Probab. Th. Rel. Fields 72, 359-386 (1986)

Probability Theory and Related Fields

Piecewise Invertible Dynamical Systems

Franz Hofbauer

Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

Summary. The aim of the paper is the investigation of piecewise monotonic maps T of an interval X. The main tool is an isomorphism of (X, T) with a topological Markov chain with countable state space which is described by a $0-1$-transition matrix M. The behavior of the orbits of points in X under T is very similar to the behavior of the paths of the Markov chain. Every irreducible submatrix of M gives rise to a T-invariant subset L of X such that L is the set $\omega(x)$ of all limit points of the orbit of an $x \in X$. The topological entropy of L is the logarithm of the spectral radius of the irreducible submatrix, which is a l^1-operator. Besides these sets L there are two T-invariant sets Y and P, such that for all $x \in X$ the set $\omega(x)$ is either contained in one of the sets L or in Y or in P. The set P is a union of periodic orbits and Y is contained in a finite union of sets $\omega(y)$ with $y \in X$ and has topological entropy zero. This isomorphism of (X, T) with a topological Markov chain is also an important tool for the investigation of T-invariant measures on X. Results in this direction, which are published elsewhere, are described at the end of the paper. Furthermore, a part of the proofs in the paper is purely topological without using the order relation of the interval X, so that some results hold for more general dynamical systems (X, T).

Introduction

The aim of this paper is the investigation of a class of topological dynamical systems (X, T) called piecewise invertible. They are a generalization of piecewise monotonic transformations on $[0, 1]$ and suggest applications to dynamical systems in higher dimensions. Mainly we shall investigate the nonwandering set of (X, T) and the topological entropy of invariant subsets of X. Some results hold only for piecewise monotonic transformations. The method we use is an oriented graph, whose one-sided paths represent the orbits of (X, T). It is used like the transition matrix of a Markov shift and hence called Markov diagram. A part of the results of this paper has been proved in [2] and [4] for piecewise monotonic transformations, but only with the additional assumption that the partition of $[0,1]$ into intervals, on which T is monotone, is a generator. This assumption excludes for example all T, which have an attracting periodic orbit. The more general case considered in this paper makes it necessary to give proofs different from those in [2] and [4], which use the order structure of piecewise monotonic transformations. Hence this paper can be read without any previous knowledge about piecewise monotonic transformations.

We call a topological dynamical system (X, T) piecewise invertible, if the compact metric space X has a finite partition $\mathfrak{Z}$ into closed sets (which are then also open), such that the continuous map $T: X \to X$ has the property that $T|Z$ is invertible for all $Z \in \mathfrak{Z}$. A special case are piecewise monotonic transformations. In this case, X is a totally ordered set, such that the topology of X is the order topology, the elements of $\mathfrak{Z}$ are closed intervals and, for $Z \in \mathfrak{Z}$, $T|Z$ is monotone and $T(Z)$ is again an interval. One gets examples of piecewise monotonic transformations, if one considers maps $T: [0,1] \to [0,1]$, where $[0,1]$ is the disjoint union of intervals J_i for $1 \leqq i \leqq N$ such that $T|J_i$ is continuous and monotone. For $1 < i \leqq N$, one substitutes the common endpoint of J_{i-1} and J_i and all its inverse images under all T^k not equal 0 or 1 by two points and extends T, such that $T|J_i$ is continuous for all i. Then T becomes a piecewise monotonic transformation and the J_i, which are now closed intervals, form the partition $\mathfrak{Z}$. As an example consider $x \mapsto 2x \pmod 1$. Here exactly those points are doubled, which have two dyadic expansions. One can generalize this to higher dimensions. Suppose $T: [0,1]^2 \to [0,1]^2$ is such that $T|J_i$ is continuous and invertible, where the J_i are pairwise disjoint, have piecewise smooth boundary, and their union is $[0,1]^2$. If a point belongs to the boundary of m different J_i, then substitute it by m different points, each of which belongs to one of the m different J_i, and do the same with all its inverse images under all T^k. Then extend T such that $T|J_i$ is continuous. In this way one gets a piecewise invertible dynamical system.

