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Probability Theory and Related Fields

Periodic Behavior of the Stochastic Brusselator in the Mean-Field Limit

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Summary. We prove a "propagation of chaos" result for the mean-field limit of a model for a trimolecular chemical reaction called "Brusselator". Then we show that the pair of nonlinear (i.e. law-dependent) stochastic differential equations describing the evolution of the concentration of the molecules at a given site in the mean field limit has a solution with a periodic law (in t). Finally we identify the limit and establish a central limit theorem for the periodic law in the case where the noise tends to zero.

1. Introduction

The "Brusselator" is a model for the trimolecular reaction

$$\begin{aligned} A &\to X \\ X+B &\to Y+C \\ 2X+Y &\to 3X \\ X &\to E \end{aligned}$$

describing the evolution of the concentration of the molecules of type X and Y. The concentrations of A and B are assumed to be constant in time (and space). Deterministic and stochastic (ordinary and partial) differential equations as well as Markov jump processes have been used to model the reaction. A nonexhaustive list of papers on such models includes [2, 4, 7, 9, 12, 15, 22].

The name "Brusselator" is due to J.J. Tyson [22] and honours the pioneering work of a group of scientists from the Université Libre in Brussels (among them Prigogine and Nicolis).

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In the well-stirred case and if stochastic fluctuations are neglected the evolution of the concentration of the reactants X and Y can be described by

$$\begin{aligned}\frac{dX(t)}{dt}&=a-(b+1)\,X(t)+X^2(t)\,Y(t)\\ \frac{dY(t)}{dt}&=bX(t)-X^2(t)\,Y(t)\end{aligned}\tag{1}$$

after some appropriate scaling, where a and b are positive constants. (1) has a unique steady state solution given by $X(t)\equiv a$, $Y(t)\equiv\frac{b}{a}$ which is asymptotically stable for $a^2\geqq b-1$ [22] and unstable for $a^2<b-1$. Furthermore if $a^2<b-1$ there exists a unique stable limit cycle surrounding the steady state [15]. Numerical simulations in that case have been carried out by Lefever and Nicolis [9]. The PDE generalization of (1) which models the spatial distribution in addition to the temporal evolution was extensively studied in the book of Nicolis and Prigogine [12]. They point out that the Brusselator is the simplest type of a chemical reaction model exhibiting a certain "interesting" (cooperative) behavior and therefore assign to it the same significance as to the harmonic oscillator as a prototype model. Apart from the PDE model, Nicolis and Prigogine also treat the Markov jump model.

As an alternative to the Markov jump approach, Dawson [2] proposed the following stochastic model in the well-stirred case:

$$\begin{aligned}dX(t)&=(a-(b+1)\,X(t)+X^2(t)\,Y(t))\,dt+g_1(X(t))\,dW_1(t)\\ dY(t)&=(bX(t)-X^2(t)\,Y(t))\,dt+g_2(Y(t))\,dW_2(t).\end{aligned}\tag{2}$$

Here W_1 and W_2 are independent Wiener processes. To include the spatial distribution in the non-well-stirred case without having to study stochastic PDEs Dawson suggested the following model

$$\begin{aligned}dX_{i,N}(t)&=(a-(b+1)\,X_{i,N}(t)+X_{i,N}^2(t)\,Y_{i,N}(t))\,dt+g_1(X_{i,N}(t))\,dW_{1,i}(t)\\ &\quad+D_1\left(\frac{1}{N-1}\sum_{\substack{j=1\\ j\neq i}}^{N}X_{j,N}(t)-X_{i,N}(t)\right)dt\\ dY_{i,N}(t)&=(bX_{i,N}(t)-X_{i,N}^2(t)\,Y_{i,N}(t))\,dt+g_2(Y_{i,N}(t))\,dW_{2,i}(t)\\ &\quad+D_2\left(\frac{1}{N-1}\sum_{\substack{j=1\\ j\neq i}}^{N}Y_{j,N}(t)-Y_{j,N}(t)\right)dt,\qquad i=1,\dots,N\end{aligned}\tag{3}$$

where $X_{i,N}$ and $Y_{i,N}$ denote the concentration of X and Y in the i-th out of N cells (small volumes), D_1 and D_2 are nonnegative constants modelling the diffusion between different cells and $W_{1,i}$, $W_{2,i}$, $i=1,\dots,N$ are independent Wiener processes. Here it is assumed that the proportion of molecules leaving cell i per time unit is proportional to the number of molecules in that cell and that they distribute equally over all other cells. We want to point out that this last assumption is a rather crude approximation in general, but it seems to be the more realistic the more nearest neighbors a fixed cell has i.e. the higher the dimension of the state space.

In this paper we study Eq. (3) in the limit $N \uparrow \infty$ (called the mean-field limit or McKean-Vlasov limit). First we will establish a "propagation of chaos" result for (3) i.e. if $(X_i(0), Y_i(0))$ $i=1,2,\ldots$ are i.i.d, $\mathbb{R}^2$-valued random variables with $\mathscr{L}(X_i(0), Y_i(0))=\mu$, ($\mathscr{L}$ denoting the law), then for any fixed k $(X_{1,N}(\cdot), Y_{1,N}(\cdot)), \ldots, (X_{k,N}(\cdot), Y_{k,N}(\cdot))$ converge in law in the space of probability measures on $C([0,\infty), \mathbb{R}^{2k})$ to k independent copies of the unique process $(X(\cdot), Y(\cdot))$ satisfying the equation

$$\begin{gathered} dX(t)=(a-(b+1)X(t)+X^2(t)Y(t))\,dt+g_1(X(t))\,dW_1(t)+D_1(EX(t)-X(t))\,dt \\ dY(t)=(bX(t)-X^2(t)Y(t))\,dt+g_2(Y(t))\,dW_2(t)+D_2(EY(t)-Y(t))\,dt \\ \mathscr{L}(X(0), Y(0))=\mu. \end{gathered} \tag{4}$$

Equation (4) is called "nonlinear" because (contrary to (3)) the corresponding Fokker-Planck equation is a nonlinear PDE. A number of propagation of chaos results have been proved in the literature [3, 11, 13, 19–21], but none of them covers Eq. (3). It turns out however that Sznitman's method can be extended to prove the result in our case. The hardest part of the proof is the pathwise uniqueness of the solution of Eq. (4) for which we require certain assumptions on the functions g_1 and g_2 as well as on the tails of the initial distribution μ. This is not too surprising since a number of nonlinear equations with coefficients satisfying a local but not a global Lipschitz condition have been shown to have more than one solution [17].

Theorem 3.4 states the main result: under certain assumptions on g_1, g_2, D_1, D_2, a and b there exists an initial law μ such that the law of the solution $(X(t), Y(t))$ of (4) is strictly periodic in t. This shows that chemical reactors interacting according to Eq. (3) are capable of cooperative behavior in the limit $N \to \infty$, whereas it is known that for finite N (3) can never have a periodic law as long as g_1 and g_2 are nondegenerate on $(0, \infty)$ [6]. Simple examples of periodic behavior of nonlinear diffusions can be found in [16]. Writing a factor $\varepsilon > 0$ before the functions g_1 and g_2 in (4), we study the limiting behavior of the periodic solutions as $\varepsilon \downarrow 0$. We identify the limit as the deterministic periodic solution and establish a central limit theorem for the fluctuations. This shows that the Brusselator is a physically motivated example, where noise and periodic behavior are simultaneously present. Even though the model treated here is a particular example, it seems that most of the results are true and can be proved in a similar way for a large class of two-dimensional systems for which the ODE has a stable limit cycle. Furthermore for the Brusselator, all proofs go through if the noise term $\begin{pmatrix} g_1(X(t))\,dW_1(t) \\ g_2(Y(t))\,dW_2(t) \end{pmatrix}$ is replaced by $G(X(t), Y(t)) \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}$ provided the Matrix G satisfies a Lipschitz condition and suitable boundary conditions and G is bounded.

2. Propagation of Chaos

Throughout the paper we will assume the following: a and b are (strictly) positive constants, g_1 and g_2 satisfy a Lipschitz condition with constant K, g_2 is bounded,

$$g_1(0)=g_2(0)=0, \quad D_1, D_2 \geqq 0.$$

Theorem 2.1. *Let μ be a probability measure on $[0,\infty)\times[0,\infty)$ such that*

$$\iint x^4\, d\mu(x,y)<\infty \quad \textit{and} \quad \iint \exp(\gamma y^2)\, d\mu(x,y)<\infty \quad \textit{for some } \gamma>0.$$

a) *For every $N\geqq 2$, $N\in\mathbb{N}$ (3) has a unique strong solution with initial law $\mu^{\otimes N}$. This solution is global (i.e. exists for all $t\geqq 0$) and concentrated on $([0,\infty)\times[0,\infty))^N$.*

b) *Equation (4) has a unique (pathwise and in law) nonnegative solution $(X(t), Y(t))$ satisfying $\mathscr{L}(X(0), Y(0))=\mu$ and $\int_0^t EX(s)+EY(s)\,ds<\infty$ for all $t\geqq 0$.*

c) *(Propagation of chaos.) For every $k\in\mathbb{N}$*

$$((X_{1,N}(\cdot), Y_{1,N}(\cdot)), \ldots, (X_{k,N}(\cdot), Y_{k,N}(\cdot)))$$

converge in law to k independent copies of solutions of Eq. (4) as $N\to\infty$ in the space of probability measures on $C([0,\infty), \mathbb{R}^{2k})$.

Proof. a) Since the drift and diffusion coefficients are locally Lipschitz continuous there exists a unique strong local solution of (3). We show that it is in fact global i.e. cannot explode in finite time.

$$Z_{i,N}(t) := X_{i,N}(t)+Y_{i,N}(t)$$

satisfies

$$\begin{aligned} dZ_{i,N}(t)= \Bigg(a-X_{i,N}(t)+D_1\Bigg(\frac{1}{N-1}\sum_{\substack{j=1\\ j\neq i}}^{N} X_{j,N}(t)-X_{i,N}(t)\Bigg)\\ +D_2\Bigg(\frac{1}{N-1}\sum_{\substack{j=1\\ j\neq i}}^{N} Y_{j,N}(t)-Y_{i,N}(t)\Bigg)\Bigg)\,dt \\ +g_1(X_{i,N}(t))\,dW_{1,i}(t)+g_2(Y_{i,N}(t))\,dW_{2,i}(t). \end{aligned} \tag{5}$$

For $m\in\mathbb{N}$ define $f_m\colon \mathbb{R}^2\to[0,\infty)$ by

$$f_m(x,y)=\begin{cases} x^2 y & \text{if } |x|, |y|\leqq m\\ \text{bounded, Lipschitz, nonnegative otherwise}\end{cases}$$

and $g_{i,m}$ $(i=1,2)$ by

$$g_{i,m}(x)=\begin{cases} g_i(x) & 0\leqq x\leqq m\\ \text{bounded, Lipschitz otherwise}\end{cases}$$

and let $(X_{i,N}^{(m)}(t), Y_{i,N}^{(m)}(t))$ denote the unique global (see [1]) solution of (3) with the terms $X_{i,N}^2(t)\,Y_{i,N}(t)$ replaced by $f_m(X_{i,N}(t), Y_{i,N}(t))$ and g_i replaced by $g_{i,m}$ $(i=1,2)$. It is easy to see that one can approximate $((X_{i,N}^{(m)}, Y_{i,N}^{(m)}),\ i=1,\ldots,N)$ by a sequence of Markov chains with state space $[0, \infty)^{2N}$ which satisfy the assumptions of Theorem 11.2.3 in [18]: let the initial distribution of the Markov chain be $\mu^{\otimes N}$ and the transition probabilities $\pi_h(z, \cdot)$, $h>0$, $z\in[0, \infty)^{2N}$ for $\|z\| \leqq \frac{1}{h}$ be the product of $2N$ uniform distributions with mean $z + h \cdot \mathrm{Drift}(z)$ and variance $hg_1^2(z_i)$ for i odd and $hg_2^2(z_i)$ for i even unless this would lead to jumps out of $[0, \infty)^{2N}$ with positive probability in which case one makes a jump to zero in the corresponding component. Furthermore, for all $\|z\| > \frac{1}{h}$ define $\pi_h(z, \cdot) = \varepsilon_z$ (unit mass in z). Theorem 11.2.3 in [18] says that the law of the (linearly interpolated and time-scaled) Markov chains converge to the solution of the martingale problem associated with the diffusion $(X_{i,N}^{(m)}, Y_{i,N}^{(m)})$. Hence $P\{X_{i,N}^{(m)}(t) \geqq 0,\ Y_{i,N}^{(m)}(t) \geqq 0$ for all $t \geqq 0\} = 1$. It is well-known [10] that the forth moments of $X_{i,N}^{(m)}$ and $Y_{i,N}^{(m)}$ exist and are bounded on finite intervals and hence $\sup_{0\leqq t\leqq T} (Z_{i,N}^{(m)}(t))^4 < \infty$ for every $T>0$. Using (5) with an index m attached to all variables and applying Itô's formula, we get

$$\begin{aligned} E(Z_{i,N}^{(m)}(t))^2 \leqq E(Z_{i,N}^{(m)}(0))^2 + 2\int_0^t a E Z_{i,N}^{(m)}(s) \\ + (D_1 + D_2) \frac{1}{N-1} \sum_{\substack{j=1 \\ j \neq i}}^{N} E Z_{i,N}^{(m)}(s)\, Z_{j,N}^{(m)}(s)\, ds \\ + \int_0^t E(g_1^2(X_{i,N}^{(m)}(s)) + g_2^2(Y_{i,N}^{(m)}(s)))\, ds. \end{aligned}$$

Due to the fact that the law of $(X_{i,N}^{(m)}(t), Y_{i,N}^{(m)}(t))$ is symmetric w.r.t. $i = 1, \ldots, N$ we have

$$\begin{aligned} E(Z_{1,N}^{(m)}(t))^2 \leqq E(Z_{1,N}^{(m)}(0))^2 + 2\int_0^t a(1 + E(Z_{1,N}^{(m)}(s))^2) \\ + (D_1 + D_2 + K^2)\, E(Z_{1,N}^{(m)}(s))^2\, ds. \end{aligned}$$

Applying the Lemma of Bellman and Gronwall (see e.g. [1]) we get

$$\begin{aligned} E(Z_{1,N}^{(m)}(t))^2 \leqq E(Z_{1,N}^{(m)}(0))^2 + 2at \\ + 2(a + D_1 + D_2 + K^2) \int_0^t e^{2(a + D_1 + D_2 + K^2)(t-s)} (E(Z_{1,N}^{(m)}(0))^2 + 2as)\, ds. \end{aligned}$$

Since $E(Z_{1,N}^{(m)}(0))^2 = EZ_{1,N}^2(0)$ and $(X_{1,N}^{(m)}(\cdot), Y_{1,N}^{(m)}(\cdot))$ converge weakly to $(X_{1,N}(\cdot), Y_{1,N}(\cdot))$ as $m \to \infty$ [18] it follows from Fatou's lemma that

$$\sup_{N\geqq 2} \sup_{0\leqq t\leqq T} E(Z_{1,N}(t))^2 < \infty \quad \text{for all } T>0 \tag{6}$$

and also that $P\{X_{i,N}(t) \geqq 0,\ Y_{i,N}(t) \geqq 0$ for all $t \geqq 0\} = 1$ proving part a) of the theorem.

b) Existence of a solution of (4) will follow from the proof of c), so we only show uniqueness. A similar problem has been treated in [5], but the results do not cover Eq. (4).

Let $(X(t), Y(t))$, $t \geqq 0$ be any nonnegative solution of (4) satisfying $\int_0^T EX(s) + EY(s)\,ds < \infty$ for all $T>0$ (otherwise (4) does not make sense), let $a(t) := EX(t)$ and $b(t) := EY(t)$ and let $(X_{(\alpha)}, Y_{(\alpha)})$ be the solution of (4) with $EX(t)$ and $EY(t)$ replaced by $a(t)$ and $b(t)$ respectively and with $h_\alpha(g_1(X(t)))$ instead of $g_1(X(t))$, where, for $\alpha > 0$

$$h_\alpha(x) := \begin{cases} x & |x| \leqq \alpha \\ \alpha & |x| \geqq \alpha \end{cases}, \qquad x \in \mathbb{R}.$$

Define

$$Z(t) := X(t) + Y(t)$$

and

$$Z_{(\alpha)}(t) := X_{(\alpha)}(t) + Y_{(\alpha)}(t).$$

$P\{X_{(\alpha)}(t) \geqq 0,\ Y_{(\alpha)}(t) \geqq 0\}$ follows as in the proof of part a). Then

$$\begin{aligned} Z_{(\alpha)}(t) = Z(0) + at - \int_0^t X_{(\alpha)}(s)\,ds \\ + D_1 \int_0^t a(s) - X_{(\alpha)}(s)\,ds + D_2 \int_0^t b(s) - Y_{(\alpha)}(s)\,ds \\ + \int_0^t h_\alpha(g_1(X_\alpha(s)))\,dW_1(s) + \int_0^t g_2(Y_{(\alpha)}(s))\,dW_2(s) \end{aligned}$$

and

$$EZ_{(\alpha)}(t) \leqq EZ(0) + at + D_1 \int_0^t a(s)\,ds + D_2 \int_0^t b(s)\,ds,$$

so $\sup_{\alpha > 0} \sup_{0 \leqq t \leqq T} EZ_{(\alpha)}(t) < \infty$ and hence

$$\begin{aligned} EZ_{(\alpha)}(t) + D_1 \int_0^t EX_{(\alpha)}(s)\,ds + D_2 \int_0^t EY_{(\alpha)}(s)\,ds \\ \leqq EZ(0) + at + D_1 \int_0^t EX(s)\,ds + D_2 \int_0^t EY(s)\,ds. \end{aligned}$$

Since, for $\alpha \to \infty$, $(X_\alpha(\cdot), Y_\alpha(\cdot))$ converge in law to $(X(\cdot), Y(\cdot))$ in the space of probability measures on $C([0,\infty), \mathbb{R}_+^2)$ according to [18], Theorem 11.1.4, it follows from Fatou's lemma (or dominated convergence) that $EZ(t) \leqq EZ(0) + at$.

