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# Rate of Convergence of Transport Processes with an Application to Stochastic Differential Equations

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**Summary.** We obtain a rate of convergence of uniform transport processes to Brownian motion, which we apply to the Wong and Zakai approximation of stochastic integrals.

## 1. Introduction

Let the real process $\{X(t),\ a \leqq t \leqq b\}$ satisfy the stochastic differential equation

$$dX(t)=\left[m(X(t),t)+\frac{1}{2}\,\sigma(X(t),t)\,\frac{\partial}{\partial x}\,\sigma(X(t),t)\right]dt$$

$$+\sigma(X(t),t)\,dW(t), \qquad a \leqq t \leqq b, \tag{1.1}$$

$$X(a)=X_a, \tag{1.2}$$

in the sense of Itô, where $W$ is one-dimensional standard Brownian motion (Wiener process). Let $\{Y_n(t),\ a \leqq t \leqq b\}$, $n=1,2,\ldots$ be continuous processes with piecewise continuous derivative, and let the corresponding processes $\{X_n(t),\ a \leqq t \leqq b\}$ satisfy the differential equations

$$dX_n(t)=m(X_n(t),\ t)\,dt+\sigma(X_n(t),t)\,dY_n(t), \qquad a \leqq t \leqq b, \tag{1.3}$$

$$X_n(a)=X_a. \tag{1.4}$$

For each $n$ and each point in sample space, (1.3) with (1.4) is a deterministic ordinary differential equation.

Assuming appropriate conditions on the coefficients $m$ and $\sigma$, (1.1) and (1.2) have a unique solution (cf., e.g., Theorem 1.1 in Friedman (1975) p. 98, and Theorem 4.6 in Liptser and Shiryayev (1977)). Wong and Zakai (1965) show that if the coefficients $m$ and $\sigma$ satisfy suitable conditions which guarantee the existence

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and uniqueness of solutions of (1.1), (1.2) and (1.3), (1.4), and if $Y_n$ converges to $W$ uniformly in $[a, b]$ with probability 1 as $n \to \infty$, then $X_n$ converges to $X$ uniformly in $[a, b]$ with probability 1 as $n \to \infty$. This result is very important in that it guarantees that numerical solutions of (1.3), (1.4) can be viewed as approximate solutions of (1.1), (1.2). Physical experiments may correspond to (1.3), (1.4) rather than (1.1), (1.2), since Brownian motion can only be realized by approximation in the physical world. Thus physical systems driven by "white noise" could lead directly to limits of $X_n(t)$ where $X_n(t)$ satisfy the Langevin equations (1.3), (1.4).

Due to Itô's stochastic calculus, the forms of the stochastic differential equation in (1.1) and the ordinary differential equation in (1.3) are different. In terms of the so-called Stratonovisch integral, a symmetrical definition for stochastic integrals, the chain rule (Itô's formula) takes the same form as in the ordinary calculus. Consequently, if we were to use Stratonovich integrals instead of Itô's, the stochastic differential equation would have the form of (1.3) instead of the present one in (1.1). For further discussion on Itô and Stratonovich calculus in the present context we refer to Wong and Zakai (1965, 1969).

Assuming $a \geqq 0$, in this paper we consider the specific processes $\{Y_n(t), t \geqq 0\}$ as follows. For each $n = 1, 2, \ldots$, let $\{Y_n(t), t \geqq 0\}$ be a stochastic process such that $Y_n(t)$ is the position on the real line at time $t$ of a particle starting from 0 with velocity $+n$ or $-n$, each with probability 1/2. The particle moves with constant velocity until a random time $\tau_1$ whose distribution function is exponential with parameter $n^2$, i.e. $E\tau_1 = n^{-2}$. At time $\tau_1$ the particle switches from velocity $\pm n$ to $\mp n$ and moves with this velocity for an additional length of time $\tau_2 - \tau_1$, independently from $\tau_1$, the time before, where $\tau_2 - \tau_1$ is again an exponential random variable (r. v.) with $E(\tau_2 - \tau_1) = n^{-2}$. At the random time $\tau_2$ it changes its velocity again. The motion continues in this manner. This process is called a (uniform) transport process. From now on $Y_n$ will be this transport process.