The paper is divided into two chapters. Chapter I contains those results, which can be proved without using the order structure of piecewise monotonic transformations. Chapter II contains those results about piecewise monotonic (X, T), the proofs of which rely on the order structure of (X, T). In §1 of Chap. I, the Markov diagram $\mathfrak{D}$ is defined and basic results about the representation of the orbits of (X, T) as onesided paths are proved. For piecewise monotonic transformations, the investigation of $\mathfrak{D}$ is continued in §1 of Chap. II. The order structure of (X, T) gives in this case a special structure of the Markov diagram. In §2 of Chap. I, the nonwandering set $\Omega(X, T)$ of (X, T) is investigated. Every irreducible subset $\mathfrak{C}$ of $\mathfrak{D}$ gives rise to a topologically transitive, T-invariant subset $L(\mathfrak{C})$ of X, if either $\mathfrak{Z}$ is a generator or (X, T) is piecewise monotonic. The other parts of $\Omega(X, T)$ are called L_∞, P and W. For L_∞ a condition is shown, which one can hope will imply that L_∞ is small in some sense. The elements of W are not in the center of (X, T). If (X, T) is piecewise monotonic, then P consists only of periodic points, and, if $\mathfrak{Z}$ is a generator, then $P = \emptyset$. In §3 of Chap. I, we write the Markov diagram as a $\mathfrak{D} \times \mathfrak{D}$-matrix M with entries 0 and 1. Then M is a $l^1(\mathfrak{D})$-operator. We express the topological entropy of certain T-invariant subsets of X in terms of the spectral radius r of submatrices of M. This and the results of § 1 of Chap. II are then used in § 2 of Chap. II to show for piecewise monotonic (X, T), that $h_{\mathrm{top}}(L(\mathfrak{C})) = \log r(M \mid \mathfrak{C})$ and that $h_{\mathrm{top}}(L_\infty) = 0$. Also a result about the growth rate of the number of inverse images of an $x \in L(\mathfrak{C})$ under $T^k \mid L(\mathfrak{C})$ is shown, which involves the topological entropy. In § 3 of Chap. I we investigate also the correspondence of closed paths in $\mathfrak{D}$ and periodic points of (X, T). In § 4 an example of a twodimensional piecewise invertible transformation is given. Finally, § 3 of Chap. II describes further applications of the method of Markov diagrams to piecewise monotonic transformations.

I. The Markov Diagram

§ 1. Imitating Markov Shifts

Let (X, T) be a piecewise invertible dynamical system and let $\mathfrak{Z}$ be the partition into closed-open sets Z, such that $T \mid Z$ is invertible. We give now the main definitions:

Successor. Suppose that D is a closed subset of some element of $\mathfrak{Z}$. The non-empty sets among $T(D) \cap Z$ for $Z \in \mathfrak{Z}$ are called successors of D. We write $D \to C$, if C is a successor of D. The successors of D are again closed subsets of elements of $\mathfrak{Z}$, so that one can iterate the formation of successors.

Markov Diagram. Let $\mathfrak{D}$ be the minimal set with $\mathfrak{Z} \subset \mathfrak{D}$ such that if $D \in \mathfrak{D}$, then $\mathfrak{D}$ contains also all successors of D. The oriented graph, which $\mathfrak{D}$ becomes, if one inserts arrows from every $D \in \mathfrak{D}$ to all successors of D, is called the Markov diagram of (X, T) with respect to $\mathfrak{Z}$.

Paths. A finite or infinite sequence $D_0 D_1 D_2 \ldots$ with $D_i \in \mathfrak{D}$ is called a path in $\mathfrak{D}$, if $D_i \to D_{i+1}$ for $i \geqq 0$. We say that an infinite path $D_0 D_1 D_2 \ldots$ represents $x \in X$, if $T^i(x) \in D_i$ for $i \geqq 0$.

We begin the investigation of $\mathfrak{D}$ with a lemma. For $k \geqq 0$ set

$$\mathfrak{Z}_k = \bigvee_{i=0}^{k} T^{-i} \mathfrak{Z} = \left\{ \bigcap_{i=0}^{k} T^{-i}(Z_i) \neq \emptyset : Z_i \in \mathfrak{Z} \right\},$$

which is again a partition of X into sets, which are closed and open, and T^k is invertible on each element of $\mathfrak{Z}_k$.

Lemma 1. *Suppose that $Z_i \in \mathfrak{Z}$ for $i \geqq 0$ and that $D \subset Z_0$. Set*

$$A_k = D \cap T^{-1}(Z_1) \cap \ldots \cap T^{-k}(Z_k) \quad \textit{for } k \geqq 0$$

and

$$D_0 = D, \quad D_k = T(D_{k-1}) \cap Z_k \quad \textit{for } k \geqq 1.$$

Then $T^k(A_k) = D_k$ for $k \geqq 0$.