For the pathwise uniqueness proof it suffices to show that there exists some $\varepsilon > 0$ and some $\tilde{\gamma} > 0$ such that

$$\sup_{0 \leqq t \leqq 1} E\exp(\tilde{\gamma} Y^2(t)) < \infty \quad \text{and} \quad \sup_{0 \leqq t \leqq 1} EZ^4(t) < \infty \tag{7}$$

for all solutions of (4) and that the solution is pathwise unique on $[0, \varepsilon]$: Define τ as the supremum over all t such that the solution is pathwise unique on $[0, t]$. Since the sample paths are continuous, pathwise uniqueness holds on $[0, \tau]$. Assuming $\tau < 1$, $\mathscr{L}(X(\tau), Y(\tau))$ satisfies the assumptions on the initial law in the theorem (once we have shown (7)) under which we will prove pathwise uniqueness on $[\tau, \tau + \varepsilon']$ which is a contradiction. By iteration we get pathwise uniqueness on $[0, \infty)$.

Step 1. First we show that there exists some $\tilde{\gamma}>0$ such that

$$\sup_{0\leqq t\leqq 1} E\exp(\tilde{\gamma} Y^2(t))<\infty$$

uniformly for all solutions of (4). Theorem 4.7 in [10] states this result for onedimensional diffusions with linearly bounded drift and bounded diffusion coefficient. Although the drift of Y is

$$\begin{aligned} &bX(t)-X^2(t)\,Y(t)+D_2(EY(t)-Y(t))\\ &=\frac{b^2}{4Y(t)}-\left(X(t)\sqrt{Y(t)}-\frac{b}{2\sqrt{Y(t)}}\right)^2+D_2(EY(t)-Y(t))\\ &\leqq\frac{b^2}{4Y(t)}+D_2(EZ(0)+at)\end{aligned}$$

which is unbounded near 0 and even though the drift of Y depends not only on Y, almost the same proof as Liptser-Shiryayev's works here. Applying Itô's formula to Y^n we get

$$\begin{aligned} Y^n(t)=Y^n(0)+\int_0^t nY^{n-1}(s)(bX(s)-X^2(s)\,Y(s)+D_2(EY(s)-Y(s)))\,ds\\ +\int_0^t nY^{n-1}(s)\,g_2(Y(s))\,dW_2(s)+\frac{n(n-1)}{2}\int_0^t Y^{n-2}\,g_2^2(Y(s))\,ds.\end{aligned}$$

Defining $\tau_\alpha:=\inf\{t\geqq 0\colon Z(t)\geqq\alpha\}$, $B:=\sup_{y\geqq 0} g_2^2(y)$ and

$$Y^{(\alpha,n)}(t):=Y^n(t\wedge\tau_\alpha)$$

we get

$$\begin{aligned} EY^{(\alpha,n)}(t)\leqq EY^n(0)+E\int_0^{t\wedge\tau_\alpha} nY^{n-1}(s)\left(\frac{b^2}{4Y(s)}+D_2(EZ(0)+\alpha s)\right)ds\\ +\frac{n(n-1)B}{2}E\int_0^{t\wedge\tau_\alpha} Y^{n-2}(s)\,ds.\end{aligned}$$

Assuming, by introduction, $\sup_{0\leqq t\leqq 1} EY^k(t)<\infty$ for all $1\leqq k\leqq n-1$ and using Fatou's lemma for $\alpha\to\infty$, we get $\sup_{0\leqq t\leqq 1} EY^n(t)<\infty$ and

$$\begin{aligned} EY^{2n}(t)\leqq EY^{2n}(0)+2n\int_0^t EY^{2n-1}(s)\left(\frac{b^2}{4Y(s)}+D_2(EZ(0)+a)\right)ds\\ +Bn(2n-1)\int_0^t EY^{2n-2}(s)\,ds.\end{aligned}$$

Defining $\bar{K}:=\max\left\{\left(B+\frac{b^2}{2}\right)^{1/2},\ D_2(EZ(0)+a)\right\}$ and using $Y^{2n-1}\leqq Y^{2n}+1$ it follows that (for $n\geqq 1$)

$$EY^{2n}(t) \leqq EY^{2n}(0) + 2n\bar{K} + 4n\bar{K}\int_0^t EY^{2n}(s)\,ds + \bar{K}^2 n(2n-1)\int_0^t EY^{2n-2}(s)\,ds$$

which is Eq. (4.142) in [10]. The rest of the proof of Step 1 (comparing the moments $EY^{2n}(t)$ with the moments of the solution of

$$dY(t) = 2\bar{K}Y(t)\,dt + \bar{K}\,dW(t))$$

is exactly the same as that of Theorem 4.7 in [10] and is therefore omitted.

Step 2. We show that $\sup_{0 \leqq t \leqq 1} EZ^4(t) < \infty$ uniformly for all solutions of (4). Let $X(t)$, $Y(t)$ be any solution of (4). As before, let $\tau_\alpha := \inf\{t \geqq 0\colon Z(t) \geqq \alpha\}$. Then, for $i = 2, 3, 4$ and $0 \leqq t \leqq 1$

$$\begin{aligned} Z^i(t) = Z^i(0) &+ i\int_0^t Z^{i-1}(s)(a - X(s) + D_1(EX(s) - X(s)) + D_2(EY(s) - Y(s)))\,ds \\ &+ \int_0^t iZ^{i-1}(s)\,g_1(X(s))\,dW_1(s) + \int_0^t iZ^{i-1}(s)\,g_2(Y(s))\,dW_2(s) \\ &+ \tfrac{1}{2}i(i-1)\int_0^t Z^{i-2}(s)(g_1^2(X(s)) + g_2^2(Y(s)))\,ds \end{aligned}$$

and, for $Z^{(\alpha,i)} := Z^i(t \wedge \tau_\alpha)$, and assuming $\sup_{0 \leqq s \leqq 1} EZ^k(s) < \infty$ for all $k \leqq i-1$ by induction

$$\begin{aligned} EZ^{(\alpha,i)}(t) \leqq EZ^i(0) &+ 4\int_0^t EZ^{i-1}(s)(a + (D_1 + D_2)(EZ(0) + a))\,ds \\ &+ 6K^2\int_0^t EZ^{(\alpha,i)}(s)\,ds \end{aligned}$$

due to the general assumptions stated at the beginning of this section. Using Gronwall's lemma and then Fatou's lemma for $\alpha \to \infty$, it follows by induction over i that

$$\sup_{0 \leqq t \leqq 1} EZ^4(t) < \infty.$$

Step 3. Let us now prove pathwise uniqueness on some interval $[0, \varepsilon]$ with $0 < \varepsilon \leqq 1$. Let $(X(t), Y(t))$ and $(\tilde{X}(t), \tilde{Y}(t))$ be two solutions of (4) on the same probability space with the same initial condition. We already proved that

$$\sup_{0 \leqq t \leqq 1} (EX^4(t) + EY^4(t) + E\tilde{X}^4(t) + E\tilde{Y}^4(t)) < \infty.$$

Let $c(s) := EY(s) - E\tilde{Y}(s)$,

$$d(s) := EX(s) - E\tilde{X}(s), \qquad \bar{X}(s) = X(s) - \tilde{X}(s), \qquad \bar{Y}(s) = Y(s) - \tilde{Y}(s)$$

and $\bar{Z}(s) = Z(s) - \tilde{Z}(s)$. Unfortunately the usual proof via Gronwall's lemma cannot be employed here due to the lack of a global Lipschitz constant. Instead we will formulate and use a "nonlinear" Gronwalltype estimate. Applying Itô's formula we get

$$\begin{aligned} d\bar{Y}^2(t) &= 2\bar{Y}(t)\,d\bar{Y}(t) + (g_2(Y(t)) - g_2(\tilde{Y}(t)))^2\,dt \\ &= 2\bar{Y}(t)(b\bar{X}(t) + \tilde{X}^2(t)\,\tilde{Y}(t) - X^2(t)\,Y(t))\,dt \\ &\quad + 2\bar{Y}(t)(g_2(Y(t)) - g_2(\tilde{Y}(t)))\,dW_2(t) \\ &\quad + 2\bar{Y}(t)(D_2\,c(t) - D_2\,\bar{Y}(t))\,dt \\ &\quad + (g_2(Y(t)) - g_2(\tilde{Y}(t)))^2\,dt. \end{aligned}$$

So

$$\begin{aligned} E\bar{Y}^2(t) \leqq{}& b\int_0^t E\bar{X}^2(s) + E\bar{Y}^2(s)\,ds + 2\int_0^t E\bar{Y}(s)(\tilde{X}^2(s)\,\tilde{Y}(s) - X^2(s)\,Y(s))\,ds \\ &+ 2D_2\int_0^t c^2(s)\,ds - 2D_2\int_0^t E\bar{Y}^2(s)\,ds + K^2\int_0^t E\bar{Y}^2(s)\,ds. \end{aligned}$$

Now

$$c^2(s) \leqq E\bar{Y}^2(s)$$

and therefore

$$E\bar{Y}^2(t) \leqq (b+K^2)\int_0^t E\bar{X}^2(s) + E\bar{Y}^2(s)\,ds + 2\int_0^t E\bar{Y}(s)(\tilde{X}^2(s)\,\tilde{Y}(s) - X^2(s)\,Y(s))\,ds.$$

Now for x, $\tilde{x}$, y, $\tilde{y} \geqq 0$

$$\begin{aligned} (y-\tilde{y})(\tilde{x}^2\,\tilde{y} - x^2\,y) &= (y-\tilde{y})[-(y-\tilde{y})\,x^2 + 2(\tilde{x}-x)\,x\tilde{y} + (\tilde{x}-x)^2\,\tilde{y}] \\ &= -(y-\tilde{y})^2\left(x - \frac{\tilde{x}-x}{y-\tilde{y}}\,\tilde{y}\right)^2 + (\tilde{x}-x)^2\,\tilde{y}y \\ &\leqq (\tilde{x}-x)^2 \cdot M^2, \end{aligned}$$

where $M := \max\{y, \tilde{y}\}$. Hence

$$E\bar{Y}^2(t) \leqq (b+K^2)\int_0^t E\bar{X}^2(s) + E\bar{Y}^2(s)\,ds + 2\int_0^t E\bar{X}^2(s)\,M^2(s)\,ds,$$

where $M(s) = \max\{Y(s), \tilde{Y}(s)\}$. Furthermore

$$\begin{aligned} d\bar{Z}^2(t) = 2\bar{Z}(t)[&-\bar{X}(t)\,dt + (g_1(X(t)) - g_1(\tilde{X}(t)))\,dW_1(t) \\ &+ (g_2(Y(t)) - g_2(\tilde{Y}(t)))\,dW_2(t) \\ &+ (D_1\,d(t) + D_2\,c(t) - D_1\,\bar{X}(t) - D_2\,\bar{Y}(t))\,dt] \\ &+ [(g_1(X(t)) - g_2(\tilde{X}(t)))^2 + (g_2(Y(t)) - g(\tilde{Y}(t)))^2]\,dt. \end{aligned}$$

Therefore

$$\begin{aligned} E\bar{Z}^2(t) \leqq{}& -2\int_0^t E\bar{X}^2(s)\,ds - 2\int_0^t E\bar{Y}(s)\,\bar{X}(s)\,ds \\ &+ 2D_1\int_0^t d^2(s)\,ds + 2(D_1 + D_2)\int_0^t d(s)\,c(s)\,ds \\ &+ 2D_2\int_0^t c^2(s)\,ds - 2D_1\int_0^t E\bar{X}^2(s)\,ds - 2(D_1+D_2)\int_0^t E\bar{X}(s)\,\bar{Y}(s)\,ds \\ &- 2D_2\int_0^t E\bar{Y}^2(s)\,ds + K^2\int_0^t E\bar{X}^2(s) + E\bar{Y}^2(s)\,ds \\ \leqq{}& (-1 + 2D_1 + 2D_2 + K^2)\int_0^t E\bar{X}^2(s)\,ds + (1 + 2D_1 + 2D_2 + K^2)\int_0^t E\bar{Y}^2(s)\,ds. \end{aligned}$$

Denoting

$$D(s) := E\bar{X}^2(s) + E\bar{Y}^2(s)$$

we have

$$D(t) = E(\bar{Z}(t) - \bar{Y}(t))^2 + E\bar{Y}^2(t) \leqq 2E\bar{Z}^2(t) + 3E\bar{Y}^2(t)$$
$$\leqq \kappa_1 \int_0^t D(s)\,ds + 6\int_0^t EM^2(s)\,\bar{X}^2(s)\,ds$$

where $\kappa_1 = 2 + 4D_1 + 4D_2 + 5K^2 + 3b$.

For $n > 0$ define $q = 1 + \frac{1}{n}$ and $p = n+1$. Then $\frac{1}{p} + \frac{1}{q} = 1$. Using Hölder's inequality twice yields for $s \leqq 1$

$$\begin{aligned} EM^2(s)\,\bar{X}^2(s) &= EM^2(s)\,\bar{X}^{2(q-1)/q}(s)\,\bar{X}^{2/q}(s) \\ &\leqq (EM^{2p}(s)\,\bar{X}^2(s))^{1/p}(E\bar{X}^2(s))^{1/q} \\ &\leqq (EM^{4p}(s))^{1/2p}(E\bar{X}^4(s))^{1/2p}(E\bar{X}^2(s))^{1/q} \\ &\leqq \kappa_2\,\alpha_p (E\bar{X}^2(s))^{1/q} \end{aligned}$$

where

$$\kappa_2 := \sup_{0 \leqq s \leqq 1} (E\bar{X}^4)^{1/2} + 1 < \infty \quad \text{and} \quad \alpha_p := \sup_{0 \leqq s \leqq 1} (EM^{4p}(s))^{1/2p}.$$

We need the following lemma, the proof of which is similar to that of Gronwall's lemma (see e.g. [10], Lemma 4.15).

Lemma. *Let* $1 > m > 0$, $A \geqq 0$, $T > 0$ *and* $f\colon [0, T] \to \mathbb{R}$ *be a nonnegative continuous function satisfying*

$$f(t) \leqq A \int_0^t f^m(s)\,ds \qquad \text{for all } t \in [0, T].$$

Then $f(t) \leqq [A(1-m)\,t]^{1/(1-m)}$ *on* $[0, T]$.

Proof. Define $z(t) = \int_0^t f^m(s)\,ds$. Then

$$\frac{dz(t)}{dt} = f^m(t) \leqq (A\,z(t))^m, \qquad z(0) = 0.$$

Let $v(\cdot)$ be the maximal solution (because of nonuniqueness at 0) of

$$\frac{dv(t)}{dt} = (A\,v(t))^m, \qquad v(0) = 0. \tag{8}$$

Then for all $t \in [0, T]$

$$z(t) \leqq v(t) = ((1-m)\,A^m\,t)^{1/(1-m)}$$

which implies

$$f(t) = \left(\frac{dz(t)}{dt}\right)^{1/m} \leqq A\,z(t) \leqq A\,v(t) = [A(1-m)\,t]^{1/(1-m)}$$

which proves the lemma.