The weak convergence of $Y_n$ to Wiener process follows from a result of Pinsky (1968), Watanabe (1968) and Griego et al. (1971). The rate of convergence of $Y_n$ to $W$ is studied by Gorostiza and Griego (1980), who obtain the following result.

**Theorem A.** *There exists a sequence of Wiener processes* $\{W_n(t), t \geqq 0\}$ *such that for all* $\varepsilon > 0$ *we have, as* $n \to \infty$

$$P\left\{\max_{a \leqq t \leqq b} |Y_n(t) - W_n(t)| > Cn^{-1/2}(\log n)^{5/2}\right\} = o(n^{-\varepsilon}),$$

*where C is a positive constant, depending on* $a, b$ *and* $\varepsilon$.

Here, in this theorem, and also later on, it is assumed without loss of generality that all random variables and stochastic processes are defined on the same probability space (cf. de Acosta (1982), Theorem A.1). We say that a function $f(x, t)$ satisfies Lipschitz condition if $|f(x, t) - f(y, t)| \leqq K|x - y|$, where $K$ is a constant, does not depend on $t$. Let $\varrho_{a,b}(X, Y)$ be the Prohorov-Lévy distance of probability measures generated by the random elements $X$, $Y$ on $C[a, b]$.

Gorostiza (1980), and Römisch and Wakolbinger (1985) deduce the following estimation for the solutions of (1.1), (1.2) and (1.3), (1.4).

**Theorem B.** *Suppose the following conditions are satisfied:*

(i) $m(x,t)$, $\sigma(x,t)$, $\partial\sigma(x,t)/\partial x$, $\partial\sigma(x,t)/\partial t$ *are continuous in* $-\infty < x < \infty$, $a \leqq t \leqq b$,

(ii) $m(x,t)$, $\sigma(x,t)$, $\sigma(x,t)\,\partial\sigma(x,t)/\partial x$ *satisfy Lipschitz conditions,*

(iii) $0 < c \leqq |\sigma(x,t)| \leqq L < \infty$ *and* $|\partial\sigma(x,t)/\partial t| \leqq K\sigma^2(x,t)$,

(iv) $X_a$ *is constant.*

*Then, as* $n \to \infty$,

$$\varrho_{a,b}(X_n, X) = O(n^{-1/2}\exp(C(\log n)^{1/2})) \tag{1.5}$$

*for some* $C > 0$.

Roughly speaking (1.5) means that $\varrho_{a,b}(X_n, X) = o(n^{-1/2+\varepsilon})$ for all $0 < \varepsilon < 1/2$.

The main aim of this paper is to give a much improved version of Theorem A. This will enable us to establish a better estimation in Theorem B.

## 2. Approximation for the Transport Process

First we introduce and study some auxiliary processes whose transforms will yield the desired results. Let $\theta_1, \theta_2, \ldots$ be independent identically distributed exponential r.v.'s with $E\theta_1 = 1$. We define

$$Z(k) = \sum_{i=1}^{k} \theta_i, \qquad k = 1, 2, \ldots,$$

$$N(t) = \inf\{k\colon Z(k) > t\}, \qquad t \geqq 0,$$

and

$$Y(t) = \sum_{i=1}^{N(t)-1} (-1)^{i+1}\theta_i.$$

**Theorem 2.1.** *We can define a Wiener process* $\{\Gamma(t), t \geqq 0\}$ *such that with* $T > 1$ *we have*

$$P\left\{\sup_{0 \leqq t \leqq T} |Y(t) - \Gamma(t)| > A_1 \log T\right\} \leqq B_1 T^{-\varepsilon}$$

*for all* $\varepsilon > 0$, *where* $A_1 = A_1(\varepsilon)$ *and* $B_1$ *are constants.*