Proof by induction. For $k=0$, the assertion is trivial. Suppose that $T^m(A_m)=D_m$ is proved. The formula $f(R\cap f^{-1}(S))=f(R)\cap S$ will be used often throughout the paper. We use it for $f=T^{m+1}$ and get

$$\begin{aligned} T^{m+1}(A_{m+1}) &= T^{m+1}(A_m\cap T^{-(m+1)}(Z_{m+1}))=T^{m+1}(A_m)\cap Z_{m+1} \\ &= T(D_m)\cap Z_{m+1}=D_{m+1}. \quad \square \end{aligned}$$

The first theorem shows that the Markov diagram can serve as a transition diagram of (X, T). The uniqueness of the representation of orbits as paths holds only partially. For $x\in X$ let $V_k(x)$ be that element of the partition $\mathfrak{Z}_k$, which contains x. We have $V_{k+1}(x)\subset V_k(x)$ for $k\geqq 0$. If $\bigcap_{k=0}^{\infty} V_k(x)=\{x\}$ for all $x\in X$, then $\mathfrak{Z}$ is called a generator.

Theorem 1. (i) *Suppose $D\in\mathfrak{D}$. Every $x\in D$ is represented by a path $D_0 D_1 D_2 \dots$ in the Markov diagram with $D_0=D$. On the other hand, every path in the Markov diagram represents an $x\in X$, which is unique, if $\bigcap_{k=0}^{\infty} V_k(x)=\{x\}$.*

(ii) *Suppose that $x\in X$ is represented by the paths $C_0 C_1 C_2 \dots$ and $D_0 D_1 D_2 \dots$ in the Markov diagram. If $C_0\cap V_k(x)=D_0\cap V_k(x)$, then $C_i=D_i$ for $i\geqq k$.*

Proof. (i) Choose $Z_i\in\mathfrak{Z}$ such that $T^i(x)\in Z_i$. Set $D_0=D$ and $D_j=T(D_{j-1})\cap Z_j$ for $j\geqq 1$. If $x\in D=D_0$, then $T^i(x)\in D_i$ for $i\geqq 0$ follows by induction. Hence $D_i\neq\emptyset$ and $D_0 D_1 D_2 \dots$ is a path in the Markov diagram representing x. On the other hand, if $D_0 D_1 D_2 \dots$ is a path in the Markov diagram, then $D_i=T(D_{i-1})\cap Y_i$ for $i\geqq 1$, where $Y_i\in\mathfrak{Z}$. Set $A_k=D_0\cap T^{-1}(Y_1)\cap\dots\cap T^{-k}(Y_k)$. Since the sets A_k are closed and decreasing, there is an $x\in\bigcap_{k=0}^{\infty} A_k$. By Lemma 1, we get $T^k(x)\in T^k(A_k)$ $=D_k$ for $k\geqq 0$, i.e. $D_0 D_1 D_2 \dots$ represents x. If $D_0 D_1 D_2 \dots$ represents also $y\in X$, then $T^i(y)\in D_i\subset Y_i$, i.e. $y\in V_k(x)=\bigcap_{i=0}^{k} T^{-i}(Y_i)$ for all k. Hence $y=x$, if $\bigcap_{k=0}^{\infty} V_k(x)$ $=\{x\}$.

(ii) Choose $Z_i\in\mathfrak{Z}$, such that $T^i(x)\in Z_i$. Since $C_0 C_1 C_2 \dots$ and $D_0 D_1 D_2 \dots$ represent x, we get $C_0\subset Z_0$, $D_0\subset Z_0$, $C_j=T(C_{j-1})\cap Z_j$ and $D_j=T(D_{j-1})\cap Z_j$ for $j\geqq 1$. Using Lemma 1, we get for $i\geqq k$ that

$$\begin{aligned} C_i &= T^i(C_0\cap T^{-1}(Z_1)\cap\dots\cap T^{-i}(Z_i))=T^i(C_0\cap V_i(x)) \\ &= T^i(D_0\cap V_i(x))=D_i. \quad \square \end{aligned}$$

The following two lemmas will be useful later.

Lemma 2. *Suppose that $D_0 D_1 \dots D_k$ is a path in the Markov diagram. Then $\bigcap_{i=0}^{k} T^{-i}(D_i)$ is contained in some element of $\mathfrak{Z}_k$. If $C_0 C_1 \dots C_k$ is a path with C_0 $=D_0$ such that $\bigcap_{i=0}^{k} T^{-i}(C_i)$ and $\bigcap_{i=0}^{k} T^{-i}(D_i)$ are contained in the same element V of $\mathfrak{Z}_k$, then $C_i=D_i$ for $0\leqq i\leqq k$.*