We proved for $t \in [0, 1]$

$$D(t) \leqq \kappa_1 \int_0^t D(s)\,ds + 6\kappa_2 \alpha_p \int_0^t (D(s))^{\frac{n}{n+1}}\,ds.$$

$D(\cdot)$ is continuous since the forth moments of Z and $\tilde{Z}$ are bounded on $[0, 1]$. Choosing $1 \geqq \varepsilon_1 > 0$ such that $D(s) \leqq 1$ for $s \in [0, \varepsilon_1]$ and applying the lemma with $m = \frac{n}{n+1}$ we get

$$0 \leqq D(t) \leqq \left[(\kappa_1 + 6\kappa_2 \alpha_{n+1}) \frac{1}{n+1} t\right]^{n+1} \quad \text{for } 0 \leqq t \leqq \varepsilon_1.$$

Obviously there exists some $\varepsilon_1 \geqq \varepsilon_2 > 0$ and a sequence $(n_k)_{k \in \mathbb{N}} \xrightarrow[k \to \infty]{} \infty$ such that the upper bound converges to 0 as $n_k \to \infty$ uniformly in $t \in [0, \varepsilon_2]$ iff

$$\liminf_{n \to \infty} \frac{\kappa_1 + 6\kappa_2 \alpha_{n+1}}{n+1} < \infty.$$

This is true exactly if

$$\liminf_{n \to \infty} n^{-1} \sup_{0 \leqq s \leqq \varepsilon_2} (EM^{2n}(s))^{1/n} < \infty.$$

Under the assumptions of the theorem, using Chebychev's inequality, for $s \in [0, \varepsilon_2]$

$$P\{M(s) \geqq \delta\} \leqq P\{Y(s) \geqq \delta\} + P\{\tilde{Y}(s) \geqq \delta\} \leqq \kappa_3 e^{-\tilde{\gamma}\delta^2}$$

where

$$\kappa_3 = \sup_{t \in [0, \varepsilon_2]} E \exp(\tilde{\gamma} Y^2(t)) + \sup_{t \in [0, \varepsilon_2]} E \exp(\tilde{\gamma} Y^2(t)) < \infty.$$

Hence for $n \in \mathbb{N}$

$$\begin{aligned}
\frac{1}{n} (EM(s)^{2n})^{1/n} &= \frac{1}{n} \left(\int_0^\infty P\{M(s)^{2n} \geqq \delta\}\,d\delta\right)^{1/n} \\
&\leqq \frac{1}{n} \left(\kappa_3 \int_0^\infty \exp(-\tilde{\gamma}\delta^{1/n})\,d\delta\right)^{1/n} \\
&= \frac{1}{n} \left(\kappa_3 \int_0^\infty e^{-\tilde{\gamma}x} n x^{n-1}\,dx\right)^{1/n} \\
&= \frac{1}{n} (\kappa_3 n!\, \tilde{\gamma}^{-n})^{1/n}
\end{aligned}$$

which is bounded since $n! \leqq n^n$ for $n \in \mathbb{N}$. So we have proved pathwise uniqueness on $[0, \varepsilon_2]$ and hence on $[0, \infty)$.

Step 4. Let us show that pathwise uniqueness implies uniqueness in law. Let $X(t)$, $Y(t)$ be a solution of (4) on some probability space and denote $a(t) := EX(t)$, $b(t) := EY(t)$. Since the coefficients are locally Lipschitz continuous it follows that the solution is strong i.e. it is measurable w.r.t. the filtration generated by the Wiener process. The results of Yamada and Watanabe [24] show that $(X(t), Y(t))$ is the only weak solution of (4) satisfying $EX(t)=a(t)$ and $EY(t)=b(t)$. If $(\tilde{X}, \tilde{Y})$ is a solution on some other space with $E\tilde{X}(t)=\tilde{a}(t)$, $E\tilde{Y}(t)=\tilde{b}(t)$, then it is also a strong solution which can be realized on the same space as (X, Y). It follows from Step 3 that $a(t)=\tilde{a}(t)$ and $b(t)=\tilde{b}(t)$ and therefore uniqueness in law.

c) For the solution $X_{i,N}$, $Y_{i,N}$ of Eq. (3), let us study $\frac{1}{N}\sum_{i=1}^{N} \varepsilon_{X_{i,N}(\cdot), Y_{i,N}(\cdot)}$ which we look upon as a random element of the space $M:=M(C([0,\infty), \mathbb{R}_+^2))$ of probability measures on the space $C([0,\infty), \mathbb{R}_+^2)$ equipped with the topology induced by the metric

$$\rho(f,g)=\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{0\leqq t\leqq n} |f(t)-g(t)|}{1+\sup_{0\leqq t\leqq n} |f(t)-g(t)|}$$

which makes $C([0,\infty), \mathbb{R}_+^2)$ and hence M (in a canonical way) a Polish space. Here ε_a denotes the measure

$$\varepsilon_a(A)=\begin{cases} 1 & a\in A \\ 0 & \text{otherwise.}\end{cases}$$

Let P_N denote the law of $\frac{1}{N}\sum_{i=1}^{N} \varepsilon_{X_{i,N}(\cdot), Y_{i,N}(\cdot)}$. So P_N is a probability measure on the space M.

Step 1. $\{P_N, N\in\mathbb{N}, N\geqq 2\}$ is relatively compact. Due to Lemma 3.2 in [19] and the remark following it, it is enough to show that $(\tilde{P}_N := \mathscr{L}(X_{1,N}, Y_{1,N}))_{N=2,3,\ldots}$ is a tight family of probability measures on $C([0,\infty), \mathbb{R}_+^2)$. Since $\mathscr{L}(X_{1,N}(0), Y_{1,N}(0))=\mu$ is independent of N it remains to show [18, Theorem 1.3.2] that for every $T>0$ and $\rho>0$

$$\lim_{\delta\downarrow 0} \sup_{N\geqq 2} \tilde{P}_N\{\sup_{\substack{0\leqq s<t\leqq T\\ t-s<\delta}} |X(t)-X(s)|+|Y(t)-Y(s)|>\rho\}=0.$$

Define

$$\bar{X}_N(t):=\frac{1}{N}\sum_{i=1}^{N} X_{i,N}(t), \qquad \bar{Y}_N(t):=\frac{1}{N}\sum_{i=1}^{N} Y_{i,N}(t) \quad \text{and} \quad \bar{Z}_N(t):=\bar{X}_N(t)+\bar{Y}_N(t).$$

To get estimates on the tails of $\sup_{0\leqq t\leqq T} \bar{Z}_N(t)$ let us show that

$$\bar{Z}_N(t)+\int_0^t \bar{Z}_N(s)\,ds$$

is a submartingale. Obviously

$$\bar{Z}_N(t)=\bar{Z}_N(0)+\int_0^t a-\bar{X}_N(s)\,ds+\int_0^t \frac{1}{N}\sum_{i=1}^{N} g_1(X_{i,N}(s))\,dW_{1,i}(s)$$
$$+\int_0^t \frac{1}{N}\sum_{i=1}^{N} g_2(Y_{i,N}(s))\,dW_{2,i}(s).$$

Since the second moments of the integrands of the stochastic integrals are bounded on $[0, T]$ it follows that

$$\bar{Z}_N(t) - \bar{Z}_N(0) - \int_0^t a - \bar{X}_N(s)\, ds$$

is a martingale and hence

$$\bar{Z}_N(t) + \int_0^t \bar{Z}_N(s)\, ds$$

is a nonnegative submartingale. The submartingale inequality [18] yields for $\alpha > 0$

$$\begin{aligned} P_N\{\sup_{0 \leqq t \leqq T} \bar{Z}_N(t) \geqq \alpha\} &\leqq P_N\left\{\sup_{0 \leqq t \leqq T} \left(\bar{Z}_N(t) + \int_0^t \bar{Z}_N(s)\, ds\right) \geqq \alpha\right\} \\ &\leqq \frac{1}{\alpha} E\left(\bar{Z}_N(T) + \int_0^T \bar{Z}_N(s)\, ds\right) \\ &\leqq \frac{1}{\alpha}\left((T+1)\, E\bar{Z}_N(0) + a\left(T + \frac{T^2}{2}\right)\right), \end{aligned}$$

where the last expression is independent of N. Furthermore

$$\begin{aligned} Z_{1,N}(t) - Z_{1,N}(0) - \int_0^t a - X_{1,N}(s) + D_1\left(\frac{1}{N-1}\sum_{i=2}^N X_{i,N}(s) - X_{1,N}(s)\right) \\ + D_2\left(\frac{1}{N-1}\sum_{i=2}^N Y_{i,N}(s) - Y_{1,N}(s)\right) ds \end{aligned}$$

is a martingale which implies that

$$Z_{1,N}(t) + (1 + D_1 + D_2) \int_0^t Z_{1,N}(s)\, ds$$

is a nonnegative submartingale. As before it follows that

$$\begin{aligned} &P_N\{\sup_{0 \leqq t \leqq T} Z_{1,N}(t) \geqq \alpha\} \\ &\quad \leqq \frac{1}{\alpha}\left((1 + T(1 + D_1 + D_2))\, EZ_{1,N}(0) + a\left(T + \frac{T^2}{2}(1 + D_1 + D_2)\right)\right). \end{aligned}$$

So we have shown that for every $T > 0$

$$\lim_{\alpha \uparrow \infty} \sup_{N \geqq 2} P_N\{\sup_{0 \leqq t \leqq T} \bar{Z}_N(t) \geqq \alpha\} = 0 \tag{9}$$

and

$$\lim_{\alpha \uparrow \infty} \sup_{N \geqq 2} P_N\{\sup_{0 \leqq t \leqq T} Z_{1,N}(t) \geqq \alpha\} = 0. \tag{10}$$

Define

$$\tau_{\alpha,N} := \inf\{t \geqq 0 \mid \bar{Z}_N(t) \geqq \alpha \text{ or } Z_{1,N}(t) \geqq \alpha\}$$

and $\tau_{\alpha,N}=\infty$ if such a t fails to exist. Obviously for any $T>0$, $\rho>0$, $\delta>0$ and $\alpha>0$

$$\begin{aligned}\tilde{P}_N\{\sup_{\substack{0\leqq s<t\leqq T\\ t-s<\delta}} |X(t)-X(s)|+|Y(t)-Y(s)|>\rho\} &\leqq P_N\{\tau_{\alpha,N}\leqq T\}\\ &+P_N\{\sup_{\substack{0\leqq s<t\leqq T\\ t-s<\delta}} |X_{1,N}(t\wedge\tau_{\alpha,N})-X_{1,N}(s\wedge\tau_{\alpha,N})|\\ &+|Y_{1,N}(t\wedge\tau_{\alpha,N})-Y_{1,N}(s\wedge\tau_{\alpha,N})|>\rho\}.\end{aligned}$$

Let $\varepsilon>0$ be given. Due to (9) and (10) we can choose some $\alpha>0$ such that $P\{\tau_{\alpha,N}\leqq T\}<\frac{\varepsilon}{2}$ for all $N\geqq 2$. So it suffices to prove that the laws of the processes $(X_{1,N}(t\wedge\tau_{\alpha,N}), Y_{1,N}(t\wedge\tau_{\alpha,N}))$ are tight, but this follows immediately from Theorem 1.4.6 in [18] since the drift and diffusion coefficients of the stopped processes are uniformly bounded for all $N\geqq 2$.

Step 2. Since $(P_N)_{N\geqq 2}$ is a tight sequence of probability measures on M it has a limit point P_∞ on M. We show that there exists a set $\bar{M}\subset M$ such that $P_\infty(\bar{M})=1$ and every $m\in\bar{M}$ solves the (nonlinear) martingale problem associated with Eq. (4). Since we already proved pathwise uniqueness and uniqueness in law, it follows that $\bar{M}$ has exactly one element [18, Corollary 8.1.6]. The idea of the proof is taken from [20], but it requires some modification because our assumptions are different.

Let $f\in C_0^\infty(\mathbb{R}^2)$, $p\in\mathbb{N}$ and $\bar{g}_1,\dots,\bar{g}_p$ be continuous and bounded functions from $\mathbb{R}^2$ to $\mathbb{R}$ and let $0\leqq s_p<\dots<s_1\leqq s<t$. Define

$$M_0:=\{m\in M:\sup_{0\leqq u\leqq t}\int x(\omega,u)+y(\omega,u)\,dm(\omega)<\infty\}$$

and for $m\in M_0$

$$\begin{aligned}F(m):=\Big\langle m, \Big(&f(x(t),y(t))-f(x(s),y(s))\\ &-\int_s^t\int Lf(x(\cdot,u),y(\cdot,u),x(\omega,u),y(\omega,u))\,m(d\omega)\,du\Big)\\ &\cdot\prod_{j=1}^p \bar{g}_j(x(\cdot,s_j),y(\cdot,s_j))\Big\rangle\end{aligned}$$

where

$$\begin{aligned}Lf(x,y,x',y')=&\frac{\partial f}{\partial x}(x,y)(a-(b+1)x+x^2y+D_1(x'-x))\\ &+\frac{\partial f}{\partial y}(x,y)(bx-x^2y+D_2(y'-y))+\frac{1}{2}g_1^2(x)\frac{\partial^2 f}{\partial x^2}(x,y)\\ &+\frac{1}{2}g_2^2(y)\frac{\partial^2 f}{\partial y^2}(x,y).\end{aligned}$$

Let $(N_k)_{k=1,2,\dots}$ be a sequence such that $P_{N_k}\xrightarrow[k\to\infty]{} P_\infty$ weakly. Obviously $F^2<\infty$ on M_0 and the P_{N_k} are concentrated on the set M_0. Also $P_\infty(M_0)=1$ due to (9).

Defining

$$\bar{X}(u)=\frac{1}{N_k}\sum_{i=1}^{N_k} X_{i,N_k}(u) \quad \text{and} \quad \bar{Y}(u)=\frac{1}{N_k}\sum_{i=1}^{N_k} Y_{i,N_k}(u)$$

it follows that

$$\begin{aligned}
\int_{M_0} F^2(m)\,dP_{N_k}(m) &= E_{N_k}\Bigg(\frac{1}{N_k}\sum_{i=1}^{N_k}\Bigg(f(X_i(t),Y_i(t))-f(X_i(s),Y_i(s)) \\
&\quad -\int_s^t Lf(X_i(u),Y_i(u),\bar{X}(u),\bar{Y}(u))\,du\Bigg)\prod_{j=1}^{p}\bar{g}_j(X_i(s_j),Y_i(s_j))\Bigg)^2 \\
&= E_{N_k}\frac{N_k-1}{N_k}\prod_{i=1}^{2}\Bigg(\Bigg(f(X_i(t),Y_i(t))-f(X_i(s),Y_i(s)) \\
&\quad -\int_s^t Lf(X_i(u),Y_i(u),\bar{X}(u),\bar{Y}(u))\,du\Bigg)\prod_{j=1}^{p}\bar{g}_j(X_i(s_j),Y_i(s_j))\Bigg) \\
&\quad +\frac{1}{N_k}E_{N_k}\Bigg(\Bigg(f(X_1(t),Y_1(t))-f(X_1(s),Y_1(s)) \\
&\quad -\int_s^t Lf(X_1(u),Y_1(u),\bar{X}(u),\bar{Y}(u))\,du\Bigg)\prod_{j=1}^{p}\bar{g}_j(X_1(s_j),Y_1(s_j))\Bigg)^2.
\end{aligned}$$

The last (quadratic) term is $o(1)$ as $k\to\infty$. Furthermore, for $1\leqq i\leqq N_k$

$$\begin{aligned}
H_i(\tau) &:= f(X_i(\tau),Y_i(\tau))-f(X_i(0),Y_i(0))-\int_0^\tau Lf(X_i(u),Y_i(u),\bar{X}(u),\bar{Y}(u))\,du \\
&\quad +\int_0^\tau \frac{\partial f}{\partial x}(X_i(u),Y_i(u))\,D_1\Bigg(\bar{X}(u)-\frac{1}{N_k-1}\sum_{\substack{j=1\\ j\neq i}}^{N_k} X_j(u)\Bigg) \\
&\quad +\frac{\partial f}{\partial y}(X_i(u),Y_i(u))\,D_2\Bigg(\bar{Y}(u)-\frac{1}{N_k-1}\sum_{\substack{j=1\\ j\neq i}}^{N_k} Y_j(u)\Bigg)\,du
\end{aligned}$$

are P_{N_k}-martingales and $\langle H_i, H_j\rangle=0$ for $i\neq j$. Using

$$\bar{X}(u)-\frac{1}{N_k-1}\sum_{j=2}^{N_k} X_j(u)=\frac{1}{N_k-1}(X_1(u)-\bar{X}(u))$$

(the corresponding equality holds for Y too) and the fact that the second moments $E_{N_k}(\bar{X}(u)+\bar{Y}(u))^2$ and $E_{N_k}(X_1(u)+Y_1(u))^2$ are bounded on $[0,t]$ uniformly for all $N\geqq 2$ (see (6)) it follows that

$$\lim_{k\to\infty}\int_{M_0} F^2(m)\,dP_{N_k}(m)=0.$$

We want to show that $\int_M F^2(m)\,dP_\infty(m)=0$. This does not follow directly since F is neither bounded nor continuous on M_0. For $\alpha>0$ define $h_\alpha\colon[0,\infty)\to\mathbb{R}$ by

$$h_\alpha(x)=\begin{cases} x & x\leqq\alpha \\ \alpha & x\geqq\alpha\end{cases}$$

and $F_\alpha(m)$ like $F(m)$ but with x' and y' replaced by $h_\alpha(x')$ and $h_\alpha(y')$ respectively in the definition of Lf. Note that F_α is bounded and continuous on M. We will proceed as follows:

(i) $\lim_{k\to\infty} \int_{M_0} F^2(m)\,dP_{N_k}(m)=0$

(ii) $\lim_{\alpha\to\infty} \sup_k \left| \int_{M_0} F_\alpha^2(m)\,dP_{N_k}(m) - \int_{M_0} F^2(m)\,dP_{N_k}(m)\right| = 0$

(iii) $\lim_{k\to\infty} \int_M F_\alpha^2(m)\,dP_{N_k}(m) = \int_M F_\alpha^2(m)\,dP_\infty(m)$

(iv) $\lim_{\alpha\to\infty} \int_M F_\alpha^2(m)\,dP_\infty(m) \geqq \int_M F^2(m)\,dP_\infty(m)$.

