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A Counter-Example to Dye's Theorem for All Non-Separable Measure Algebras

S.J. Eigen

Department of Mathematics, Northeastern University, 360 Huntington Avenue, Boston, MA 02115, USA

Summary. In 1959, H. Dye showed that any two ergodic, measure-preserving automorphisms of a Lebesgue measure algebra were weakly equivalent. In this paper, we study weak equivalence, for ergodic measure-preserving automorphisms on non-separable measure algebras. It is shown that, in general, Dye's Theorem does not hold, and in particular, it holds only on separable, i.e. Lebesgue, measure algebras.

In 1959, H. Dye [3] showed that any two ergodic, measure-preserving automorphisms of a Lebesgue measure algebra were weakly equivalent. The separability of the measure algebra was not one of his initial assumptions. In fact, he does not impose it until two-thirds through the paper – just before he proves the above theorem bearing his name. He imposes it there because the proof is essentially a constructive one which employs the separability of the measure algebra to build a sequence of transformations whose limit yields the desired conjugacy (see p. 155 [3]). This raises the question is the theorem true in general, i.e., are any two ergodic, measure-preserving automorphisms on a normalized, measure algebra weakly equivalent? Since a non-atomic, measure algebra is homogeneous if and only if it carries ergodic automorphisms [2] we will restrict ourselves to the homogeneous ones and show, not only that Dye's Theorem in general is false but, that it is only true in the Separable case.

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Let I denote the unit interval, and D an uncountable index set. We put $Z = \Pi I_d$, $d \in D$, where $I_d = I$ for all d. The product measure will be m, and the measure algebra $\mathbb{B}$. Thus, $(\mathbb{B}, m)$ is a homogeneous measure algebra of Maharam type the cardinality of D, with Z a representation space. By a theorem of D. Maharam [5], every homogeneous measure algebra has a space representation as above and so we may restrict ourselves to such throughout this paper. If $J \subset D$ then $\mathbb{B}(J)$ will denote the measure algebra generated by the "cylinder sets" based on the coordinates designated by J. In general, we will be a bit lax in putting in the phrase "modulo sets of measure zero" and other such related termonology. Definitions and basic knowledge regarding homogeneous measure algebras may be found in [2, 5, 6]. An easily accessible proof of Dye's Theorem may be found in the paper of A. Hajian, Y. Ito and S. Kakutani [5]. We refer the reader there, as well as to [3], for the definition of the full group and weak equivalence. Two immediate and well-known observations which we will use throughout are a follows.

i) If $\bar{A}\in\mathbb{B}$, $m(\bar{A})>0$ then there is a countable $J\subset D$ with $\bar{A}\in\mathbb{B}(J)$.

ii) If J is countable then $(\mathbb{B}(J),m)$ is a Lebesgue measure algebra. In such a case we will put $X=\Pi I_d$, $d\in J$ (which is again isomorphic to the unit interval, and visualize $Z=X\times\Pi I_d$, $d\in D\setminus J$.

Let $\bar{S}$ be an automorphism of $\mathbb{B}$, then we will say $J\subset D$ is $\bar{S}$-invariant if $\bar{S}(\mathbb{B}(J))=\mathbb{B}(J)$. If in addition, J is countable, then we will visualize $\bar{S}$ as a skew product over $X=\Pi I_d$, $d\in J$. In particular, we would denote this as $\bar{S}=S\times\phi_x$, where S is the point transformation induced by $\bar{S}$ on X, and for each $x\in X$, ϕ_x is an automorphism of ΠI_d, $d\in D\setminus J$. We will denote the fibre $x\times\Pi I_d$, $d\in D\setminus J$ by f_x. So $\bar{S}$ takes fibres to fibres, i.e. $x\times\Pi I_d\to Sx\times\phi_x(\Pi I_d)$.

We begin with a lemma originally due to D. Maharam [5].