*Proof.* By Corollary 4.2 in Csörgö et al. (1987b) (cf. also Theorem in Csörgö et al. (1986)) we obtain

$$P\{N(T) > A_{1,1}T\} \leqq B_{1,1}T^{-\varepsilon}, \tag{2.1}$$

where $A_{1,1} = A_{1,1}(\varepsilon)$ and $B_{1,1}$ are constants. One can easily see

$$\sum_{i=1}^{N(t)-1} (-1)^{i+1}\theta_i = \sum_{i=1}^{[(N(t)-1)/2]} (\theta_{2i-1} - \theta_{2i}) + \theta_t^*, \tag{2.2}$$

with

$$\theta_t^* = \begin{cases} 0 & \text{if} \quad (N(t)-1) \quad \text{is even} \\ \theta_{2[(N(t)-1)/2]+1} & \text{if} \quad (N(t)-1) \quad \text{is odd.} \end{cases}$$

By (2.1) we obtain

$$\begin{aligned} P\left\{\sup_{0\leqq t\leqq T}\theta_t^* > A_{1,2}\log T\right\} &\leqq P\left\{\max_{1\leqq i\leqq N(T)}\theta_i > A_{1,2}\log T\right\} \\ &\leqq P\left\{\max_{1\leqq i\leqq A_{1,1}T}\theta_i > A_{1,2}\log T\right\} + B_{1,1}T^{-\varepsilon} \\ &\leqq B_{1,2}T^{-\varepsilon}. \end{aligned} \tag{2.3}$$

Next we let

$$\xi_i = \theta_{2i-1} + \theta_{2i}, \quad \eta_i = \theta_{2i-1} - \theta_{2i}$$

and define the sums

$$A(t) = \sum_{1\leqq i\leqq t}\xi_i, \quad D(t) = \sum_{1\leqq i\leqq t}\eta_i.$$

We also introduce the renewal of $A(t)$

$$B(t) = \inf\{s\colon A(s) > t\},$$

and observe that

$$B(t) = \begin{cases} N(t)/2 & \text{if } N(t) \text{ is even} \\ [N(t)/2]+1 & \text{if } N(t) \text{ is odd.} \end{cases} \tag{2.4}$$

Consequently

$$\sum_{i=1}^{[(N(t)-1)/2]}(\theta_{2i-1} - \theta_{2i}) = D(B(t)) + R(t), \tag{2.5}$$

where by (2.3)

$$P\left\{\sup_{0\leqq t\leqq T}|R(t)| > A_{1,3}\log T\right\} \leqq B_{1,3}T^{-\varepsilon}. \tag{2.6}$$

It follows from the Komlós et al. (1976) embedding theorem (cf. Theorem 2.6.2 in Csörgö and Révész (1981)) that we have

$$P\left\{\sup_{0\leqq t\leqq T}\left|\sum_{1\leqq i\leqq t}(\theta_{2i}-1) - W_1(t)\right| > A_{1,4}\log T\right\} \leqq B_{1,4}T^{-\varepsilon} \tag{2.7}$$

and

$$P\left\{\sup_{0\leqq t\leqq T}\left|\sum_{1\leqq i\leqq t}(\theta_{2i-1}) - W_2(t)\right| > A_{1,4}\log T\right\} \leqq B_{1,4}T^{-\varepsilon}, \tag{2.8}$$

where $W_1$ and $W_2$ are suitably constructed independent Wiener processes. We define the independent Wiener processes $W_3 = 2^{-1/2}(W_1 + W_2)$ and $W_4 = 2^{-1/2}(W_2 - W_1)$. Then, from (2.7) and (2.8), we get

$$P\left\{\sup_{0\leqq t\leqq T}\left|(A(t) - 2t) - 2^{1/2}W_3(t)\right| > 2A_{1,4}\log T\right\} \leqq 2B_{1,4}T^{-\varepsilon} \tag{2.9}$$

and

$$P\left\{\sup_{0\leqq t\leqq T}|D(t)-2^{1/2}W_4(t)|>2A_{1,4}\log T\right\}\leqq 2B_{1,4}T^{-\varepsilon}. \tag{2.10}$$