We already proved (i). (iii) follows from the definition of weak convergence. Note that for $m\in M_0$

$$F_\alpha(m)=F(m)+\Big\langle m, \int_s^t \int \frac{\partial f}{\partial x}(x(\cdot,u),y(\cdot,u)) \cdot D_1(x(\omega,u)-h_\alpha(x(\omega,u)))+\frac{\partial f}{\partial y}(x(\cdot,u),y(\cdot,u)) \cdot D_2(y(\omega,u)-h_\alpha(y(\omega,u)))\,dm(\omega)\,du \prod_{j=1}^p \bar{g}_j(x(\cdot,s_j),y(\cdot,s_j))\Big\rangle$$

which implies (iv) due to Fatou's lemma.

A calculation similar to the one before shows (ii) provided one is able to prove that

$$\lim_{\alpha\to\infty} \sup_{0\leqq u\leqq t} \sup_k E_{N_k}(X_1(u)-h_\alpha(X_1(u)))^2=0$$

and the corresponding result for Y_1 (all other required estimates reduce to these if one exploits the symmetry of the law and Hölder's inequality). To show this, note that $\sup_{N\geqq 2} \sup_{0\leqq u\leqq t} EZ_{1,N}^4(u)<\infty$ which we did not prove, but which follows in the same way as the corresponding result for the second moment in the proof of part a) of the theorem if, in addition, one employs a stopping argument like in Step 2 of the proof of part b). Then

$$E(X_1(u)-h_\alpha(X_1(u)))^2 \leqq E\,\frac{X_1^2(u)}{\alpha^2}(X_1(u)-h_\alpha(X_1(u)))^2 \leqq \frac{1}{\alpha^2}\,EX_1^4(u)$$

which proves (ii).

(i)–(iii) imply

$$\lim_{\alpha\to\infty} \int_M F_\alpha^2(m)\,dP_\infty(m)=0.$$

So (iv) gives $\int_M F^2(m)\,dP_\infty(m)=0$ and hence $F(m)=0$ P_∞-a.s.

Note that P_∞-a.s. the projection of m at $t=0$ is equal to μ by the law of large numbers. Therefore P_∞-a.a. m solve the nonlinear martingale problem associated with (4). By Lemma 3.1 in [19] this implies propagation of chaos and so part c) is proved. □

Remark. The theorem remains true if the initial laws μ_N of (3) are not necessarily $\mu^{\otimes N}$, but only assumed to be symmetric and "μ-chaotic" i.e.

$$\langle \mu_N, \phi_1 \otimes \ldots \otimes \phi_k \otimes 1 \ldots \otimes 1 \rangle \xrightarrow[N\to\infty]{} \prod_{i=1}^{k} \langle \mu, \phi_i \rangle$$

for all $\phi_1, \ldots, \phi_k \in C_b([0, \infty) \times [0, \infty))$ and if the marginals $\mu^{(N)}$ satisfy the moment conditions of Theorem 2.1 uniformly in N. Note that the only place we made use of the independence was the very last part of the proof, which is obviously true for symmetric and μ-chaotic initial conditions μ_N.

3. Periodic Behavior of the Brusselator

Our next aim is to show that (4) has a strictly periodic law in t if $a^2 < b-1$ (in this case Eq. (1) has a stable limit cycle) for a suitable initial law μ^*. The main idea of the proof will be an application of Tihonov's fixed point theorem. For this we have to find a suitable weakly compact subset $\mathcal{M}$ of the probability measures on $[0, \infty)^2$ such that for any solution $(X(t), Y(t))$ of (4), $\mathcal{L}(X(0), Y(0)) \in \mathcal{M}$ implies $\mathcal{L}(X(t), Y(t)) \in \mathcal{M}$ for certain $t>0$ to be defined later. This requires uniform estimates of the moments of the solution of (4) which will be established in the following lemmas. We will always assume that initial laws μ satisfy the moment conditions stated in Theorem 2.1.

Lemma 3.1. *Fix* $T>0$, $c>0$, $\bar{c}>0$, $w_1 \geqq 0$, $w_2 \geqq 0$ *and define*

$$\tilde{Z}(t) := g(X(t)) + Y(t)$$

where

$$M := \bar{c} + aT + 1$$

and

$$g(x) := \begin{cases} M & 0 \leqq x \leqq M-1 \\ x & x \geqq M+1 \\ \text{arbitrary otherwise, but such that } g \in C^2[0, \infty) \text{ and } 0 \leqq \dfrac{dg}{dx} \leqq 1. \end{cases}$$

Then for all $n \in \mathbb{N}$

$$M_n := \sup_{EZ(0) \leqq \bar{c}} \; \sup_{E\tilde{Z}^n(0) \leqq c} \; \sup_{D_1 \geqq 0} \; \sup_{w_1 D_1 + w_2 \geqq D_2 \geqq 0} \; \sup_{0 \leqq t \leqq T} E\tilde{Z}^n(t) < \infty.$$

Proof. By Itô's lemma, for $0 \leqq t \leqq T$

$$\begin{aligned} d\tilde{Z}^n(t) &= n\tilde{Z}^{n-1}(t)(dY(t) + g'(X(t))\,dX(t) + \tfrac{1}{2} g''(X(t))\, g_1^2(X(t))\,dt) \\ &\quad + \tfrac{1}{2} n(n-1)\, \tilde{Z}^{n-2}(t)(g_2^2(Y(t)) + g'(X(t))^2\, g_1^2(X(t)))\,dt. \end{aligned}$$

Assume $E\tilde{Z}^n(0) \leqq c$ and $EZ(0) \leqq \bar{c}$.

Let $\tau^{(\alpha)} := \inf\{t \geqq 0 \colon \tilde{Z}(t) \geqq \alpha\}$ and let

$$1_\alpha(t) := \begin{cases} 0 & \text{if } \sup\limits_{0 \leqq s \leqq t} \tilde{Z}(s) \geqq \alpha \\ 1 & \text{otherwise} \end{cases}$$

and

$$\tilde{Z}_{(\alpha)}(t) := \tilde{Z}(t \wedge \tau_\alpha).$$

Then

$$\begin{aligned} E1_\alpha(t)\,\tilde{Z}^n(t) \leqq E\tilde{Z}^n_{(\alpha)}(t) = E\tilde{Z}^n(0) + n\int_0^t E1_\alpha(s)\,\tilde{Z}^{n-1}(s) \\ \cdot\,[bX(s) - X^2(s)\,Y(s) + D_2(EY(s) - Y(s)) + g'(X(s))(a-(b+1)\,X(s) \\ + X^2(s)\,Y(s) + D_1(EX(s) - X(s))) + \tfrac{1}{2}g''(X(s))\,g_1^2(X(s))]\,ds \\ + \tfrac{1}{2}n(n-1)\int_0^t E1_\alpha(s)\,\tilde{Z}^{n-2}(s)(g_2^2(Y(s)) + g'(X(s))^2\,g_1^2(X(s)))\,ds. \end{aligned} \tag{11}$$

Case 1. $D_1 \geqq D_2$.

Substituting $D_2(EY(s) - Y(s))$ by $-D_2\tilde{Z}(s) + D_2\,g(X(s)) + D_2\,EY(s)$ we get

$$E1_\alpha(t)\,\tilde{Z}^n(t) = E\tilde{Z}^n(0) - nD_2\int_0^t E1_\alpha(s)\,\tilde{Z}^n(s)\,ds + \int_0^t R(s)\,ds + R_0(t) \tag{12}$$

where all remaining terms and the (negative) difference of the right and the left hand side of (11) are collected in the functions R and R_0 respectively. Solving this integral equation and assuming $M_{n-1} < \infty$ by induction (note that $M_1 \leqq M + \bar{c} + aT$) we get

$$\begin{aligned} E1_\alpha(t)\,\tilde{Z}^n(t) &\leqq e^{-D_2nt}\left(E\tilde{Z}^n(0) + \int_0^t R(s)\,e^{D_2ns}\,ds\right) \\ &\leqq E\tilde{Z}^n(0) + \int_0^t (c_1D_2 + c_2)\,e^{-D_2(t-s)}\,ds \\ &\quad + n(n-1)\,K^2\int_0^t E1_\alpha(s)\,\tilde{Z}^n(s)\,e^{-D_2n(t-s)}\,ds \\ &\leqq c + \frac{c_1}{n} + c_2T + n(n-1)\,K^2\int_0^t E1_\alpha(s)\,\tilde{Z}^n(s)\,ds \end{aligned}$$

where c_1 and c_2 are constants not depending on D_1, D_2 and N but possibly depending on $c, \bar{c}, n$ and T and where we used the fact that

$$0 \geqq D_2\,g'(X(s))(EX(s) - X(s)) \geqq D_1\,g'(X(s))(EX(s) - X(s)),$$

since $g'(x) \neq 0$ implies

$$EX(s) - x \leqq \bar{c} + aT - (M-1) = 0 \quad \text{for } 0 \leqq s \leqq T.$$

Applying Gronwall's lemma and then Fatou's lemma for $\alpha \to \infty$ we get the result under the additional restriction $D_1 \geqq D_2$.

Case 2. $D_2 \geqq D_1 \geqq 0$ and $D_2 \leqq w_1D_1 + w_2$.

In (11) add the term

$$0 = -nD_1\int_0^t E1_\alpha(s)\,\tilde{Z}^n(s)\,ds + nD_1\int_0^t E1_\alpha(s)\,\tilde{Z}^{n-1}(s)(Y(s) + g(X(s)))\,ds.$$

Then we have

$$E1_\alpha(t)\,\tilde{Z}^n(t)=E\tilde{Z}^n(0)-nD_1\int_0^t E1_\alpha(s)\,\tilde{Z}^n(s)\,ds+\int_0^t \bar{R}(s)\,ds+R_0(t). \tag{13}$$

Similarly as before it follows that

$$\begin{aligned}E1_\alpha(t)\,\tilde{Z}^n(t)\leqq E\tilde{Z}^n(0)&+\int_0^t(\bar{c}_1 D_1+\bar{c}_2)\,e^{-D_1 n(t-s)}\,ds\\&+n(n-1)\,K^2\int_0^t E1_\alpha(s)\,\tilde{Z}^n(s)\,ds\end{aligned}$$

where $\bar{c}_1$ and $\bar{c}_2$ are constants independent of D_1, D_2 and N but possibly depending on $c, \bar{c}, n, T, w_1$ and w_2 and where we used

$$D_2(EY(s)-Y(s))\leqq(w_1 D_1+w_2)\,EY(s)-D_1\,Y(s).$$

The rest of the proof follows as in Case 1. □

Remark. The lemma is also true if $\tilde{Z}$ is replaced by Z because $\tilde{Z}-M\leqq Z\leqq\tilde{Z}$. The reason for introducing $\tilde{Z}$ will become apparent in the proof of the next lemma which is not true if $\tilde{Z}$ is replaced by Z.

Lemma 3.2. *Let the assumptions of Lemma* 3.1 *be satisfied. Fix* $n\in\mathbb{N}\backslash\{1\}$, $\alpha_{n-1}>0$ *and* $\bar{N}>K^2(n-1)$. *There exists a number* $\bar{M}_n$ *such that if* $\infty>E\tilde{Z}^n(0)\geqq\bar{M}_n$, $E\tilde{Z}^{n-1}(0)\leqq\alpha_{n-1}$ *and* $EZ(0)\leqq\bar{c}$, *then* $E\tilde{Z}^n(t)\leqq E\tilde{Z}^n(0)$ *for all* $t\in[0,T]$ *and all* D_1, D_2 *satisfying* $\bar{N}\leqq D_2\leqq w_1 D_1+w_2$ *and* $D_1\geqq\bar{N}$.

Proof. Assume $E\tilde{Z}^n(0)<\infty$ and $E\tilde{Z}^{n-1}(0)\leqq\alpha_{n-1}$. Using (12) and (13) and noting that the remainder terms $R(s)$ and $\bar{R}(s)$ can be estimated by

$$|R(s)|\leqq n(n-1)\,K^2\,E1_\alpha(s)\,\tilde{Z}^n(s)+c_1 D_2+c_2$$

and

$$|\bar{R}(s)|\leqq n(n-1)\,K^2\,E1_\alpha(s)\,\tilde{Z}^n(s)+\bar{c}_1 D_1+\bar{c}_2$$

where c_1, c_2, $\bar{c}_1$ and $\bar{c}_2$ are chosen as in the proof of Lemma 3.1 and independently of $E\tilde{Z}^n(0)$, we get for any $\alpha>0$:

$$E1_\alpha(t)\,\tilde{Z}^n(t)\leqq E\tilde{Z}^n(0)+\int_0^t \beta D+\gamma-n(D-K^2(n-1))\,E1_\alpha(s)\,\tilde{Z}^n(s)\,ds$$

where $D=\min\{D_1, D_2\}\geqq\bar{N}$ and $\beta>0$ and $\gamma>0$ are constants not depending on D_1, D_2 and $E\tilde{Z}^n(0)$ but possibly depending on w_1, w_2, T, n and α_{n-1}. For $\alpha\uparrow\infty$

$$E\tilde{Z}^n(t)\leqq E\tilde{Z}^n(0)+\int_0^t \beta D+\gamma-n(D-K^2(n-1))\,E\tilde{Z}^n(s)\,ds.$$

More generally:

$$E\tilde{Z}^n(t)\leqq E\tilde{Z}^n(u)+\int_u^t \beta D+\gamma-n(D-K^2(n-1))\,E\tilde{Z}^n(s)\,ds \tag{14}$$

whenever $0\leqq u\leqq t\leqq T$.

Define

$$\bar{M}_n := \frac{\beta \bar{N} + \gamma}{n(\bar{N} - K^2(n-1))} = \left(\frac{n}{\beta}\left(1 - \frac{K^2(n-1)}{\bar{N}}\right)\right)^{-1} + \frac{\gamma}{n(\bar{N} - K^2(n-1))} \geqq \frac{\beta D + \gamma}{n(D - K^2(n-1))}.$$

Let $E\tilde{Z}^n(0) \geqq \bar{M}_n$ and assume there exists some $t \leqq T$ with $E\tilde{Z}^n(t) > E\tilde{Z}^n(0)$. Let $u := \sup\{\tau \leqq t: E\tilde{Z}^n(\tau) \leqq E\tilde{Z}^n(0)\}$. Because of $E\tilde{Z}^n(s+h) - E\tilde{Z}^n(s) \leqq (\beta D + \gamma) h$ for all $0 \leqq s \leqq s+h \leqq T$ it follows that $u < t$ and $E\tilde{Z}^n(u) \leqq E\tilde{Z}^n(0)$ which contradicts (14). □

The following lemma states, that the function $(EX(t), EY(t))$ stays close to the solution of Eq. (1) with the same initial condition provided D_1 is large.

Lemma 3.3. *Fix $w_1 \geqq 0$, $w_2 \geqq 0$, $T > 0$ and $c > 0$. Then there exist constants $C \geqq 0$ and $\alpha \geqq 0$ such that for any $\delta > 0$*

$$(EX(t) - f_1(t))^2 + (EY(t) - f_2(t))^2 \leqq C\delta$$

for all $0 \leqq t \leqq T$ and all initial conditions satisfying $EZ^4(0) \leqq c$ and $E(X(0) - EX(0))^2 \leqq \delta$, provided $0 \leqq D_2 \leqq w_1 D_1 + w_2$ and $D_1 \geqq \max\left\{0, \frac{\alpha}{2\delta} - (b+1)\right\}$, where (f_1, f_2) is the solution of (1) with $f_1(0) = EX(0)$, $f_2(0) = EY(0)$.