Lemma 1. *Let $\bar{S}$ be an automorphism of $\mathbb{B}$. Let $\bar{A}_n$, $n=1,2,\ldots$, be any measurable sets. Then there exists a countable J which is $\bar{S}$-invariant and $\bar{A}_n\in\mathbb{B}(J)$ for all n.*

Proof. Each $\bar{A}_n$ is essentially based on a countable number of coordinates, so there exists J^1 countable with $\bar{A}_n\in\mathbb{B}(J^1)$ for all n. Let $\bar{B}^1_m$, $m=1,2,\ldots$, be a basis for $\mathbb{B}(J^1)$. By the same reasoning the sets $\bar{S}^k\bar{B}^1_m k=0,\ \pm1,\ \pm2,\ \ldots,\ m=1,$ $2,\ \ldots$, are contained in $\mathbb{B}(J^2)$ where J^2 is also countable. Let $\bar{B}^2_m$, $m=1,2,\ldots$, be a basis for $\mathbb{B}(J^2)$. By induction, we obtain J^1, J^2, J^3, each countable, and put $J=\bigcup J^n$, $n=1,2,\ldots$.

We will assume the reader is familiar with the notion of entropy for a measure-preserving transformation (see [7]). We point out, that the definition of the entropy of a measure preserving automorphism $h(\bar{T})$ as the sup of the entropy $h(\bar{T},P)$ along all finite partitions P carries straight through for the homogeneous measure algebra cases. In addition, $h(\bar{T})=\sup h(T)$ where T is $\bar{T}$ restricted to an invariant Lebesgue algebra. We will be using the following well known fact. If all the images of the partition P under T are stochastically independent, then $h(T)\geqq h(T,P)=H(P)$.

Lemma 2. *There exists an ergodic measure preserving automorphism $\bar{U}$ of $\mathbb{B}$ with $h(\bar{U})=0$.*

Proof. Let U be a mixing transformation of I with $h(U)=0$. Define $\bar{U}=\Pi U_d$, $d\in D$, $U_d=U$ for all d. That $\bar{U}$ is ergodic may be found in [2] - in fact, it is mixing. Let $\mathbb{L}$ be any invariant Lebesgue algebra for $\bar{U}$. From Lemma 1, there are countable coordinates $J\subset D$ with $\mathbb{L}\subset B(J)$, and J is $\bar{U}$-invariant. $\bar{U}$ induces on $X=\Pi I_d$, $d\in J$, $V=\Pi U_d$, $d\in J$. As a countable product of 0-entropy transformations, V has 0-entropy. $\bar{U}$ on $\mathbb{L}$ is a factor of V, hence the entropy of $\bar{U}$

restricted to $\mathbb{L}$ is 0. Since $\mathbb{L}$ was any invariant Lebesgue algebra, we conclude $h(\bar{U})=0$.

Let T be any ergodic, invertible, measure-preserving transformation on the unit interval with $h(T)>0$. Put $\bar{T}=\Pi T_d$, $d\in D$. We will show

Theorem 1. *Any ergodic $\bar{S}$ in the full group $[\bar{T}]$, or conjugate to anything in said full group has positive entropy.*

Corollary. *The $\bar{U}$ as previously defined in Lemma 2 is not in $[\bar{T}]$ nor is anything conjugate to $\bar{U}$ in $[\bar{T}]$. This negates Dye's Theorem, i.e. the full groups of $\bar{U}$ and $\bar{T}$ are not conjugate.*

Let $\bar{S}$ be ergodic in $[\bar{T}]$. Let Z_n be the sets where $\bar{S}=\bar{T}^n$. By Lemma 1, we can find a countable $J\subset D$ invariant under $\bar{S}$ with $Z_n\in\mathbb{B}(J)$. Now, we can realize $Z=X\times\Pi I_d$, $d\in D\backslash J$, $\bar{S}=\hat{S}\times\phi_x$ and $\bar{T}=\hat{T}\times\Pi T_d$, $d\in D\backslash J$, where $\hat{T}=\Pi T_d$, $d\in J$.