Applying Theorem 4.1 of Csörgö et al. (1987b) in combination with (2.9), we can define a Wiener process $W_5$, which is independent of $W_4$, so that

$$P\left\{\sup_{0\leqq t\leqq T}|(B(t)-t/2)-2^{-1}W_5(t)|>A_{1,5}\log T\right\}\leqq B_{1,5}T^{-\varepsilon}. \tag{2.11}$$

On using (2.1) and (2.4) together with (2.10) and (2.11) we obtain

$$\begin{aligned}
&P\left\{\sup_{0\leqq t\leqq T}|D(B(t))-2^{1/2}W_4(t/2+2^{-1}W_5(t))|>3A_{1,6}\log T\right\}\\
&\leqq P\left\{\sup_{0\leqq t\leqq T}|D(B(t))-2^{1/2}W_4(B(t))|>A_{1,6}\log T\right\}\\
&\quad+P\left\{\sup_{0\leqq t\leqq T}|W_4(B(t))-W_4(t/2+2^{-1}W_5(t))|>A_{1,6}\log T\right\}\\
&\leqq P\left\{\sup_{0\leqq t\leqq A_{1,1}T}|D(t)-2^{1/2}W_4(t)|>A_{1,6}\log T\right\}+2B_{1,1}T^{-\varepsilon}\\
&\quad+P\left\{\sup_{0<t<A_{1,1}T}\ \sup_{0\leqq s\leqq A_{1,5}\log T}|W_4(t+s)-W_4(t)|>A_{1,6}\log T\right\}+B_{1,5}T^{-\varepsilon}\\
&\leqq B_{1,6}T^{-\varepsilon},
\end{aligned} \tag{2.12}$$

where in the last estimation we used Lemma 1.2.1 of Csörgö and Révész (1981) on the increments of the Wiener process. Using now Lemma 3 of Csörgö et al. (1987a), we can define a Wiener process $\Gamma$ such that

$$P\left\{\sup_{0\leqq t\leqq T}|2^{1/2}W_4(t/2+2^{-1}W_5(t))-\Gamma(t)|>A_{1,7}\log T\right\}\leqq B_{1,7}T^{-\varepsilon}. \tag{2.13}$$

Now the statement of Theorem 2.1 follows immediately by (2.2), (2.3), (2.6), (2.12) and (2.13).

*Remark 2.1.* Csörgö et al. (1987a) study the approximation of stopped sums when the stopping process is independent from the summands. The rate of approximation they obtain is $\log T$, which is best possible. Their result cannot be applied under the conditions of Theorem 2.1 due to the dependence of $N(t)$ on the summands $\theta_i$ in the definition of $Y(t)$. We should note that the rate of approximation in Theorem 2.1 is the same as the optimal rate of approximation when the summands and the stopping process are independent.

We can now prove the main result of this section.

**Theorem 2.2.** *We can define a sequence of Wiener processes* $\{\Gamma_n(t), t\geqq 0\}$ *such that*

$$P\left\{\sup_{0\leqq t\leqq 1}|Y_n(t)-\Gamma_n(t)|>A_2n^{-1}\log n\right\}\leqq B_2n^{-\varepsilon}$$

*for all* $\varepsilon>0$, *where* $A_2=A_2(\varepsilon)$ *and* $B_2$ *are constants.*

*Proof.* Let $\tau$ be a random variable with $P\{\tau=1\}=P\{\tau=-1\}=1/2$ and independent of $\{\theta_i, i\geqq 1\}$ and $\{\Gamma(t), t\geqq 0\}$ of Theorem 2.1. We observe that with $\tau_0=0$ we have

$$\{n^2\tau_i n^2(\tau_i-\tau_{i-1}), i\geqq 1\} \stackrel{\mathscr{D}}{=} \{\theta_i, i\geqq 1\} \tag{2.14}$$

for each $n=1,2,\ldots$, and consequently also

$$\{Y_n(t), 0\leqq t\leqq 1\} \tag{2.15}$$
$$\stackrel{\mathscr{D}}{=} \{n^{-1}\tau(Y(n^2t)+(n^2t-Z(N(n^2t)-1))(-1)^{N(n^2t)+1}), 0\leqq t\leqq 1\}.$$