Proof. Define

$$A(t) := X(t) - EX(t), \qquad B(t) := Y(t) - EY(t).$$

By Itô's lemma,

$$dA^2(t) = (-2(D_1 + b + 1) A^2(t) + 2A(t)(X^2(t) Y(t) - EX^2(t) Y(t)) + g_1^2(X(t)))\, dt + 2A(t)\, g_1(X(t))\, dW_1(t).$$

Assuming $EZ^4(0) \leqq c$ we know from Lemma 3.1 and the remark following it, that all of the following expected values, as well as $EA^2(t)\, g_1^2(X(t))$ are bounded on $[0, T]$. Hence, taking expectations and solving the integral equation for $EA^2(t)$,

$$EA^2(t) = EA^2(0)\, e^{-2(D_1+b+1)t} + \int_0^t e^{-2(D_1+b+1)(t-s)} (2EA(s)\, X^2(s)\, Y(s) + E g_1^2(X(s)))\, ds.$$

According to Lemma 3.1 (and the remark following it)

$$\alpha := \sup_{EZ^4(0) \leqq c}\ \sup_{D_1 \geqq 0}\ \sup_{0 \leqq D_2 \leqq w_1 D_1 + w_2}\ \sup_{0 \leqq s \leqq T} 2E|A(s)\, X^2(s)\, Y(s) + g_1^2(X(s))| < \infty.$$

Therefore

$$EA^2(t) \leqq EA^2(0)\, e^{-2(D_1+b+1)t} + \frac{\alpha}{2(D_1+b+1)} (1 - e^{-2(D_1+b+1)t}) \leqq \max\left\{EA^2(0), \frac{\alpha}{2(D_1+b+1)}\right\}. \tag{15}$$

Assuming $EA^2(0) \leqq \delta$, we get $EA^2(t) \leqq \delta$. Now

$$(EX(t)-f_1(t))^2 = \int_0^t 2(EX(s)-f_1(s))(-(b+1)(EX(s)-f_1(s)) + EX^2(s)\,Y(s) - f_1^2(s)\,f_2(s))\,ds$$

$$(EY(t)-f_2(t))^2 = \int_0^t 2(EY(s)-f_2(s))(b(EX(s)-f_1(s)) - (EX^2(s)\,Y(s) - f_1^2(s)\,f_2(s))\,ds.$$

Defining $D(t) := (EX(t)-f_1(t))^2 + (EY(t)-f_2(t))^2$ we have

$$D(t) \leqq \int_0^t -2(b+1)(EX(s)-f_1(s))^2 + bD(s) + 2(|EX(s)-f_1(s)| + |EY(s)-f_2(s)|)\,|EX^2(s)\,Y(s) - f_1^2(s)\,f_2(s)|\,ds.$$

Now

$$EX^2(t)\,Y(t) = E(A(t)+EX(t))^2(B(t)+EY(t)) = EA^2(t)\,B(t) + EA^2(t)\,EY(t) + 2EA(t)\,B(t)\,EX(t) + (EX(t))^2\,EY(t)$$

and

$$\begin{aligned}
&|(EX(t))^2\,EY(t) - f_1^2(t)\,f_2(t)| \\
&= |(EX(t))^2(EY(t)-f_2(t)) + f_2(t)((EX(t))^2 - f_1^2(t))| \\
&\leqq (EX(t))^2\,|EY(t)-f_2(t)| + f_2(t)(EX(t)+f_1(t))\,|EX(t)-f_1(t)|.
\end{aligned}$$

Writing $h(t) := EA^2(t)\,B(t) + EA^2(t)\,EY(t) + 2EA(t)\,B(t)\,EX(t)$, and noting that f_1, f_2, EX and EY are uniformly bounded on $[0, T]$ for all initial conditions satisfying $EZ^4(0) \leqq c$, there exists some $C_1 \geqq 0$ such that for $0 \leqq t \leqq T$

$$\begin{aligned}
D(t) &\leqq C_1 \int_0^t D(s)\,ds + 2\int_0^t (|EX(s)-f_1(s)| + |EY(s)-f_2(s)|)\,|h(s)|\,ds \\
&\leqq C_1 \int_0^t D(s)\,ds + 2\int_0^t D(s)\,ds + \int_0^t h^2(s)\,ds.
\end{aligned}$$

By Gronwall's lemma

$$D(t) \leqq \int_0^t h^2(s)\,ds + (C_1+2)\int_0^t e^{(C_1+2)(t-s)} \left(\int_0^s h^2(u)\,du\right) ds. \tag{16}$$

Using

$$\begin{aligned}
|h(u)| &= |EA(u)(A(u)\,B(u) + A(u)\,EY(u) + 2B(u)\,EX(u))| \\
&\leqq (EA^2(u))^{1/2}(E(A(u)\,B(u) + A(u)\,EY(u) + 2B(u)\,EX(u))^2)^{1/2} \leqq C_3\,\delta^{1/2}
\end{aligned}$$

for some constant C_3, the assertion follows. □

Remark. Note that only the variance of $X(0)$ is required to be small, not the variance of $Y(0)$!

We are now in a position to formulate a theorem on the periodic behavior of the solution of (4). Recall that (as mentioned in the introduction) (1) has a unique stable limit cycle whenever $a^2 < b-1$.

Theorem 3.4. *Assume $a^2 < b-1$. For fixed $w_1 \geqq 0$, and $w_2 \geqq 0$ there exists a number N^* such that for all $D_1 \geqq N^*$ and $3K^2+1 \leqq D_2 \leqq w_1 D_1 + w_2$ there exists a probability measure μ^* on $[0,\infty)\times[0,\infty)$ and some $\tau^*>0$ such that if $\mathscr{L}(X(0), Y(0)) = \mu^*$, then $\mathscr{L}(X(\tau^*), Y(\tau^*)) = \mu^*$ but $\mathscr{L}(X(s), Y(s)) \neq \mu^*$ for $0<s<\tau^*$ i.e. (4) has a periodic distribution.*

Proof. The main idea of the proof is an application of Tihonov's fixed point theorem [8] on a certain weakly compact and convex subset of the probability measures on $[0,\infty)\times[0,\infty)$.

Let (f_1^*, f_2^*) be the unique periodic solution of (1) with initial condition $f_1^*(0)=a$, $f_2^*(0) > \frac{b}{a}$ (see [15]). Fix

$$c_2 > f_2^*(0) > c_1 > \frac{b}{a}$$

and define

$$T := 2 \max_{c_1 \leqq c \leqq c_2} \min \left\{ u>0: f_1(u)=a, f_2(u) > \frac{b}{a},\; f_1(0)=a, f_2(0)=c \text{ and } (f_1, f_2) \text{ solve } (1) \right\}$$

i.e. T is twice the maximal time a solution of (1) starting on the line segment $f_1(0)=a$, $f_2(0)\in[c_1,c_2]$ needs to return to that segment (because the limit cycle is stable). Let $(f_1^{(i)}, f_2^{(i)})$ be the solution of (1) starting at $f_1^{(i)}(0)=a$, $f_2^{(i)}(0)=c_i$, $i=1,2$ and define

$$\bar{t} := \min_{c_1 \leqq c \leqq c_2} \min \left\{ u \geqq 0: f_1(u)=a, f_2(u) < \frac{b}{a},\; f_1(0)=a, f_2(0)=c, (f_1, f_2) \text{ solve } (1) \right\}$$

$$\bar{c} := c_2 + a$$

$$\bar{N} := 3K^2 + 1$$

$$\varepsilon_1 := \min_{0 \leqq t \leqq T} \left\{ \left((f_1^{(1)}(t)-a)^2 + \left(f_2^{(1)}(t) - \frac{b}{a} \right)^2 \right)^{1/2} \right\}$$

$$\varepsilon_2 := \operatorname{dist}\left(\left\{ (a,c); \frac{b}{a} \leqq c \leqq c_1 \right\}, \{ (f_1^{(1)}(t), f_2^{(1)}(t)); \bar{t} \leqq t \leqq T \} \right)$$

$$\varepsilon_3 := \operatorname{dist}(\{(a,c); c \geqq c_2\}, \{(f_1^{(2)}(t), f_2^{(2)}(t)); \bar{t} \leqq t \leqq T\}).$$

$\varepsilon_1 > 0$ because $\{(f_1^{(1)}(t), f_2^{(1)}(t)), 0 \leqq t \leqq T\}$ is a compact set which does not contain $\left(a, \frac{b}{a}\right)$ due to the uniqueness of the solutions of (1). ε_2 is the distance between two compact sets. We show that $\varepsilon_2 > 0$ i.e. the two sets are disjoint. Note that for the solution (X, Y) of (1) $\frac{dX}{dt} > 0$ if $X=a$ and $Y > \frac{b}{a}$. Assume for some $\bar{t} \leqq t \leqq T\; f_1^{(1)}(t) = a$ and $\frac{b}{a} \leqq f_2^{(1)}(t) \leqq c_1$. Take the smallest such t. Then the (closed) set enclosed by the curve $\{(f_1^{(1)}(s), f_2^{(1)}), 0 \leqq s \leqq t\}$ and the line segment $\{(a, u), c_1 \geqq u \geqq f_1^{(1)}(t)\}$ is invariant under (1), not identical with $\left\{\left(a, \frac{b}{a}\right)\right\}$ and does not contain the limit cycle (f_1^*, f_2^*) which is impossible, since the limit cycle is globally stable (except for the steady state) (see [15]), so $\varepsilon_2 > 0$. An analogous reasoning shows that $\varepsilon_3 > 0$.

For $n = 2, 3, 4$ successively choose values $\bar{M}_n$ satisfying the conclusion of Lemma 3.2 with $\alpha_{n-1} = \bar{M}_{n-1}$, $n = 3, 4$ and $\alpha_1 = \bar{c} + aT$. Define $c := \bar{M}_4$, let C and C_3 be the constants in Lemma 3.3 and fix $\delta > 0$ satisfying $(C\delta)^{1/2} < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and $\delta^{1/2}\; C_3 < a^2 c_1 - ab$. Now let $N^* \geqq \bar{N}$ be so large that

$$\begin{aligned}\alpha := \sup_{EZ^4(0) \leqq c}\ \sup_{D_1 \geqq 0}\ \sup_{0 \leqq D_2 \leqq w_1 D_1 + w_2}\ \sup_{0 \leqq s \leqq T} 2E(A(s) X^2(s) Y(s) + g_1^2(X(s)))\\ \leqq 2(b + 1 + N^*)\delta.\end{aligned}$$

From (15) it follows that $\sup_{0 \leqq t \leqq T} EA^2(t) \leqq EA^2(0)$ if $EA^2(0) \geqq \delta$, $E\tilde{Z}^4(0) \leqq c$ and $D_1 \geqq N^*$. Lemma 3.2 implies that $\sup_{0 \leqq t \leqq T} E\tilde{Z}^4(t) \leqq c$ if $E\tilde{Z}^i(0) \leqq \bar{M}_i$, $i = 2, 3, 4$ and $E\tilde{Z}(0) \leqq \bar{c} + aT$. We will show that there exist constants $\gamma > 0$ and $\eta > 1$ such that $\sup_{0 \leqq t \leqq T} E\exp(\gamma Y^2(t)) \leqq \eta$ whenever $E\exp(\gamma Y^2(0)) \leqq \eta$. Let $\mathcal{M}_1([0, \infty)^2)$ be the set of probability measures on $[0, \infty)^2$ and define

$$\begin{aligned}\mathcal{M} := \{\mu \in \mathcal{M}_1([0, \infty)^2)\colon E_\mu X(0) = a,\ c_1 \leqq E_\mu Y(0) \leqq c_2,\ E_\mu A^2(0) \leqq \delta,\\ E_\mu \tilde{Z}^n(0) \leqq \bar{M}_n \text{ for } n = 2, 3, 4 \text{ and } E_\mu \exp(\gamma Y^2(0)) \leqq \eta\}\end{aligned}$$

and for $\mu \in \mathcal{M}$

$$\tau(\mu) := \inf\left\{t > 0\colon E_\mu X(t) = a,\ E_\mu Y(t) > \frac{b}{a},\ \exists\, t \geqq s \geqq 0\colon E_\mu Y(s) < \frac{b}{a}\right\}$$

and

$$\begin{aligned}S\colon \mathcal{M} \to \mathcal{M}_1([0, \infty)^2)\\ S(\mu) = \mathcal{L}(X(\tau(\mu)), Y(\tau(\mu))).\end{aligned}$$

Since every $\mu \in \mathcal{M}$ satisfies the assumptions of Theorem 2.1, there exists a unique solution with initial condition μ. Obviously $\mathcal{M}$ is convex and weakly compact. Furthermore it follows from the first condition on δ and Lemma 3.3 that $0 < \tau(\mu) < T$ and that $c_1 \leqq E_\mu Y(\tau(\mu)) \leqq c_2$. We want to show that $S\colon \mathcal{M} \to \mathcal{M}$ and that S is weakly continuous. Once we have established these facts it follows from Tihonov's fixed point theorem [8] that S has a fixed point $\mu^* \in \mathcal{M}$ which is the initial law of a solution of (4) such that $\mathcal{L}(X(t), Y(t))$ is periodic with period $\tau(\mu^*)$.

To prove that S maps $\mathcal{M}$ into $\mathcal{M}$ all that remains to show is that there exist constants $\gamma > 0$ and $\eta > 1$ such that $\sup_{0 \leqq t \leqq T} E\exp(\gamma Y^2(t)) \leqq \eta$ whenever $E\exp(\gamma Y^2(0)) \leqq \eta$.

Define $B:=\sup_{y\geqq 0} g_2^2(y)$, $\gamma:=\frac{\bar{N}}{4B}$ and assume $E\tilde{Z}^4(0)\leqq c$ and $E\exp(\gamma Y^2(0))<\infty$. As in Step 1 of the proof of Theorem 2.1.b it follows that $\sup_{0\leqq t\leqq T} EY^{2n}(t)<\infty$ for all $n\geqq 1$ and for $\bar{M}:=c^{1/4}$

$$EY^{2n}(t)\leqq EY^{2n}(0)+n\int_0^t EY^{2n-2}(s)\left(\frac{b^2}{2}+(2n-1)B\right) + 2D_2(\bar{M}EY^{2n-1}(s)-EY^{2n}(s))\,ds.$$

Therefore

$$\begin{aligned} E\sum_{n=0}^{N}\frac{\gamma^n Y^{2n}(t)}{n!} \leqq & E\exp(\gamma Y^2(0)) \\ & +\sum_{n=1}^{N}\frac{\gamma^n}{n!}\int_0^t EY^{2n}(s)\left(\left(\frac{b^2}{2}+(2n+1)B\right)\gamma-nD_2\right)ds \\ & +\sum_{n=1}^{N}\frac{\gamma^n}{n!}\int_0^t D_2 nEY^{2n-1}(s)(2\bar{M}-Y(s))\,ds \\ & +\gamma\left(\frac{b^2}{2}+B\right)t-\frac{\gamma^{N+1}}{N!}\left(\frac{b^2}{2}+(2N+1)B\right)\int_0^t EY^{2N}(s)\,ds. \end{aligned}$$

Because of $\gamma=\frac{\bar{N}}{4B}\leqq\frac{D_2}{4B}$ the first integrand is negative for sufficiently large n. The second integrand is at most $D_2 n(2\bar{M})^{2n}$ and

$$\sum_{n=1}^{N}\frac{\gamma^n}{n!}D_2 n(2\bar{M})^{2n}t\leqq\gamma D_2(2\bar{M})^2 t\exp(\gamma(2\bar{M})^2).$$

Therefore the right hand side of (17) is bounded as $N\to\infty$ (uniformly for all $0\leqq t\leqq T$) and, writing the sum of the second and forth term on the right hand side of (17) as

$$\left(\frac{b^2}{2}+B\right)\gamma\sum_{n=0}^{N}\frac{\gamma^n}{n!}\int_0^t EY^{2n}(s)\,ds+\gamma(2B\gamma-D_2)\sum_{n=1}^{N}\frac{\gamma^{n-1}}{(n-1)!}EY^{2n-2}(s)\cdot Y^2(s)\,ds$$

and using $2B\gamma-D_2\leqq-\frac{1}{2}D_2$, we get

$$E\exp(\gamma Y^2(t))\leqq E\exp(\gamma Y^2(0))+\left(\frac{b^2}{2}+B\right)\gamma\int_0^t E\exp(\gamma Y^2(s))\,ds -\tfrac{1}{2}\gamma D_2\int_0^t EY^2(s)\exp(\gamma Y^2(s))\,ds+4\gamma D_2\bar{M}^2 t\exp(\gamma(2\bar{M})^2).$$

Obviously $EY^2\exp(\gamma Y^2)-\alpha E\exp(\gamma Y^2)\to\infty$ as $E\exp(\gamma Y^2)\to\infty$ for any $\alpha>0$ and uniformly for all distributions of Y^2. Hence it is possible to choose $\eta>1$ such that

$$\tfrac{1}{2}D_2 EY^2\exp(\gamma Y^2)-\left(\frac{b^2}{2}+B\right)E\exp(\gamma Y^2)>4D_2\bar{M}^2\exp(\gamma(2\bar{M})^2)$$

whenever $E\exp(\gamma Y^2) \geqq \eta$ uniformly for all $D_2 \geqq \bar{N}$. The same argument as used at the end of Lemma 3.2 for $\tilde{Z}^n(s)$ instead of $\exp(\gamma Y^2(s))$ shows that $\sup_{0 \leqq t \leqq T} E\exp(\gamma Y^2(t)) \leqq \eta$ whenever $E\exp(\gamma Y^2(0)) \leqq \eta$.

It remains to show that S is weakly continuous. Pick a sequence $\mu_n \in \mathcal{M}$ converging to $\mu \in \mathcal{M}$ and let v_n and v be the laws of the solutions of (4) on $C([0, T], \mathbb{R}_+^2)$ with initial conditions μ_n and μ respectively. Since the forth moments are uniformly bounded, the functions $E_{\mu_n} X(t)$ and $E_{\mu_n} Y(t)$ are continuously differentiable and equicontinuous on $[0, T]$.

So there exists a subsequence $(\mu_{n_k})_{k=1,2,\ldots}$ and continuous functions $a(t)$, $b(t)$ such that

$$(E_{\mu_{n_k}} X(t), E_{\mu_{n_k}} Y(t)) \to (a(t), b(t))$$

uniformly on $[0, T]$. According to Theorem 11.1.4 in [18] v_{n_k} converge weakly to the solution $\bar{v}$ of the martingale problem associated with Eq. (4) with $(EX(t), EY(t))$ replaced by $(a(t), b(t))$. Obviously $a(t) = E_\mu X(t)$ and $b(t) = E_\mu Y(t)$ which implies $\bar{v} = v$ and $v_n \xrightarrow[n\to\infty]{} v$ weakly.