We take a disintegration of Z with respect to $X\times\Pi I_d$, $d\in J$. That is, we have λ the projection of m onto X which is Lebesgue, and for each $x\in X$ we have μ_x on $f_x=x\times\Pi I_d$ which is the fibre measure. The relation between the various measures is for $\bar{E}\in B$

$$m(\bar{E})=\int_X \mu_x(\bar{E}\cap f_x)\,d\lambda(x).$$

Since $h(T)>0$, we know by Sinai's Theorem (see [7]) that every Bernoulli shift with entropy less than $h(T)$ is isomorphic to some factor of T. Hence, there exists a two set partition $P_0=\{P_0,P_0'\}$ of I such that $T^k P_0$, $k=0,\pm1,\pm2,\ldots$, are stochastically independent and $h(T)\geqq h(T,P_0)=H(P_0)>0$.

Pick a coordinate $b\in D\backslash J$ and let $\bar{P}_0=\{\bar{P}_0,\bar{P}_1\}$ be the inverse of the canonical projection onto I_b. Then, $T^k\bar{P}_0$, $k=0,\pm1,\pm2,\ldots$, are stochastically independent with the base algebra $\mathbb{B}(J)$. We wish to show that $\bar{S}^k\bar{P}_0$ are also stochastically independent. This would imply that $h(\bar{S})\geqq h(\bar{S},\bar{P}_0)=H(\bar{P}_0)>0$.

Look at the partition induced on f_x by $\bar{T}^k\bar{P}_0$, i.e. $\bar{T}^k\bar{P}_0\cap f_x$. These are stochastically independent on f_x with respect to μ_x (for almost all x with respect to λ).

Now look at $\bar{S}^k\bar{P}_0\cap f_x$. We can assume x is a generic point for $\hat{T}$ and S. For each k, we have some $n(k)$ such that

$$\bar{S}^k\bar{P}_0\cap f_x=\bar{T}^{n(k)}\bar{P}_0\cap f_x,$$

and if $k\neq j$ then $n(k)\neq n(j)$ (if $n(k)=n(j)$ then $S^k x=S^j x$, but by x generic and S ergodic this is only possible if $k=j$). Hence we conclude that the partitions $\bar{S}^k\bar{P}_0\cap f_x$, $k=0,\pm1,\pm2,\ldots$, are independent with respect to μ_x. The integration formula now yields that the $\bar{S}^k\bar{P}_0$ are also independent with respect to m. This completes the proof of the theorem.

As a further remark, observe that the above also yields that the $\bar{S}^k\bar{P}_0$, $k=0,\pm1,\pm2,\ldots$, together with the base algebra $\mathbb{B}(J)$ are stochastically independent. We will use this in the next theorem.

With the negation of Dye's Theorem for non-separable homogeneous measure algebras, many questions arise regarding the weak equivalence of various ergodic automorphisms. One such, is whether a factorizable automorphism is always weakly equivalent to a non-factorizable (see [1] for definition of non-factorizable) and vice-versa. In the next theorem we give a partial answer to this question by presenting a factorizable automorphism which is weakly equivalent only to other factorizables. The examples is a variation of that presented earlier, i.e., it and all the ergodics in its full group have positive entropy. Thus it remains an open question as to whether the non-factorizable automorphism [1] and the 0-entropy automorphism constructed in Lemma 2 are weakly equivalent.

Let $\mathscr{B}$ on I be isomorphic to the Bernoulli two-shift. So we have a two set partition $P_0=\{P_0, P_1\}$ on I such that

i) $\mathscr{B}^k P_0$, $k=0, \pm 1, \pm 2, \ldots$, are stochastically independent, and

ii) the total collection of sets in $\mathscr{B}^k P_0$, $k=0, \pm 1, \pm 2, \ldots$, generate the measure algebra on I.

