log h_{n+1}^{-1})^{-1}\sup_{(x,t,\in R\times[0,t']}\left|h_n^{\frac{1}{2}}N(h_n,x,t)-\sqrt{\frac{2}{\pi}}\,\eta(x,t)\right|=0 \qquad \text{a.s.}$$

Let now $h_{n+1}\leqq h<h_n$. Then our latter two relationships and the trivial inequality

$$(1-1/n)h_n^{\frac{1}{2}}N(h_n,x,t)\leqq h^{\frac{1}{2}}N(h,x,t)\leqq(1+1/n)h_{n+1}^{\frac{1}{2}}N(h_{n+1},x,t)$$

imply Theorem 1.

Acknowledgement. This work was partly done while the authors were visiting the University of Leiden, Department of Mathematics and Computer Science. They are grateful to this department and especially to Willem van Zwet for their hospitality.

References

Csörgő, M., Révész, P.: Strong approximations in probability and statistics. Budapest: Akadémiai Kiadó and New York: Academic Press 1981

Csörgő, M., Révész, P.: On strong invariance for local time of partial sums. In: Tech. Rep. Ser. Lab. Res. Statist. Probab. **37**, 25–72 (1984) Carleton University, and Stochastic Process. Appl. **20**, 59–84 (1985)

Itô, K., McKean, H.P.: Diffusion processes and their sample paths. Berlin-Heidelberg-New York: Springer 1965

Knight, F.B.: Brownian local time and taboo processes. Trans. Am. Math. Soc. **143**, 173–185 (1969)

Knight, F.B.: Essentials of Brownian motion and diffusion. Math. Survey No. 18. Amer. Math. Soc. Providence, Rhode Island 1981

Perkins, E.: A global intrinsic characterization of Brownian local time. Ann. Probab. **9**, 800–817 (1981)

Rényi, A.: Probability theory. Budapest: Akadémiai Kiadó 1970

Received December 6, 1984; In revised form September 26, 1985

Note added in proof. Recently Csörgő, Horváth and Révész (How big must be the difference between local time and mesure du voisinage of Brownian motion? Statist. Probab. Lett. **4** (1986), in press) showed that the rate of convergence in Theorem 1 is optimal except the log term.