Moreover, Theorem 11.1.4 in [18] shows that the mapping $(\mu, t) \mapsto \mathcal{L}_\mu(X(t), Y(t))$ is (jointly) continuous on $\mathcal{M} \times [0, T]$, where $\mathcal{L}_\mu(X(t), Y(t))$ denotes the law of the solution of (4) with initial condition $\mathcal{L}(X(0), Y(0)) = \mu$.

So once we have established that $\tau\colon \mathcal{M} \to [0, T]$ is continuous, it will follow that S is weakly continuous on $\mathcal{M}$. We show that $\tau\colon \mathcal{M} \to [0, T]$ is continuous: For $\mu \in \mathcal{M}$

$$\begin{aligned}
\frac{d}{dt} EX(\tau(\mu)) &= a - (b+1) EX(\tau(\mu)) + EX^2(\tau(\mu))\, Y(\tau(\mu)) \\
&= -ab + E(A(\tau(\mu)) + a)^2\, Y(\tau(\mu)) \\
&= -ab + a^2\, EY(\tau(\mu)) + EA(\tau(\mu))(A(\tau(\mu))\, B(\tau(\mu)) \\
&\quad + A(\tau(\mu))\, EY(\tau(\mu)) + 2B(\tau(\mu))\, a) \\
&\geqq -ab + a^2 c_1 - \delta^{1/2} C_3 =: \beta > 0.
\end{aligned}$$

Fix $\mu \in \mathcal{M}$ and choose $t_0 > 0$ such that $\frac{d}{dt} E_\mu X(t) > \frac{\beta}{2}$ for all $t \in [\tau(\mu) - t_0, \tau(\mu) + t_0]$. Fix $0 < \varepsilon \leqq \frac{\beta}{2} t_0$, let $\mu_n \xrightarrow[n\to\infty]{} \mu$ $(\mu_n \in \mathcal{M})$ and choose n_0 such that $\sup_{0 \leqq t \leqq T} |E_{\mu_n}(X(t)) - E_\mu(X(t))| < \varepsilon$ for all $n \geqq n_0(\varepsilon)$. Then $|\tau(\mu) - \tau(\mu_n)| \leqq \frac{2}{\beta} \varepsilon (\leqq t_0)$ for $n \geqq n_0(\varepsilon)$ which shows that τ is continuous on $\mathcal{M}$ and hence the theorem is proved. □

Corollary. *Assume* $a^2 < b - 1$, $w_1 \geqq 0$, $w_2 \geqq 0$, *and let* $(D_{1,n}, D_{2,n})_{n \in \mathbb{N}}$ *be a sequence such that* $D_{1,n}$ *and* $D_{2,n}$ *satisfy the assumptions of Theorem* 3.4 *for every* $n \in \mathbb{N}$ *and* $D_{1,n} \xrightarrow[n\to\infty]{} \infty$. *Then there exists a sequence* μ_n^* *of probability measures on* $[0, \infty) \times [0, \infty)$ *such that* (4) *with* $D_1 := D_{1,n}$, $D_2 := D_{2,n}$ *and* $\mathcal{L}(X(0), Y(0)) = \mu_n^*$ *has a periodic distribution and*

a) $(E_{\mu_n^*} X(t), E_{\mu_n^*} Y(t)) \xrightarrow[n\to\infty]{} (f_1^*(t), f_2^*(t))$ *uniformly on compact intervals*

b) $\mathcal{L}_{\mu_n^*}(X(\cdot)) \xrightarrow[n\to\infty]{} \varepsilon_{f_1^*(\cdot)}$.

Proof. In the proof of Theorem 3.4 let $c_1=c_1(n)$ and $c_2=c_2(n)$ converge to $f_2^*(0)$ in such a way that $D_{1,n}\geqq N^*=N^*(n)$. Then $\delta=\delta(n)$ converges to zero and part a) follows. Furthermore,

$$dA(t)=(-(D_{1,n}+b+1)\,A(t)+X^2(t)\,Y(t)-EX^2(t)\,Y(t))\,dt+g_1(X(t))\,dW_1(t)$$

and hence, solving for $A(t)$ (see [1], p. 142, the proof given there also works for nondeterministic coefficients),

$$A(t)=e^{-K_nt}A(0)+\int_0^t e^{-K_n(t-s)}(X^2(x)\,Y(s)-EX^2\,Y(s))\,ds$$
$$+\int_0^t e^{-K_n(t-s)}g_1(X(s))\,dW_1(s)$$

with the abbreviation $K_n=D_{1,n}+b+1$, and so using Chebychev's inequality

$$P\{\sup_{0\leqq t\leqq T}|A(t)|\geqq 3R\}\leqq P\{|A(0)|\geqq R\}$$
$$+P\left\{\int_0^T e^{-K_n(t-s)}|X^2(s)\,Y(s)-EX^2(s)\,Y(s)|\,ds\geqq R\right\}$$
$$+P\left\{\sup_{0\leqq t\leqq T}\left|\int_0^t e^{-K_n(t-s)}g_1(X(s))\,dW_1(s)\right|\geqq R\right\}$$
$$\leqq\frac{1}{R}\,E|A(0)|+\frac{1}{R}\frac{\beta}{K_n}+P\{\sup_{0\leqq t\leqq T}|H_n(t)|\geqq R\}$$

where

$$H_n(t):=\int_0^t e^{-K_n(t-s)}g_1(X(s))\,dW_1(s)$$

and

$$\beta=\sup_{0\leqq t\leqq T}E|X^2(t)\,Y(t)-EX^2(t)\,Y(t)|.$$

Since

$$E|A(0)|\leqq(EA^2(0))^{1/2}\leqq\delta^{1/2}(n)\xrightarrow[n\to\infty]{}0$$

all we have to show to prove part b) is

$$P\{\sup_{0\leqq t\leqq T}|H_n(t)|\geqq R\}\xrightarrow[n\to\infty]{}0\qquad\text{for every } R>0.$$

Again by [1], p. 142 it follows that

$$dH_n(t)=-K_nH_n(t)\,dt+g_1(X(t))\,dW_1(t),\qquad H_n(0)=0.$$

Hence, for

$$f(x)=\begin{cases}x^4 & |x|\leqq R^4\\ \text{bounded, } C^2(\mathbb{R})\end{cases}$$

$$f(H_n(t))+K_n\int_0^t f'(H_n(s))\,H_n(s)\,ds-\frac{1}{2}\int_0^t f''(H_n(s))\,g_1^2(X(s))\,ds$$

is a martingale w.r.t. the filtration induced by (W_1, W_2). Defining the stopping time

$$\tau:=\inf\{t\geqq 0:|H_n(t)|=R\}\wedge T,$$

we have, by Chebychev's inequality

$$P\{\sup_{0\leqq t\leqq T}|H_n(t)|\geqq R\}=P\{f(H_n(\tau))\geqq R^4\}\leqq\frac{1}{R^4}\,Ef(H_n(\tau))$$

$$=\frac{1}{R^4}\,E\int_0^\tau 6H_n^2(s)\,g_1^2(X(s))-4K_n H_n^4(s)\,ds.$$

For $-R^4\leqq x\leqq R^4$,

$$6x^2\,g_1^2(X(s))-4K_n x^4=-4K_n\left(x^2-\frac{3g_1^2(X(s))}{4K_n}\right)^2+\frac{9g_1^4(X(s))}{4K_n}$$

and therefore

$$6x^2\,g_1^2(X(s))-4K_n x^4\leqq\begin{cases}\dfrac{9g_1^4(X(s))}{4K_n} & \text{if } g_1^2(X(s))\leqq R^8 K_n\\ 6R^8\,g_1^2(X(s))-4K_n R^{16} & \text{if } g_1^2(X(s))>R^8 K_n\end{cases}$$

which implies

$$P\{\sup_{0\leqq t\leqq T}|H_n(t)|\geqq R\}\leqq\frac{1}{R^4}\,E\int_0^T 1_{\{g_1^2(X(s))\leqq R^8 K_n\}}\cdot\frac{9g_1^4(X(s))}{4K_n}$$

$$+1_{\{g_1^2(X(s))>R^8 K_n\}}\cdot\frac{6}{K_n}\,g_1^4(X(s))\,ds\xrightarrow[n\to\infty]{}0$$

since $\sup_n E\int_0^T g_1^4(X(s))\,ds<\infty$. □

Remarks. In the Corollary we do not require that $D_{2,n}$ converges as $n\to\infty$, so the corresponding processes $Y_n(\cdot)$ need not converge. If however $\lim_{n\to\infty} D_{2,n}=D_2$ and $g_2(x)>0$ for all $x>0$, then it is easy to see that $(Y_n(\cdot))$ converges in law to the unique solution of

$$dY(t)=bf_1^*(t)-(f_1^*(t))^2\,Y(t)+D_2(f_2^*(t)-Y(t))+g_2(Y(t))\,dW_2(t)$$

having a periodic law. The uniqueness can be established by considering the irreducible and ergodic Markov chain $\bar{Y}_k:=Y(k\tau)$, where τ is the period of (f_1^*,f_2^*).

A heuristic explanation of the corollary is the following: If D_1 is very large, then, because of the term $D_1(EX(t)-X(t))$, there is a strong force driving the solution X of Eq. (4) towards its expectation, so in the limit $D_1\to\infty X$ becomes deterministic.

4. Small Noise Limit and Fluctuations

We will now study the behavior of the Brusselator for fixed D_1 and D_2 as the noise converges to zero. First (Theorem 4.1) we identify the periodic solution of (1) as the limit of the periodic solution of the stochastic Brusselator as the noise converges to zero, then (Theorem 4.3) we study the fluctuations. Finally (Lemma 4.5) we derive the asymptotic difference of the expected value functions of the periodic solutions and the deterministic periodic solution.

Theorem 4.1. *Let the assumptions of Theorem* 3.4 *be satisfied. There exist numbers* N_1 *and* N_2 *such that, if in addition* $D_1 \geqq N_1$ *and* $D_2 \geqq N_2$, *then for all* $1 \geqq \varepsilon \geqq 0$, *there exists a probability measure* μ_ε *on* $[0, \infty)^2$ *such that*

$$\begin{aligned} dX(t) &= (a-(b+1)X(t)+X^2(t)Y(t)+D_1(EX(t)-X(t)))\,dt+\varepsilon g_1(X(t))\,dW_1(t) \\ dY(t) &= (bX(t)-X^2(t)Y(t)+D_2(EY(t)-Y(t)))\,dt+\varepsilon g_2(Y(t))\,dW_2(t), \end{aligned} \tag{18}$$

$\mathscr{L}(X(0), Y(0)) = \mu_\varepsilon$ *has a periodic distribution with* $\sup_{1 \geqq \varepsilon \geqq 0} \sup_{t \geqq 0} E_{\mu_\varepsilon} Z^8(t) < \infty$, $c_2 \geqq E_{\mu_\varepsilon} Y(0) \geqq c_1$, $E_{\mu_\varepsilon} X(0) = a$ *and period at most* T, *where* c_1, c_2 *and* T *are defined as in the proof of Theorem* 3.4.

Furthermore, denoting a periodic solution of (18) *with these properties by* $(X_\varepsilon, Y_\varepsilon)$, $(X_\varepsilon, Y_\varepsilon) \xrightarrow[\varepsilon \downarrow 0]{} (f_1^*, f_2^*)$ *weakly on* $C([0, \infty), \mathbb{R}^2)$.

Proof. The first part of the theorem is obvious, since all estimates in the proof of Theorem 3.4 depend on g_1 and g_2 only through an upper bound K of their Lipschitz constant and an upper bound B of g_2^2. The proof of Theorem 3.4 shows only $\sup_{t \geqq 0} EZ^4(t) < \infty$ but, using Lemma 3.2, one can easily see that there exists a periodic solution satisfying $\sup_{t \geqq 0} EZ^8(t) < \infty$.

Now let $\varepsilon_n \downarrow 0$. Since $EX_{\varepsilon_n}(0) = a$ and $EY_{\varepsilon_n}(0) \leqq c_2$, the family $\{\mu_{\varepsilon_n}\}_{n \in \mathbb{N}}$ is tight. The same proof showing that S is weakly continuous in the proof of Theorem 3.4 can be employed to show that there exists a weakly convergent subsequence of $(X_{\varepsilon_n}, Y_{\varepsilon_n})$ converging to a solution (X, Y) of

$$\begin{aligned} \dot{X}(t) &= a-(b+1)X(t)+X^2(t)Y(t)+D_1(EX(t)-X(t)), \qquad EX(0)=a \\ \dot{Y}(t) &= bX(t)-X^2(t)Y(t)+D_2(EY(t)-Y(t)), \qquad c_2 \geqq EY(0) \geqq c_1. \end{aligned} \tag{19}$$

Let τ_ε be the period of $(X_\varepsilon, Y_\varepsilon)$ and, for a given sequence $\varepsilon_n \downarrow 0$ $(0 < \varepsilon_n \leqq 1)$, take a subsequence ε_{n_k}, $k = 1, 2, 3, \ldots$ such that $\mathscr{L}(X_{\varepsilon_{n_k}}, Y_{\varepsilon_{n_k}})$ converges to a solution of (19) such that $\tau = \lim_{k \to \infty} \tau_{\varepsilon_{n_k}}$ exists. Theorem 11.1.4 in [18] implies that the mapping $(\mu, \varepsilon, t) \mapsto \mathscr{L}_{\mu,\varepsilon}(X(t), Y(t))$ from $\mathscr{M} \times [0,1] \times [0,T]$ to $\mathscr{M}_1(C([0,T], \mathbb{R}_+^2))$ is (jointly) continuous, where $\mathscr{L}_{\mu,\varepsilon}(X(t), Y(t))$ denotes the law of the solution of (18) with $\mathscr{L}(X(0), Y(0)) = \mu$ at t.

Hence, denoting $\mu := \lim_{k \to \infty} \mu_{\varepsilon_{n_k}}$, it follows that $\mathscr{L}_{\mu,0}(X(\tau), Y(\tau)) = \mu$ i.e. the limit (X, Y) of the periodic processes $(X_{\varepsilon_{n_k}}, Y_{\varepsilon_{n_k}})$ has a periodic law. Note that it cannot be constant since its expectation is nonconstant. To prove the theorem, it is enough to show that the only solution of (19) with a periodic distribution is (f_1^*, f_2^*). Note that the randomness enters the dynamics of (19) only via the initial condition. Defining $\tilde{Z}(t) := g(X(t)) + Y(t)$, $0 \leqq t \leqq T$ with g defined as in Lemma 3.1 with $\bar{c} := a + c_2$, it follows that

$$\begin{aligned} \frac{d\tilde{Z}}{dt}(t) &= -X^2(t)Y(t)(1-g'(X(t)))+D_1(EX(t)-X(t))g'(X(t)) \\ &\quad + bX(t)(1-g'(X(t)))-g'(X(t))(X(t)-a)+D_2(EY(t)-Y(t)) \\ &\leqq bX(t)(1-g'(X(t)))-g'(X(t))(X(t)-a)+D_2(EY(t)-Y(t)) \end{aligned} \tag{20}$$

since $g'(x)=0$ for all $x \leqq \sup_{0 \leqq t \leqq T} EX(t) \leqq \bar{c}+aT$. The right hand side of (20) is negative provided either X or Y is sufficiently large i.e. there exists some $\gamma>0$ such that $\frac{d\tilde{Z}}{dt}(t)<0$ whenever $\tilde{Z}(t) \geqq \gamma$, showing that the support of $\mathscr{L}(X(t), Y(t))$, being periodic, is contained in

$$\{(x, y) \mid x \geqq 0, y \geqq 0, x+y \leqq \gamma\}$$

for every $t \geqq 0$.

Writing $A := X-EX$ and $B := Y-EY$ we have (dropping t)

$$X^2 Y=(A+EX)^2 Y=A(AY+2YEX)+Y(EX)^2.$$

Therefore, since (w.p.1) $0 \leqq X+Y \leqq \gamma$ and hence $0 \leqq EX$, $EY \leqq \gamma$ and $-\gamma \leqq A$, $B \leqq \gamma$,

$$EAX^2 Y \leqq 3\gamma^2 EA^2+\gamma^2 E|AB| \leqq \tfrac{7}{2}\gamma^2 EA^2+\tfrac{1}{2}\gamma^2 EB^2$$
$$EBX^2 Y \geqq -3\gamma^2 E|AB| \geqq -\tfrac{3}{2}\gamma^2 EA^2-\tfrac{3}{2}\gamma^2 EB^2$$

and hence

$$\begin{aligned} \frac{d}{dt} EA^2 &= -2(b+1+D_1) EA^2+2EAX^2 Y \\ &\leqq (-2(b+1+D_1)+7\gamma^2) EA^2+\gamma^2 EB^2 \\ \frac{d}{dt} EB^2 &= -2D_2 EB^2+2b\,EAB-2EBX^2 Y \\ &\leqq (b+3\gamma^2) EA^2+(-2D_2+b+3\gamma^2) EB^2 \end{aligned}$$

which implies $\frac{d}{dt} E(A^2(t)+B^2(t))<0$ whenever $E(A^2(t)+B^2(t))>0$ provided D_1 and D_2 are large enough (note that γ may depend on w_1 and w_2, but not on D_1 and D_2). Since $EA^2(t)$ and $EB^2(t)$ are periodic, it follows that $EA^2(t) \equiv EB^2(t) \equiv 0$ i.e. $(X(t), Y(t))$ is deterministic. The assertion follows since (1) has only one periodic solution with $X(0)=a$ and $Y(0) \geqq c_1$. □

Our next aim is to study the fluctuations of the periodic solutions as $\varepsilon \downarrow 0$. To prove the main result, we need the following lemma.