Put $\bar{\mathfrak{B}}=\Pi\mathscr{B}_d$, $d\in D$, $\mathscr{B}_d=\mathscr{B}$ for all d. This product Bernoulli has infinite entropy and is isomorphic to any other product Bernoulli. That is, if $\mathscr{R}_d$ is any Bernoulli on I_d, not necessarily the same for different $d\in D$, then $\bar{\mathfrak{R}}=\mathscr{R}_d$, $d\in D$, is isomorphic to the above $\bar{\mathfrak{B}}$. Further, if we put $W=\Pi Z_n$, $-\infty<n<\infty$, $Z_n=Z$ for all n and define $\bar{V}$ on W as the shift $(\bar{V}w)_n=w_{n+1}$, then $\bar{V}$ also is isomorphic to $\bar{\mathfrak{B}}$.

We will prove

Theorem 2. *Any ergodic $\bar{S}$ with $[\bar{S}]=[\bar{\mathfrak{B}}]$ factorizes.*

Let Z_n be where $\bar{S}=\bar{\mathfrak{B}}^n$. Let Z_m^* be where $\bar{\mathfrak{B}}=\bar{S}^m$. By Lemma 1, we find a countable $J\subset D$, $\bar{S}$-invariant and Z_n, $Z_m^*\in B(J)$ for all n, m.

So we have $Z=X\times\Pi I_d$, $d\in D\backslash J$, $\bar{S}=S\times\phi_x$, and $\bar{\mathfrak{B}}=T\times\bar{\mathfrak{R}}$ where $\bar{\mathfrak{R}}=\mathscr{B}_d$, $d\in D\backslash J$ and $T=\Pi\mathscr{B}_d$, $d\in J$. Notice that, T on the Lebesgue space X is again a Bernoulli automorphism. It is of finite entropy if J is finite, otherwise it is the infinite entropy Bernoulli.

Let P_d on I_d be the two-set generating partition of I_d. Let $\bar{P}_d=\pi_d^{-1}(P_d)$, where π_d is the canonical projection on the d coordinate. Put $\bar{P}_0=\bigvee\bar{P}_d$, $d\in D\backslash J$ the σ-algebra generated by the $\bar{P}_d$. The following is straight forward.

i) $\bar{\mathfrak{B}}^k\bar{P}_0$, $-\infty<k<\infty$, are stochastically independent.

ii) $\bar{\mathfrak{B}}^k\bar{P}_0$, $-\infty<k<\infty$, is independent of $\mathbb{B}(J)$.

iii) $\bar{\mathfrak{B}}^k\bar{P}_0\ \mathbb{B}(J)=\mathbb{B}$.

We will show, that conditions i), ii), iii) also hold for $\bar{S}$ in place of $\bar{\mathfrak{B}}$. Thus the factorization suggested by $\mathbb{B}(J)$, and the $\bar{S}^k\bar{P}_0$ will yield a factorization for $\bar{S}$.

Conditions i) and ii) are evident from the previous argument for Theorem 1. We need to show that for any n,

$$\overline{\mathbb{B}P_0}\subset\mathbb{B}(J)\vee\bar{S}^k\bar{P}_0, \qquad -\infty<k<\infty.$$

We start with $\overline{\mathfrak{B}P_0}$. We have $\bar{\mathfrak{B}}=\bar{S}^m$ on $Z_m^*\ \mathbb{B}(J)$. Hence, $\overline{\mathfrak{B}P_0}$ can be pieced together in $\mathbb{B}(J)\vee\bar{S}^k\bar{P}_0$. Next, $\bar{\mathfrak{B}}\in[\bar{S}]$ implies that $\bar{\mathfrak{B}}^n\in[\bar{S}]$, and the elements where $\bar{\mathfrak{B}}^n=\bar{S}^k$ are also in $\mathbb{B}(J)$. Thus, the same reasoning goes through and we are done.

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