Proof. Choose $Z_i \in \mathfrak{Z}$ such that $D_0 \subset Z_0$ and $D_i = T(D_{i-1}) \cap Z_i$ for $1 \leqq i \leqq k$. Then $\bigcap_{i=0}^{k} T^{-i}(D_i) \subset V := \bigcap_{i=0}^{k} T^{-i}(Z_i) \in \mathfrak{Z}_k$. This shows the first assertion. If $Y_i \in \mathfrak{Z}$ is such that $C_0 \subset Y_0$ and $C_i = T(C_{i-1}) \cap Y_i$ for $1 \leqq i \leqq k$, then V is also $\bigcap_{i=0}^{k} T^{-i}(Y_i)$, since $\bigcap_{i=0}^{k} T^{-i}(C_i) \subset V$. Since $\mathfrak{Z}$ is a partition, we get $Z_i = Y_i$ for $1 \leqq i \leqq k$. Since $C_0 = D_0$, we get now $C_i = D_i$ for $1 \leqq i \leqq k$ by the definition of a successor. □

Lemma 3. *If (X, T) is piecewise monotonic, then all $D \in \mathfrak{D}$ are closed intervals with endpoints in $\bigcup_{i=0}^{\infty} T^i(K)$, where K is the set of endpoints of the intervals in $\mathfrak{Z}$.*

Proof by induction. For $D \in \mathfrak{Z}$, the assertion is trivial. If it holds for D, then it holds also for $T(D) \cap Z$, where $Z \in \mathfrak{Z}$, i.e. for all successors of D, since D is a closed subinterval of some $Y \in \mathfrak{Z}$, which implies that $T(D)$ is again a closed interval by the monotonicity of $T|Y$ and the fact that $T(Y)$ is an interval. □

§ 2. The Nonwandering Set

An $x \in X$ is called wandering, if x has a neighbourhood U with $T^k(U) \cap U = \emptyset$ for all $k \geqq 1$. This is equivalent with the existence of a neighbourhood U of x and a $k_0 \geqq 1$ with $T^k(U) \cap U = \emptyset$ for $k \geqq k_0$. We denote the set of all $x \in X$, which are not wandering, by $\Omega(X, T)$. For $z \in X$ let $\omega(z)$ be the set of limit points of the orbit $\{T^i(z) : i \geqq 0\}$ of z. Then $\omega(z)$ is contained in $\Omega(X, T)$ and called the ω-limit of z.

We begin with the investigation of correspondences between subsets of $\mathfrak{D}$ and subsets of X. If $\mathfrak{A} \subset \mathfrak{D}$, we set $H(\mathfrak{A}) = \bigcup_{D \in \mathfrak{A}} D$ and $v(\mathfrak{A}) = \{D \in \mathfrak{D} : \exists\, C \in \mathfrak{A}$ with $C \to D\}$. We call a subset $\mathfrak{A}$ of $\mathfrak{D}$ *closed*, if $C \in \mathfrak{A}$ and $C \to D$ imply $D \in \mathfrak{A}$. We call $\mathfrak{A} \subset \mathfrak{D}$ *perfect*, if $\mathfrak{A}$ is closed and if for every $C \in \mathfrak{A}$ there is a $D \in \mathfrak{A}$ with $D \to C$.

Lemma 4. (i) $\mathfrak{A}$ *closed* $\Leftrightarrow v(\mathfrak{A}) \subset \mathfrak{A}$, $\mathfrak{A}$ *perfect* $\Leftrightarrow v(\mathfrak{A}) = \mathfrak{A}$.

(ii) $\mathfrak{A}$ *closed* $\Rightarrow v(\mathfrak{A})$ *is closed.*

(iii) $H(v(\mathfrak{A})) = T(H(\mathfrak{A}))$.

(iv) *If* $\mathfrak{A}$ *is closed, then* $T(H(\mathfrak{A})) \subset H(\mathfrak{A})$. *If* $\mathfrak{A}$ *is perfect, then* $T(H(\mathfrak{A})) = H(\mathfrak{A})$.

Proof. (i) and (ii) are direct consequences of the definitions. (iii) follows from the fact that, if $D \in \mathfrak{D}$, then $T(D)$ is the union of the successors of D. (iv) follows from (i) and (iii). □

We say that a path leads from C to D in the Markov diagram, if there is a path $C_0 C_1 \dots C_k$ with $C_0 = C$ and $C_k = D$. We call $\mathfrak{C} \subset \mathfrak{D}$ *irreducible*, if for every pair C, D of elements of $\mathfrak{C}$, a path leads from C to D, and if every subset of $\mathfrak{D}$, which contains $\mathfrak{C}$ strictly, does not have this property.