Lemma 4.2. *Assume $a^2<b-1$ and fix nonnegative numbers $w_1, w_2, w_3, w_4 \geqq 0$. Then for $D_2 \leqq w_1 D_1+w_2$ and $D_1 \leqq w_3 D_2+w_4$ and D_1 and D_2 sufficiently large $\sup_{0<\varepsilon \leqq 1} \frac{\gamma_\varepsilon}{\varepsilon}<\infty$, where*

$$\gamma_\varepsilon := \varepsilon \sup_{t \geqq 0} \{(EA_\varepsilon^4(t))^{1/4}, (EB_\varepsilon^4(t))^{1/4}\},$$

$$A_\varepsilon(t) := \varepsilon^{-1}(X_\varepsilon(t)-EX_\varepsilon(t)), \qquad B_\varepsilon(t) := \varepsilon^{-1}(Y_\varepsilon(t)-EY_\varepsilon(t))$$

and $(X_\varepsilon, Y_\varepsilon)$ is a periodic solution of (4) *with the properties stated in Theorem* 4.1.

Proof. Note that $\sup_{0<\varepsilon \leqq 1} \gamma_\varepsilon<\infty$ and $\lim_{\varepsilon \downarrow 0} \gamma_\varepsilon=0$ according to Theorem 4.1. It remains to prove $\gamma_\varepsilon=O(\varepsilon)$ for $\varepsilon \downarrow 0$. Let us define a family of Lyapunov-type functions

$$V_\varepsilon(x, y)=\lambda_\varepsilon(x+y)\,\phi(x, y)+(1-\lambda_\varepsilon(x+y))\,\psi(x, y), \qquad x, y \geqq -\frac{M}{\varepsilon}$$

where

$$M := \sup_{1 \geqq \varepsilon > 0,\, t \geqq 0,\, D_1,\, D_2} (EX_\varepsilon(t) + EY_\varepsilon(t)),$$

$$\phi(x, y) = x^4 + y^4, \qquad \psi(x, y) = 17(x+y)^4,$$

$$\lambda \in C^2([0, \infty), \mathbb{R}) \text{ satisfies } \frac{d\lambda}{dv} \leqq 0 \quad \text{and} \quad \lambda(v) = \begin{cases} 1 & v \leqq 1 \\ 0 & v \geqq 2 \end{cases}$$

and

$$\lambda_\varepsilon(v) = \lambda(\alpha^{-1} \varepsilon v), \qquad \text{where } \alpha = M \cdot \max\{w_1, w_3, 1\}.$$

Our aim is to prove that

$$\sup_{t \geqq 0} \sup_{1 \geqq \varepsilon > 0} EV_\varepsilon(A_\varepsilon(t), B_\varepsilon(t)) < \infty$$

from which the assertion follows easily. Let $L_{t,\varepsilon}$ be the generator of the diffusion $(A_\varepsilon, B_\varepsilon)$. Then

$$\begin{aligned} L_{t,\varepsilon} V_\varepsilon(x, y) = {} & \lambda_\varepsilon L_{t,\varepsilon} \phi + (1-\lambda_\varepsilon) L_{t,\varepsilon} \psi + \alpha^{-1} \varepsilon \lambda'_\varepsilon (\phi - \psi)(-(D_1+1)x - D_2 y) \\ & + \tfrac{1}{2} \alpha^{-2} \varepsilon^2 \lambda''_\varepsilon (\phi - \psi)(g_1^2 + g_2^2) + \alpha^{-1} \varepsilon \lambda'_\varepsilon ((\phi_x - \psi_x) g_1^2 + (\phi_y - \psi_y) g_2^2), \end{aligned}$$

where we have dropped the arguments $x+y$ of λ_ε, (x, y) of ϕ and ψ, $\varepsilon x + EX_\varepsilon(t)$ of g_1^2 and $\varepsilon y + EY_\varepsilon(t)$ of g_2^2 for notational simplicity.

Let us first consider the case $x + y \geqq \frac{2\alpha}{\varepsilon}$ which implies $\lambda_\varepsilon = 0$ and hence $L_{t,\varepsilon} V_\varepsilon(x, y) = L_{t,\varepsilon} \psi(x, y)$.

In case $x + y \geqq \frac{\alpha}{\varepsilon}$

$$\begin{aligned} \tfrac{1}{17} L_{t,\varepsilon} \psi(x, y) &= 4(x+y)^3(-(D_1+1)x - D_2 y) \\ &\quad + 6(x+y)^2 (g_1^2(\varepsilon x + EX_\varepsilon(t)) + g_2^2(\varepsilon y + EY_\varepsilon(t))) \\ &\leqq 4(x+y)^3(-(D_1+1)x - D_2 y) \\ &\quad + 6(x+y)^2 K^2((\varepsilon x + M)^2 + (\varepsilon y + M)^2). \end{aligned}$$

Now, for $x, y \geqq -\frac{M}{\varepsilon}$, $x + y \geqq \frac{\alpha}{\varepsilon}$ and $D_2 \geqq 1$

$$\begin{aligned} D_1 x + (D_2 - 1) y &\geqq \min\left\{ D_1 \left(-\frac{M}{\varepsilon}\right) + (D_2 - 1)\left(x + y + \frac{M}{\varepsilon}\right), \right. \\ &\qquad \left. D_1 \left(x + y + \frac{M}{\varepsilon}\right) + (D_2 - 1)\left(-\frac{M}{\varepsilon}\right) \right\} \\ &\geqq \varepsilon^{-1} \min\{-(w_3 D_2 + w_4) M + (D_2 - 1)(\alpha + M), \\ &\qquad D_1(\alpha + M) - (w_1 D_1 + w_2 - 1) M\} \\ &= \varepsilon^{-1} \min\{(D_2 - 1)(\alpha + M - w_3 M) - M(w_3 + w_4), \\ &\qquad D_1(\alpha + M - w_1 M) - M(w_2 - 1)\} \\ &\geqq 0 \end{aligned}$$

if D_1 and D_2 are sufficiently large. Hence

$$-(D_1+1)x-D_2 y \leqq -(x+y).$$

Furthermore, since $\varepsilon x+M\geqq 0$ and $\varepsilon y+M\geqq 0$,

$$\begin{aligned}6(x+y)^2 K^2((\varepsilon x+M)^2+(\varepsilon y+M)^2) &\leqq 6(x+y)^2 K^2(\varepsilon x+M+\varepsilon y+M)^2\\ &\leqq 6(x+y)^2 K^2(\varepsilon(x+y)+2\alpha)^2\\ &\leqq 6(x+y)^4 K^2\cdot 9\varepsilon^2 \leqq (x+y)^4\end{aligned}$$

for all $x+y\geqq\frac{\alpha}{\varepsilon}$ and $\varepsilon\leqq(54K^2)^{-1/2}$. Therefore

$$\tfrac{1}{17}L_{t,\varepsilon}\psi(x,y)\leqq -3(x+y)^4 = -\tfrac{3}{17}\psi(x,y)$$

if ε is sufficiently small.

Let us now assume $x+y\leqq\frac{2\alpha}{\varepsilon}$. Then

$$\begin{aligned}L_{t,\varepsilon}\phi(x,y) &= 4x^3(-(D_1+b+1)x+h(\varepsilon,x,y,t))+4y^3(bx-D_2 y-h(\varepsilon,x,y,t))\\ &\quad +6x^2 g_1^2(\varepsilon x+EX_\varepsilon(t))+6y^2 g_2^2(\varepsilon y+EY_\varepsilon(t))\\ &\leqq -4(D_1+b+1)x^4+4bxy^3-4D_2 y^4+4(|x|^3+|y|^3)\,|h(\varepsilon,x,y,t)|\\ &\quad +6(x^2+y^2)K^2(2\alpha+2M)^2,\end{aligned} \tag{21}$$

where

$$\begin{aligned}h(\varepsilon,x,y,t) &:= \varepsilon^{-1}((\varepsilon x+EX_\varepsilon(t))^2(\varepsilon y+EY_\varepsilon(t))-EX_\varepsilon^2 Y_\varepsilon(t))\\ &= \varepsilon^2(x^2 y-EA_\varepsilon^2(t)B_\varepsilon(t))+\varepsilon EY_\varepsilon(t)(x^2-EA_\varepsilon^2(t))\\ &\quad +2\varepsilon EX_\varepsilon(t)(xy-EA_\varepsilon(t)B_\varepsilon(t))\\ &\quad +2x\,EX_\varepsilon(t)\,EY_\varepsilon(t)+y(EX_\varepsilon(t))^2.\end{aligned} \tag{22}$$

There exists a constant c_3 such that

$$(|x|^3+|y|^3)(\varepsilon^2 x^2|y|+\varepsilon Mx^2+2\varepsilon M|xy|+2|x|M^2+|y|M^2)\leqq c_3(x^4+y^4).$$

Furthermore, defining $\gamma_\varepsilon(t):=\varepsilon\max\{(EA_\varepsilon^4(t))^{1/4},(EB_\varepsilon^4(t))^{1/4}\}$,

$$\begin{aligned}\varepsilon^2 E|A_\varepsilon^2(t)B_\varepsilon(t)| &= \varepsilon^{-1}E(\varepsilon A_\varepsilon(t))^2(\varepsilon|B_\varepsilon(t)|)\\ &\leqq\frac{1}{\varepsilon}(E(\varepsilon A_\varepsilon(t))^4)^{1/2}(E(\varepsilon B_\varepsilon(t))^2)^{1/2}\leqq\frac{\gamma_\varepsilon^3(t)}{\varepsilon}.\end{aligned} \tag{23}$$

In the same way it follows that

$$\varepsilon EA_\varepsilon^2(t)\leqq\frac{\gamma_\varepsilon^2(t)}{\varepsilon}\quad\text{and}\quad 2\varepsilon E|A_\varepsilon(t)B_\varepsilon(t)|\leqq 2\,\frac{\gamma_\varepsilon^2(t)}{\varepsilon} \tag{24}$$

which implies

$$\begin{aligned}&(|x|^3+|y|^3)(\varepsilon^2 E|A_\varepsilon^2(t)B_\varepsilon(t)|+\varepsilon EA_\varepsilon^2(t)+2\varepsilon E|A_\varepsilon(t)B_\varepsilon(t)|)\\ &\quad\leqq(|x|^3+|y|^3)\,4\,\frac{\gamma_\varepsilon^2(t)}{\varepsilon}\leqq x^4+y^4+2\left(4\,\frac{\gamma_\varepsilon^2(t)}{\varepsilon}\right)^4\end{aligned}$$

if ε is so small that $\gamma_\varepsilon\leqq 1$.

Furthermore,

$$-4(D_1+b+1)x^4+4bxy^3-D_2y^4+6(x^2+y^2)K^2(2\alpha+2M)^2 \leqq -(4c_3+4+2)(x^4+y^4)+c_4$$

for D_1 and D_2 sufficiently large and c_4 a suitable constant.

Together we have shown that there exist constants c_4 and c_5 such that

$$L_{t,\varepsilon}\phi(x,y) \leqq -2(x^4+y^4)+c_5\gamma_\varepsilon^8(t)\,\varepsilon^{-4}+c_4$$

provided D_1 and D_2 are sufficiently large and ε is sufficiently small.

We still have to consider the three remaining terms in $L_{t,\varepsilon}V_\varepsilon(x,y)$ which contain derivatives of λ_ε.

Since $\lambda' \leqq 0$, $-(D_1+1)x-D_2y \leqq 0$ whenever $\lambda'_\varepsilon \neq 0$ (as shown before) and

$$\phi(x,y)=x^4+y^4 \leqq \left(-\frac{M}{\varepsilon}\right)^4+\left(x+y+\frac{M}{\varepsilon}\right)^4 \leqq (x+y)^4+(2(x+y))^4$$
$$=17(x+y)^4=\psi(x,y) \quad \text{for } \frac{2\alpha}{\varepsilon} \geqq x+y \geqq \frac{\alpha}{\varepsilon}$$

(note that for fixed z, $x^4+(z-x)^4$ attains its maximum on an interval at the boundary) we have $\alpha^{-1}\varepsilon\lambda'_\varepsilon(\phi-\psi)(-(D_1+1)x-D_2y) \leqq 0$.

Furthermore, since $g_1^2(\varepsilon x+EX_\varepsilon(t))$ and $g_2^2(\varepsilon y+EY_\varepsilon(t))$ are bounded on $x+y \leqq \frac{2\alpha}{\varepsilon}$ uniformly for all $\varepsilon>0$, there exists a constant c_6 such that the last two terms of $L_{t,\varepsilon}$ can be estimated by $\min(x^4+y^4,\ 17(x+y)^4)+c_6$ provided ε is small enough. So we have shown that for $c_7:=c_4+c_6$

$$\begin{aligned} L_{t,\varepsilon}V_\varepsilon(x,y) &\leqq -2\lambda_\varepsilon(x+y)\,\phi(x,y)-3(1-\lambda_\varepsilon(x+y))\,\psi(x,y) \\ &\quad +\min(\phi(x,y),\psi(x,y))+c_4+c_6+c_5\gamma_\varepsilon^8(t)\cdot\varepsilon^{-4} \\ &\leqq -V_\varepsilon(x,y)+c_7+c_5\gamma_\varepsilon^8(t)\,\varepsilon^{-4} \end{aligned}$$

for all $x, y \geqq -\frac{M}{\varepsilon}$. Hence

$$\begin{aligned} \frac{d}{dt}EV_\varepsilon(A_\varepsilon(t),B_\varepsilon(t)) &= EL_{t,\varepsilon}V_\varepsilon(A_\varepsilon(t),B_\varepsilon(t)) \\ &\leqq -EV_\varepsilon(A_\varepsilon(t),B_\varepsilon(t))+c_7+c_5\gamma_\varepsilon^4\max\{EA_\varepsilon^4(t),EB_\varepsilon^4(t)\} \\ &\leqq -EV_\varepsilon(A_\varepsilon(t),B_\varepsilon(t))+c_7+c_5\gamma_\varepsilon^4 EV_\varepsilon(A_\varepsilon(t),B_\varepsilon(t)). \end{aligned}$$

Since $EV_\varepsilon(A_\varepsilon(t),B_\varepsilon(t))$ is periodic and $\lim_{\varepsilon\downarrow 0}\gamma_\varepsilon^4=0$ it follows that

$$\sup_{1\geqq\varepsilon\geqq 0}\ \sup_{t\geqq 0}\ \sup_{D_1,D_2} EV_\varepsilon(A_\varepsilon(t),B_\varepsilon(t))<\infty$$

provided D_1 and D_2 satisfy the assumptions above and are sufficiently large. This implies the assertion of the lemma. □

Theorem 4.3. *Let the assumptions of Lemma 4.2 be satisfied. Then for D_1 and D_2 sufficiently large, as $\varepsilon\downarrow 0$, $(A_\varepsilon, B_\varepsilon)$ converge weakly to a Gaussian process (A_0, B_0) which is the unique solution of*

$$\begin{pmatrix} dA_0(t) \\ dB_0(t) \end{pmatrix} = \begin{pmatrix} -D_1 - b - 1 + 2f_1^*(t) f_2^*(t) & (f_1^*(t))^2 \\ b - 2f_1^*(t) f_2^*(t) & -D_2 - (f_1^*(t))^2 \end{pmatrix} \cdot \begin{pmatrix} A_0(t) \\ B_0(t) \end{pmatrix} dt + \begin{pmatrix} g_1(f_1^*(t))\, dW_1(t) \\ g_2(f_2^*(t))\, dW_2(t) \end{pmatrix} \tag{25}$$

with a periodic or time-invariant distribution. (Note that (25) *is equation* (4) *linearized around the periodic solution of* (1)).

Proof. Lemma 4.2 implies that for any sequence $\varepsilon_n\downarrow 0$, the sequence $\mathscr{L}(A_{\varepsilon_n}(0), B_{\varepsilon_n}(0))$ is tight. Furthermore

$$dA_\varepsilon(t) = (-(D_1 + b + 1) A_\varepsilon(t) + h(\varepsilon, A_\varepsilon(t), B_\varepsilon(t), t))\, dt + g_1(\varepsilon A_\varepsilon(t) + EX_\varepsilon(t))\, dW_1(t)$$

$$dB_\varepsilon(t) = (-D_2 B_\varepsilon(t) + bA_\varepsilon(t) - h(\varepsilon, A_\varepsilon(t), B_\varepsilon(t), t))\, dt + g_2(\varepsilon B_\varepsilon(t) + EY_\varepsilon(t))\, dW_2(t)$$

where h is defined in (22). Now

$$\sup_{t\geqq 0} \varepsilon^2 EA_\varepsilon^2(t)\,|B_\varepsilon(t)| + \varepsilon EA_\varepsilon^2(t) + 2\varepsilon E|A_\varepsilon(t) B_\varepsilon(t)| = o(1) \quad \text{as} \quad \varepsilon\downarrow 0$$

due to (23), (24) and Lemma 4.2. Using again Theorem 11.1.4 in [18], $(A_\varepsilon, B_\varepsilon)$ converge weakly to the solution of (25) provided we can show that (25) has at most one solution satisfying $\mathscr{L}(A_0(\bar\tau), B_0(\bar\tau)) = \mathscr{L}(A_0(0), B_0(0))$ for some $\bar\tau > 0$ (cf. the proof of Theorem 4.1).