By definition we have

$$Z(N(t))=Z(N(t)-1)+\theta_{N(t)}\geqq t. \tag{2.16}$$

Hence

$$\theta_{N(t)}\geqq t-Z(N(t)-1) \tag{2.17}$$

Similar to the proof of (2.3) we obtain

$$P\left\{\max_{0\leqq t\leqq T}\theta_{N(t)}>A_{2,1}\log T\right\}\leqq B_{2,1}T^{-\varepsilon}. \tag{2.18}$$

It follows immediately from (2.17) and (2.18) that we have

$$P\left\{\sup_{0\leqq t\leqq 1}|(n^2t-Z(N(n^2t)-1))(-1)^{N(n^2t)+1}|>A_{2,2}\log n^2\right\}\leqq B_{2,2}n^{-\varepsilon}. \tag{2.19}$$

Clearly, the stochastic process $\{n^{-1}\tau\Gamma(n^2t), 0\leqq t\leqq 1\}$ is a Wiener process for each $n$. Hence Theorem 2.2 follows from Theorem 2.1 via (2.15) and (2.19).

*Remark 2.2.* We wish to point out the crucial role of (2.14) which results in the time transformed equality in distribution of (2.15) for each $n$. Namely, if we were to apply the Komlós et al. (1976) inequality directly to the partial sums of $\{\tau_i, i\geqq 1\}$, then the constants of this inequality would depend on $n$ due to the dependence of the distribution of these r.v.'s on $n$. Consequently, such a procedure cannot yield a result like that of Theorem 2.2.

## 3. Rate of Convergence of the Solutions of the Stochastic Differential Equations (1.3), (1.4)

Applying Theorem 2.2 we obtain the following improvement of Theorem B.

**Theorem 3.1.** *Under the conditions of Theorem B we have*

$$\varrho_{a,b}(X_n, X)=O(n^{-1}\exp(C(\log n)^{1/2})) \tag{3.1}$$

*with some $C>0$, where $X_n$ is solution of* (1.3), (1.4), *and $X$ is solution of* (1.1), (1.2).

*Proof.* We take the sequence of Wiener processes $\Gamma_n$ of Theorem 2.2 and consider the differential equation of (1.1), (1.2) in terms of it with corresponding solution, say $X_n^*$. Thus we have

$$dX_n^*(t) = \left[ m(X_n^*(t), t) + \frac{1}{2} \sigma(X_n^*(t), t) \frac{\partial}{\partial x} \sigma(X_n^*(t), t) \right] dt$$
$$+ \sigma(X_n^*(t), t) d\Gamma_n(t), \quad a \leqq t \leqq b, \tag{3.2}$$

$$X_n^*(a) = X_a, \tag{3.3}$$

where $X_a$ is the same constant as that of (1.2). Using unicity of the solution of (3.2), (3.3) (cf. Theorem 1.1 in Friedman 1975, p. 98) we get

$$\{X_n^*(t), a \leqq t \leqq b\} \overset{\mathscr{D}}{=} \{X(t), a \leqq t \leqq b\}, \tag{3.4}$$

where $\{X(t), a \leqq t \leqq b\}$ is the solution of (1.1), (1.2). By Theorem 3 of Römisch and Wakolbinger (1985) we obtain, with some constants $C_1$ and $C_2$,

$$\sup_{a \leqq t \leqq b} |X_n^*(t) - X_n(t)|$$
$$\leqq C_1 \exp\left( C_2 \max\left( \sup_{a \leqq t \leqq b} |\Gamma_n(t)|, \sup_{a \leqq t \leqq b} |Y_n(t)| \right)\right) \sup_{a \leqq t \leqq b} |\Gamma_n(t) - Y_n(t)|. \tag{3.5}$$