Lemma 5. *Suppose $\mathfrak{C} \subset \mathfrak{D}$ is irreducible. Let $\bar{\mathfrak{C}}$ be the set of all $D \in \mathfrak{D}$ for which a path leads from some element of $\mathfrak{C}$ to D. Then $\bar{\mathfrak{C}}$ is perfect and $\tilde{\mathfrak{C}} := \bar{\mathfrak{C}} \setminus \mathfrak{C}$ is closed.*

Proof. That $\bar{\mathfrak{C}}$ is closed, follows from the definition of $\bar{\mathfrak{C}}$. That $\bar{\mathfrak{C}}$ is perfect, follows from the irreducibility of $\mathfrak{C}$. Now suppose $D \in \tilde{\mathfrak{C}}$ and $D \to C$. As $D \in \bar{\mathfrak{C}}$, we get $C \in \bar{\mathfrak{C}}$, since $\bar{\mathfrak{C}}$ is closed. If $C \in \mathfrak{C}$, we get $D \in \mathfrak{C}$ by the definition of irreducibility and of $\bar{\mathfrak{C}}$, a contradiction to $D \in \tilde{\mathfrak{C}}$. Hence $C \in \bar{\mathfrak{C}} \setminus \mathfrak{C} = \tilde{\mathfrak{C}}$ and $\tilde{\mathfrak{C}}$ is closed. $\square$

If $\mathfrak{A}$ and $\mathfrak{B}$ are closed subsets of $\mathfrak{D}$ with $\mathfrak{B} \subset \mathfrak{A}$, then set

$$\Psi(\mathfrak{A}, \mathfrak{B}) = \bigcap_{i=0}^{\infty} \overline{H(\mathfrak{A}) \setminus T^{-i}(H(\mathfrak{B}))}.$$

For an irreducible subset $\mathfrak{C}$ of $\mathfrak{D}$, we set $\Omega(\mathfrak{C}) = \Psi(\bar{\mathfrak{C}}, \tilde{\mathfrak{C}})$. By Lemma 5, $\bar{\mathfrak{C}}$ and $\tilde{\mathfrak{C}}$ are closed. This definition of $\Omega(\mathfrak{C})$ is introduced in [2] and a bit more convenient than those used in [4].

Lemma 6. *Suppose $\mathfrak{B} \subset \mathfrak{A} \subset \mathfrak{D}$ and that $\mathfrak{A}$ and $\mathfrak{B}$ are closed.*

(i) *$x \notin \Psi(\mathfrak{A}, \mathfrak{B})$ if and only if x has a neighbourhood U with $T^i(U \cap H(\mathfrak{A})) \subset H(\mathfrak{B})$ for some i.*

(ii) *$\Psi(\mathfrak{A}, \mathfrak{B})$ is T-invariant.*

(iii) *If $\mathfrak{F} \subset \mathfrak{D}$ is closed, then $\Psi(\mathfrak{A} \cup \mathfrak{F}, \mathfrak{B} \cup \mathfrak{F}) \subset \Psi(\mathfrak{A}, \mathfrak{B})$.*

Proof. (i) $x \notin \Psi(\mathfrak{A}, \mathfrak{B})$ is equivalent with $x \notin \overline{H(\mathfrak{A}) \setminus T^{-i}(H(\mathfrak{B}))}$ for some i, and this is equivalent with $T^i(U \cap H(\mathfrak{A})) \subset H(\mathfrak{B})$ for some i and some neighbourhood U of x.

(ii) If $T(x) \notin \Psi(\mathfrak{A}, \mathfrak{B})$, then, by (i), $T^i(U \cap H(\mathfrak{A})) \subset H(\mathfrak{B})$ for some i and some neighbourhood U of $T(x)$. Since T is continuous, $T^{-1}(U)$ is a neighbourhood of x and $T^{i+1}(T^{-1}(U) \cap H(\mathfrak{A})) \subset H(\mathfrak{B})$, since $T(H(\mathfrak{A})) \subset H(\mathfrak{A})$ by (iv) of Lemma 4. By (i), we get $x \notin \Psi(\mathfrak{A}, \mathfrak{B})$.