Any solution (A_0, B_0) of (25) with these properties can be represented in the form

$$\begin{bmatrix} A_0(t) \\ B_0(t) \end{bmatrix} = \Phi(t) \left[\begin{bmatrix} A_0(0) \\ B_0(0) \end{bmatrix} + \int_0^t \Phi(s)^{-1} Q(s) \begin{pmatrix} dW_1(s) \\ dW_2(s) \end{pmatrix} \right]$$

(see [1]), where

$$Q(s) := \begin{pmatrix} g_1(f_1^*(s)) & 0 \\ 0 & g_1(f_2^*(s)) \end{pmatrix}$$

and $\Phi(t)$ is the fundamental matrix of the corresponding deterministic system, which, according to Floquet's theorem ([23], p. 194), can be represented in the form $\phi(t) = P(t)\, e^{Ct}$, where $P(t)$ is a matrix-valued function satisfying $P(t+\tau) = P(t)$ for all $t \geqq 0$, τ is the period of (f_1^*, f_2^*) and C is a constant 2×2-matrix. We assume that D_1 and D_2 are so large that the eigenvalues of C (the characteristic exponents) have negative real parts.

Define $A_0^{(n)}(t) := A_0(t + n\tau)$, $B_0^{(n)}(t) := B_0(t + n\tau)$, $t \geqq 0$, and let $\overline{W} = (\overline{W}_1, \overline{W}_2)$ be a pair of independent Wiener processes on $\mathbb{R}$ with $\overline{W}_1(0) = \overline{W}_2(0) = 0$. Then $(A_0^{(n)}, B_0^{(n)})$ converge in law in the space $\mathscr{M}_1(C[0, \infty), \mathbb{R}^2)$ to the process

$$G(t)=\phi(t)\int_{-\infty}^{t}\phi(s)^{-1}Q(s)\,d\overline{W}(s)$$
$$=P(t)\int_{-\infty}^{t}e^{C(t-s)}P^{-1}(s)\,Q(s)\,d\overline{W}(s)$$
$$=P(t+n\tau)\int_{-\infty}^{t+n\tau}e^{C(t+\tau n-s)}P^{-1}(s)\,Q(s)\,d\overline{W}(s-n\tau)$$
$$=\Phi(t+n\tau)\int_{-\infty}^{t+n\tau}\phi^{-1}(s)\,Q(s)\,d\overline{W}(s-n\tau) \tag{26}$$

which is a process having a periodic Gaussian law as can be seen from (26). Since the law of (A_0, B_0) is (not necessarily strictly) periodic for all $n\in\mathbb{N}$, it follows that $\mathscr{L}(A_0(\tau), B_0(\tau))=\mathscr{L}(A_0(0), B_0(0))$ and hence the laws of $(A_0^{(n)}, B_0^{(n)})$ and (A_0, B_0) coincide which implies that also the laws of (A_0, B_0) and G coincide proving uniqueness of a solution of (25) with a (not necessarily strictly) periodic law. □

Remark. If g_1 and g_2 vanish on the range of f_1^* and f_2^* respectively, then $A_0(t)\equiv B_0(t)\equiv 0$.

Theorem 4.3 describes the fluctuations of the periodic processes $(X_\varepsilon, Y_\varepsilon)$ around $(EX_\varepsilon, EY_\varepsilon)$ as $\varepsilon\downarrow 0$. We know from Theorem 4.1 that $(EX_\varepsilon, EY_\varepsilon)\to(f_1^*, f_2^*)$ as $\varepsilon\downarrow 0$. Hence one may pose the question whether in the limit $\varepsilon\downarrow 0$ the fluctuations around (f_1^*, f_2^*) are the same as around $(EX_\varepsilon, EY_\varepsilon)$. This question is closely related to the rate of convergence of $(EX_\varepsilon, EY_\varepsilon)$ towards (f_1^*, f_2^*) which will be established in Lemma 4.5. To do this however we need to know that the solutions of the deterministic system (1) converge to the periodic solution with exponential speed, which we will show in the following Lemma.

Lemma 4.4. *Let $f^{(c)}$ be the solution of* (1) *with $f_1^{(c)}(0)=a$, $f_2^{(c)}(0)=c>\frac{b}{a}$ and*

$$\tau_c := \min\left\{t>0: f_1^{(c)}(t)=a,\ f_2^{(c)}(t)>\frac{b}{a}\right\}.$$

Then there exists some $\delta>0$ and $0\leqq q<1$ such that

$$|f_2^{(c)}(\tau_c)-f_2^*(0)|\leqq q\,|c-f_2^*(0)|$$

whenever $|c-f_2^*(0)|<\delta$ (i.e. f^* is "exponentially stable").

Proof. Ponzo and Wax [15] used the transformation $X=\frac{a}{1+(b-1)x}$, $t=\frac{(b-1)^{1/2}}{a}\tau$ which maps the solution (X, Y) of (1) to the solution of the equation

$$\ddot{x}+\mu\left[2x-1+\frac{1}{\mu^2(x+\lambda)^2}\right]\dot{x}+\frac{x}{x+\lambda}=0,\qquad \mu=(B-1)^{3/2}/A,\qquad \lambda=\frac{1}{B-1},$$

which is equivalent to the "Lienard" system

$$\dot{x}=\mu[y-F(x)], \quad \dot{y}=-\frac{g(x)}{\mu} \tag{27}$$

where $F(x)=x^2-x+\frac{1}{\mu^2}\left[\frac{1}{\lambda}-\frac{1}{x+\lambda}\right]$ and $g(x)=\frac{x}{x+\lambda}$. Uniqueness and stability of a limit cycle are established in the appendix of [14]. Denoting the unique periodic solution by $\bar{f}$ with the initial condition $\bar{f}_1(0)=0$, $\bar{f}_2(0)>0$, choosing $A<B$ both sufficiently close to $\bar{f}_2(0)$ and defining $\bar{A}:=\bar{f}_2^{(A)}(\tau_A)$ and $\bar{B}:=\bar{f}_2^{(B)}(\tau_B)$, where $\bar{f}^{(A)}$ and $\bar{f}^{(B)}$ are the solutions of (27) starting in $(0,A)$ and $(0,B)$ respectively and $\tau_A:=\min\{t>0:\bar{f}_1^{(A)}(t)=0\}$ and $\tau_B:=\min\{t>0:\bar{f}_1^{(B)}(t)=0\}$, Ponzo and Wax show that

$$\bar{B}^2-\bar{A}^2<B^2-A^2-2M_A(B-A+\bar{A}-\bar{B}), \tag{28}$$

where $M_A>0$ is the value of $\bar{f}_2^{(A)}(t)$ at the (unique) intersection of $\bar{f}^{(A)}$ with the curve $y=F(x)$ for $0<t<\tau_A$. Since $B-A>0$ and $\bar{A}-\bar{B}>0$ it follows that $\bar{B}^2-\bar{A}^2<B^2-A^2$ and, because the same arguments can be repeated for the next "half-revolution" they get stability of the limit cycle. We show that their proof can be extended to prove even exponential stability: (28) implies

$$\frac{\bar{B}^2-\bar{A}^2}{B^2-A^2}<1-\frac{2M_A}{B+A}\leqq q_1<1$$

for all A, B in a suitable neighborhood of $\bar{f}_2(0)$. Defining $\bar{\bar{A}}:=\bar{f}_2^{(A)}(\bar{\tau}_A)$, $\bar{\bar{B}}:=\bar{f}_2^{(B)}(\bar{\tau}_B)$ with $\bar{\tau}_A=\min\{t>\tau_A\colon \bar{f}_1^{(A)})(t)=0\}$ and $\bar{\tau}_B=\min\{t>\tau_B\colon \bar{f}_1^{(B)}(t)=0\}$ we get, for a suitable constant $q_2<1$

$$q_2\geqq\frac{\bar{\bar{B}}^2-\bar{\bar{A}}^2}{B^2-A^2}=\frac{(\bar{\bar{B}}-\bar{\bar{A}})(\bar{\bar{B}}+\bar{\bar{A}})}{(B-A)(B+A)}$$

and hence $\bar{\bar{B}}-\bar{\bar{A}}\leqq q_3(B-A)$ for a constant $q_3<1$ and A, B close to $\bar{f}_2(0)$, since $(\bar{\bar{B}}+\bar{\bar{A}})(B+A)^{-1}$ is close to one in small neighborhoods of $\bar{f}_2(0)$. Transforming back to the original coordinates, the assertion follows. □

Lemma 4.5. *Fix $w_1, w_2, w_3, w_4\geqq 0$ and $a^2<b-1$. Let $0<\varepsilon\leqq 1$, $D_2\leqq w_1D_1+w_2$ and $D_1\leqq w_3D_2+w_4$ and D_1 and D_2 be sufficiently large and define*

$$\phi_{1,\varepsilon}(t):=\frac{EX_\varepsilon(t)-f_1^*(t)}{\varepsilon^2}, \qquad \phi_{2,\varepsilon}(t):=\frac{EY_\varepsilon(t)-f_2^*(t)}{\varepsilon^2}.$$

Then $(\phi_{1,\varepsilon},\phi_{2,\varepsilon})\xrightarrow[\varepsilon\downarrow 0]{}(\phi_1,\phi_2)$ uniformly on compact intervals, where (ϕ_1,ϕ_2) is the unique periodic (or constant) function satisfying

$$\begin{pmatrix}\dfrac{d\phi_1(t)}{dt}\\[2ex] \dfrac{d\phi_2(t)}{dt}\end{pmatrix}=\begin{pmatrix}-b-1+2f_1^*(t)f_2^*(t) & (f_1^*(t))^2\\ b-2f_1^*(t)f_2^*(t) & -(f_1^*(t))^2\end{pmatrix}\begin{pmatrix}\phi_1(t)\\ \phi_2(t)\end{pmatrix}$$
$$+\begin{pmatrix}k_1(t)f_2^*(t)+2k_2(t)f_1^*(t)\\ -k_1(t)f_2^*(t)-2k_2(t)f_1^*(t)\end{pmatrix} \tag{29}$$

with $\phi_1(0)=0$, where $k_1(t)=EA_0^2(t)$ and $k_2(t)=EA_0(t)B_0(t)$.

Remarks. $k_1(t)$ and $k_2(t)$ can be calculated by solving a linear system of three ordinary differential equations with periodic coefficients provided (f_1^*, f_2^*) is known explicitely (see [1]).

Note that the homogeneous part of (29) (i.e. for $k_1 = k_2 = 0$) is the same as Eq. (1), linearized around the periodic solution (f_1^*, f_2^*).

Proof. Since the homogeneous part of (29) is exponentially stable because of the last remark and Lemma 4.4, uniqueness of a periodic or constant solution follows as in the proof of Theorem 4.3. Obviously

$$\begin{aligned}\frac{d}{dt}\phi_{1,\varepsilon}(t) &= -(b+1)\,\phi_{1,\varepsilon}(t) + \varepsilon^{-2}(EX_\varepsilon^2(t)\,Y_\varepsilon(t) - (f_1^*(t))^2 f_2^*(t))\\ &= -(b+1)\,\phi_{1,\varepsilon}(t) + \varepsilon^{-2}\,E(\varepsilon A_\varepsilon(t)\\ &\quad + EX_\varepsilon(t))^2(\varepsilon B_\varepsilon(t) + EY_\varepsilon(t)) - (f_1^*(t))^2 f_2^*(t))\\ &= -(b+1)\,\phi_{1,\varepsilon}(t) + \varepsilon E A_\varepsilon^2(t)\,B_\varepsilon(t) + (EY_\varepsilon(t))\,EA_\varepsilon^2(t)\\ &\quad + 2EX_\varepsilon(t)\,EA_\varepsilon(t)\,B_\varepsilon(t) + \varepsilon^{-2}((EX_\varepsilon(t))^2\,EY_\varepsilon(t) - (f_1^*(t))^2 f_2^*(t)).\end{aligned}$$

Now

$$\begin{aligned}&\varepsilon^{-2}((EX_\varepsilon(t))^2\,EY_\varepsilon(t) - (f_1^*(t))^2 f_2^*(t))\\ &\quad = (EX_\varepsilon(t))^2\,\frac{EY_\varepsilon(t) - f_2^*(t)}{\varepsilon^2} + f_2^*(t)\,\frac{(EX_\varepsilon(t))^2 - (f_1^*(t))^2}{\varepsilon^2}\\ &\quad = (EX_\varepsilon(t))^2\,\phi_{2,\varepsilon}(t) + f_2^*(t)(EX_\varepsilon(t) + f_1^*(t))\,\phi_{1,\varepsilon}(t).\end{aligned}$$

Since $(EX_\varepsilon(t),\ EY_\varepsilon(t)) \to (f_1^*(t),\ f_2^*(t))$ uniformly on compact intervals, and because of Lemma 4.2 and Theorem 4.3, all we have to prove to apply Theorem 11.1.4 in [18] is $\limsup_{\varepsilon \downarrow 0} |\phi_{2,\varepsilon}(0)| < \infty$.

From (16) and Lemma 4.2 it follows that there exists a constant c_9 such that

$$\sup_{0 \leqq t \leqq T} ((EX_\varepsilon(t) - f_{1,\varepsilon}(t))^2 + (EY_\varepsilon(t) - f_{2,\varepsilon}(t))^2)^{1/2} \leqq c_9\,\varepsilon^2$$

for all $0 < \varepsilon \leqq 1$, where $(f_{1,\varepsilon}, f_{2,\varepsilon})$ is the solution of (1) with $f_{1,\varepsilon}(0) = a$, $f_{2,\varepsilon}(0) = EY_\varepsilon(0)$. Let $\tau^{(\varepsilon)} := \inf\left\{t > 0\colon f_{1,\varepsilon}(t) = a,\ f_{2,\varepsilon}(t) > \frac{b}{a}\right\}$ and let τ_ε be the period of $EX_\varepsilon(t)$. According to Lemma 4.4 there exists some $q < 1$ such that, for ε sufficiently small,

$$\begin{aligned}\varepsilon^2\,|\phi_{2,\varepsilon}(0)| = |f_{2,\varepsilon}(0) - f_2^*(0)| &\leqq \frac{1}{1-q}\,|f_{2,\varepsilon}(\tau^{(\varepsilon)}) - f_{2,\varepsilon}(0)|\\ &\leqq \frac{1}{1-q}\,(|f_{2,\varepsilon}(\tau_\varepsilon) - EY_\varepsilon(\tau_\varepsilon)| + |f_{2,\varepsilon}(\tau_\varepsilon) - f_{2,\varepsilon}(\tau^{(\varepsilon)})|)\\ &\leqq \frac{1}{1-q}\,(c_9\,\varepsilon^2 + c_{10}\,|\tau_\varepsilon - \tau^{(\varepsilon)}|)\end{aligned}$$

where c_{10} is an upper bound of the derivative of the second component in (1) in the bounded set $\{(f_{1,\varepsilon}(t),\ f_{2,\varepsilon}(t)),\ t \geqq 0,\ 0 < \varepsilon \leqq 1\}$. There exist constants δ_1, $\delta_2 > 0$ such that $\frac{d}{dt} f_{1,\varepsilon}(t) > \delta_1$ if $a - \delta_2 \leqq f_{1,\varepsilon}(t) \leqq a + \delta_2$ uniformly for all

$0<\varepsilon\leqq 1$. Hence

$$|\tau_\varepsilon-\tau^{(\varepsilon)}|\leqq\frac{1}{\delta_1}\sup_{0\leqq t\leqq T}|EX_\varepsilon(t)-f_{1,\varepsilon}(t)|\leqq\frac{c_9}{\delta_1}\varepsilon^2$$

for ε sufficiently small. Consequently $\limsup_{\varepsilon\downarrow 0}|\phi_{2,\varepsilon}(0)|<\infty$ and the assertion follows from Theorem 11.1.4 in [18]. □

Corollary. *Let the assumptions of Lemma* 4.5 *be satisfied and define*

$$\psi_{1,\varepsilon}(t):=\frac{X_\varepsilon(t)-f_1^*(t)}{\varepsilon},\qquad \psi_{2,\varepsilon}(t):=\frac{Y_\varepsilon(t)-f_2^*(t)}{\varepsilon}.$$

Then as $\varepsilon\downarrow 0$, $(\psi_{1,\varepsilon},\psi_{2,\varepsilon})$ *converge to* (A_0,B_0) *i.e. the same limit as* $(A_\varepsilon,B_\varepsilon)$.

Proof. This follows immediately from Lemma 4.5 since

$$\psi_{1,\varepsilon}(t)=A_\varepsilon(t)+\varepsilon\phi_{1,\varepsilon}(t),\qquad \psi_{2,\varepsilon}(t)=B_\varepsilon(t)+\varepsilon\phi_{2,\varepsilon}(t)$$

and $(\varepsilon\phi_{1,\varepsilon}(t),\ \varepsilon\phi_{2,\varepsilon}(t))\to(0,0)$ uniformly. □

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