Since $\Gamma_n(t)$ is a Wiener process for each $n \geqq 1$ we have

$$P\left\{ \sup_{a \leqq t \leqq b} |\Gamma_n(t)| > C_3 (\log n)^{1/2} \right\} \leqq n^{-2}, \tag{3.6}$$

and hence, on account of Theorem 2.2, we have also

$$P\left\{ \sup_{a \leqq t \leqq b} |Y_n(t)| > C_4 (\log n)^{1/2} \right\} \leqq n^{-2}, \tag{3.7}$$

with some constants $C_3$ and $C_4$. Consequently by Theorem 2.2, (3.5), (3.6) and (3.7) we conclude

$$P\left\{ \sup_{a \leqq t \leqq b} |X_n^*(t) - X_n(t)| > C_5 n^{-1} \exp(C_6 (\log n)^{1/2}) \right\} \leqq C_7 n^{-2}, \tag{3.8}$$

with some constants $C_5$, $C_6$ and $C_7$. Applying now the Strassen-Dudley estimation of the Prohorov-Lévy distance (cf. Dudley (1968)) we obtain

$$\varrho_{a,b}(X_n, X) \leqq \inf_{\varepsilon > 0} \left( \varepsilon + P\left\{ \sup_{a \leqq t \leqq b} |X_n^*(t) - X_n(t)| > \varepsilon \right\}\right),$$

and hence also Theorem 3.1 by (3.8).

We note also that if $\sigma(x, t)$ is not a function of $x$, then (1.1) reduces to

$$dX(t) = m(X(t), t) dt + \sigma(t) dW(t), \quad a \leqq t \leqq b. \tag{3.9}$$

Consider now

$$dX_n^*(t) = m(X_n^*(t), t) dt + \sigma(t) d\Gamma_n(t), \quad a \leqq t \leqq b, \tag{3.10}$$

$$X_n^*(a) = X_a, \tag{3.11}$$

instead of (3.2), (3.3). Using the Lipschitz condition for $m$,

$$|m(x,t)-m(y,t)| \leqq L|x-y|.$$

An integration by parts gives

$$\begin{aligned}\sup_{t_1 \leqq t \leqq t_2} |X_n^*(t)-X_n(t)| \leqq L(t_2-t_1) \sup_{t_1 \leqq s \leqq t_2} |X_n(s)-X_n^*(s)| \\ + |X_n(t_1)-X_n^*(t_1)| \\ + \sup_{a \leqq t \leqq b} |\Gamma_n(t)-Y_n(t)| \int_a^b |\sigma'(s)|\,ds,\end{aligned}$$

and hence if we take $(t_2-t_1)$ so that $(t_2-t_1)L \leqq 1/2$, we obtain

$$\begin{aligned}\sup_{t_1 \leqq t \leqq t_2} |X_n^*(t)-X_n(t)| \leqq 2|X_n(t_1)-X_n^*(t_1)| \\ +2 \sup_{a \leqq t \leqq b} |\Gamma_n(t)-Y_n(t)| \int_a^b |\sigma'(s)|\,ds.\end{aligned}$$

Using (1.4) and (3.11) we get

$$\begin{aligned}&\sup_{a \leqq t \leqq b} |X_n^*(t)-X_n(t)| \\ &\leqq (2^{2L(b-a)+2}-2) \sup_{a \leqq t \leqq b} |\Gamma_n(t)-Y_n(t)| \int_a^b |\sigma'(s)|\,ds,\end{aligned}$$

and hence Theorem 2.2 combined with the Strassen-Dudley estimation of the Prohorov-Lévy distance gives the following result.

**Corollary 3.1.** *Assume the conditions of Theorem B and that $\sigma$ does not depend on $x$. Then*

$$\varrho_{a,b}(X_n, X) = O(n^{-1}\log n),$$

*where $X_n$ is solution of* (1.3), (1.4), *$X$ is solution of* (1.1), (1.2).

Finally we note that the transport process $Y_n$ is constructed from $n^2$ independent random variables. Another possibility for constructing approximate solutions of (1.1), (1.2) is in terms of partial sums of independent random variables. If we want to compare such an approximate solution to the transport process, then we should take partial sums of $n^2$ random variables. This would again result in an approximation like that of Theorem 3.1 (cf. Example 2 in Römisch and Wakolbinger (1985)).

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