(iii) If $x \notin \Psi(\mathfrak{A}, \mathfrak{B})$, then $T^i(U \cap H(\mathfrak{A})) \subset H(\mathfrak{B})$ for some i and some neighbourhood U of x, by (i). As $T(H(\mathfrak{F})) \subset H(\mathfrak{F})$ by (iv) of Lemma 4, this implies $T^i(U \cap H(\mathfrak{A} \cup \mathfrak{F})) \subset H(\mathfrak{B} \cup \mathfrak{F})$, and hence $x \notin \Psi(\mathfrak{A} \cup \mathfrak{F}, \mathfrak{B} \cup \mathfrak{F})$. $\square$

Examples of piecewise monotonic transformations can be found in [3] and [4]. We give here an example, where $\mathfrak{Z}$ is not a generator. Let T be the transformation shown by Fig. 1. The points a and b are doubled (cf. the introduction). Set $I = [0, a-]$, $J = [a+, b-]$, $K = [b+, 1]$, $J_1 = [T(a), a-]$, $J_2 = [T^2(a), 1]$ and $K_j = [a+, T^j(1)]$ for $j \geqq 1$. Since $T(a) = T^3(a)$ we get the following Markov diagram

$$\begin{tikzcd}
I \arrow[r] \arrow[d, shift left] & J \arrow[r] \arrow[loop above] \arrow[dl] & J_1 \arrow[r, shift left] & J_2 \arrow[l, shift left] \arrow[dll] & & \\
K \arrow[r] \arrow[u, shift left] & K_1 \arrow[r] \arrow[ur] & K_2 \arrow[r] \arrow[u] & K_3 \arrow[r] \arrow[ul] & K_4 \arrow[r] \arrow[ull] & \cdots
\end{tikzcd}$$

We have the irreducible subsets $\mathfrak{C}_1 = \{I, J, K\}$ and $\mathfrak{C}_2 = \mathfrak{D} \setminus \mathfrak{C}_1$. This gives $\bar{\mathfrak{C}}_1 = \mathfrak{D}$, $\tilde{\mathfrak{C}}_1 = \bar{\mathfrak{C}}_2 = \mathfrak{C}_2$ and $\tilde{\mathfrak{C}}_2 = \emptyset$. Hence

$$H(\bar{\mathfrak{C}}_1) = [0, 1], \qquad H(\tilde{\mathfrak{C}}_1) = H(\bar{\mathfrak{C}}_2) = [T(a), x) \cup [T^2(a), 1], \qquad \Omega(\mathfrak{C}_2) = H(\bar{\mathfrak{C}}_2)$$

and $\Omega(\mathfrak{C}_1)$ is a Cantor set.

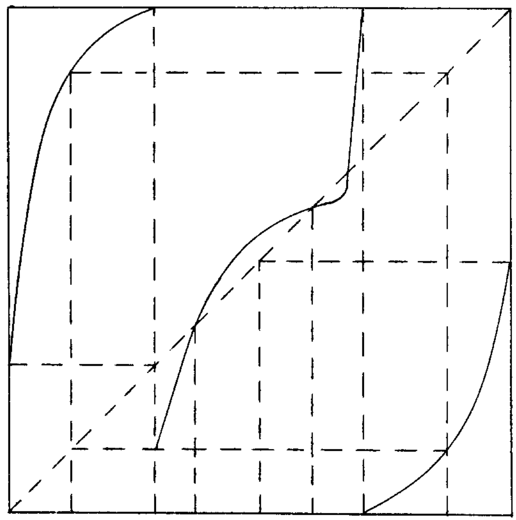

Fig. 1

The next two lemmas give the method, how the Markov diagram is used for the study of T-invariant subsets of X. We shall use them several times in the sequel.

Lemma 7. *Suppose that $\mathfrak{B} \subset \mathfrak{A} \subset \mathfrak{D}$ and that $\mathfrak{A}$ and $\mathfrak{B}$ are closed. Suppose $\mathfrak{A}' \subset \mathfrak{A}$ and $V \in \mathfrak{Z}_k$ are such that $H(\mathfrak{A}') \supset (V \cap H(\mathfrak{A})) \backslash H(\mathfrak{B})$. If $V \cap \Psi(\mathfrak{A}, \mathfrak{B}) \neq \emptyset$, then there is a path $D_0 D_1 \ldots D_k$ in $\mathfrak{A} \backslash \mathfrak{B}$ with $D_0 \in \mathfrak{A}'$ and $D_k \not\subset H(\mathfrak{B})$ such that $\bigcap_{i=0}^{k} T^{-i}(D_i) \subset V$.*

Proof. Choose $Z_i \in \mathfrak{Z}$ such that $V = \bigcap_{i=0}^{k} T^{-i}(Z_i)$. For every $D \in \mathfrak{A}'$ set $D_0 = D$ and $D_i = T(D_{i-1}) \cap Z_i$ for $1 \leq i \leq k$. For $V_D := V \cap D$ we have

$$H(\mathfrak{B}) \cup \bigcup_{D \in \mathfrak{A}'} V_D \supset H(\mathfrak{B}) \cup (V \cap H(\mathfrak{A}')) \supset V \cap H(\mathfrak{A}).$$

If $V_D \neq \emptyset$, then $D \subset Z_0$ as $D \cap Z_0 \neq \emptyset$, and $T^k(V_D) = D_k$ by Lemma 1. If for all $D \in \mathfrak{A}'$ with $V_D \neq \emptyset$ we have $D_k \subset H(\mathfrak{B})$, then

$$T^k(V \cap H(\mathfrak{A})) \subset T^k(H(\mathfrak{B}) \cup \bigcup_{D \in \mathfrak{A}'} V_D) \subset T^k(H(\mathfrak{B})) \cup \bigcup_{D \in \mathfrak{A}'} D_k \subset H(\mathfrak{B}),$$

since $H(\mathfrak{B})$ is T-invariant by Lemma 4. As V is open, we get $V \cap \Psi(\mathfrak{A}, \mathfrak{B}) = \emptyset$ by (i) of Lemma 6. Hence there is a $D \in \mathfrak{A}'$ with $V_D \neq \emptyset$ and $D_k \not\subset H(\mathfrak{B})$. This implies that $D_k \notin \mathfrak{B}$ and hence $D_i \notin \mathfrak{B}$ for $0 \leq i \leq k$, as $\mathfrak{B}$ is closed. Furthermore, $D_k = T^k(V_D) \neq \emptyset$ as $V_D \neq \emptyset$ and hence $D_i \neq \emptyset$ for $0 \leq i \leq k$ as $D_k \subset T^{k-i}(D_i)$. Since $D_0 = D \in \mathfrak{A}' \subset \mathfrak{A}$ and $\mathfrak{A}$ is closed, the path $D_0 D_1 \ldots D_k$ is in $\mathfrak{A} \backslash \mathfrak{B}$. From $D_i \subset Z_i$ (for $i = 0$ this follows from $D_0 \cap Z_0 \supset V_D \neq \emptyset$, as every element of $\mathfrak{D}$ is contained in an element of the partition $\mathfrak{Z}$) we get $\bigcap_{i=0}^{k} T^{-i}(D_i) \subset V$, finishing the proof. □

Lemma 8. *Let $\mathfrak{C}$ be an irreducible subset of $\mathfrak{D}$ such that $H(\bar{\mathfrak{C}})\setminus H(\tilde{\mathfrak{C}})\neq\emptyset$. Then every infinite path in $\mathfrak{C}$ represents an $x\in\Omega(\mathfrak{C})$.*

Proof. We write F for $H(\bar{\mathfrak{C}})$ and G for $H(\tilde{\mathfrak{C}})$. As $F\setminus G\neq\emptyset$, we find a $y\in F\setminus G$. Hence there is a $C\in\bar{\mathfrak{C}}\setminus\tilde{\mathfrak{C}}=\mathfrak{C}$ with $y\in C$. Suppose $D_0 D_1 D_2 \dots$ is an infinite path in $\mathfrak{C}$. Since $\mathfrak{C}$ is irreducible, we find for every i an $m\geqq i$ and a finite path $C_i C_{i+1} \dots C_m$ in $\mathfrak{C}$ with $C_i=D_i$ and $C_m=C$. Choose $Z_j\in\mathfrak{Z}$ such that $D_j=T(D_{j-1})\cap Z_j$ for $1\leqq j\leqq i$ and $C_j=T(C_{j-1})\cap Z_j$ for $i<j\leqq m$. Set $A_i=D_0\cap T^{-1}(Z_1)\cap\dots\cap T^{-i}(Z_i)$. By Lemma 1 we get $T^m(A_i\cap T^{-(i+1)}(Z_{i+1})\cap\dots\cap T^{-m}(Z_m))=C$. Hence we find an $x_i\in A_i$ with $T^m(x_i)=y$. The points x_i have a limit point x in $\bigcap_{i=0}^{\infty} A_i$, since every A_i is closed. Then $T^i(x)\in T^i(A_i)=D_i$ for all i by Lemma 1 and $D_0 D_1 D_2 \dots$ represents x. Furthermore, $T^m(x_i)=y\notin G$, i.e. $x_i\in F\setminus T^{-m}(G)$. Since $m\geqq i$ and the sets $\overline{F\setminus T^{-m}(G)}$ decrease to $\Omega(\mathfrak{C})$ for $m\to\infty$, we get $x\in\Omega(\mathfrak{C})$. Remark that $T(G)\subset G$ by Lemmas 4 and 5. □

The sets $V_k(x)$ defined above are open neighbourhoods of $x\in X$. They define a topology on X, which has less open sets than the original one. We call limit points with respect to this topology $\mathfrak{Z}$-limit points. If the two topologies coincide, i.e. if $\bigcap_{k=0}^{\infty} V_k(x)=\{x\}$ for all $x\in X$, then $\mathfrak{Z}$ is called a generator of (X, T).

Theorem 2. *Suppose $\mathfrak{C}\subset\mathfrak{D}$ is irreducible.*

(i) *If $x\in\operatorname{int}(H(\bar{\mathfrak{C}})\setminus H(\tilde{\mathfrak{C}}))$ and $x\notin\Omega(\mathfrak{C})$ then x is wandering.*

(ii) *If $H(\bar{\mathfrak{C}})\setminus H(\tilde{\mathfrak{C}})\neq\emptyset$, then there is a path in $\mathfrak{C}$, which contains every finite path of $\mathfrak{C}$ and which represents a $z\in\Omega(\mathfrak{C})$. Furthermore, $\omega(z)\subset\Omega(\mathfrak{C})$ and $\Omega(\mathfrak{C})$ is a subset of the set of $\mathfrak{Z}$-limit points of the orbit of z.*

Proof. (i) We can choose the neighbourhood U of x in (i) of Lemma 6 such that $U\subset H(\bar{\mathfrak{C}})\setminus H(\tilde{\mathfrak{C}})$ and get $T^j(U)\cap U=\emptyset$ for $j\geqq i$, since $H(\tilde{\mathfrak{C}})$ is T-invariant by Lemmas 4 and 5.

(ii) As $\mathfrak{C}$ is irreducible, we find an infinite path in $\mathfrak{C}$, which contains every finite path of $\mathfrak{C}$. By Lemma 8, it represents a $z\in\Omega(\mathfrak{C})$. By (ii) of Lemma 6, $\Omega(\mathfrak{C})$ is T-invariant, and by definition, $\Omega(\mathfrak{C})$ is closed. Hence the limit points of the orbit of $z\in\Omega(\mathfrak{C})$ are in $\Omega(\mathfrak{C})$. On the other hand consider some $V_k(x)$ of an $x\in\Omega(\mathfrak{C})$. Since $x\in V_k(x)\cap\Omega(\mathfrak{C})$, we find by Lemma 7 applied to $\mathfrak{A}=\mathfrak{A}'=\bar{\mathfrak{C}}$ and $\mathfrak{B}=\tilde{\mathfrak{C}}$ a path $D_0 D_1 \dots D_k$ in $\mathfrak{C}$ with $\bigcap_{i=0}^{k} T^{-i}(D_i)\subset V_k(x)$. As $D_0 D_1 \dots D_k$ is contained in the path in $\mathfrak{C}$ representing z, there is an m with $T^{m+i}(z)\in D_i$ for $0\leqq i\leqq k$. This implies $T^m(z)\in V_k(x)$. Since k was arbitrary, this says that x is a $\mathfrak{Z}$-limit point of the orbit of z. □

Theorem 2 clarifies, which points of $\operatorname{int}(H(\bar{\mathfrak{C}})\setminus H(\tilde{\mathfrak{C}}))$ belong to $\Omega(X, T)$, if $\mathfrak{Z}$ is a generator. The next theorem deals with $H(\mathfrak{A})$ for a closed set $\mathfrak{A}$, e.g. $\mathfrak{A}=\tilde{\mathfrak{C}}$. For a sequence $\mathfrak{A}_1\supsetneqq\mathfrak{A}_2\supsetneqq\mathfrak{A}_3\supsetneqq\dots$ of closed sets $\mathfrak{A}_i\subset\mathfrak{D}$ set $\mathfrak{A}_\infty=\bigcap_{i=1}^{\infty}\mathfrak{A}_i$ and $\Omega((\mathfrak{A}_j)_{j\geqq 1})=\bigcap_{i=1}^{\infty}\Psi(\mathfrak{A}_i, \mathfrak{A}_\infty)$. One easily checks that $\mathfrak{A}_\infty$ is a closed set. We say that a subset Y of X is *represented at infinity* in the Markov diagram, if for every $x \in Y$, for every finite subset $\mathfrak{F}$ of $\mathfrak{D}$, and for every $k \in \mathbb{N}$ there is a path $D_0 D_1 \dots D_k$ in $\mathfrak{D} \setminus \mathfrak{F}$ with $\bigcap_{i=0}^{k} T^{-i}(D_i) \subset V_k(x)$. The Markov diagram for piecewise monotonic transformations (cf. §1 of Chap. II below) suggests, that there is not "enough space" for "large" sets at infinity in the Markov diagram. Hence representation at infinity can be considered as a kind of smallness condition.

Theorem 3. *Suppose $(\mathfrak{A}_i)_{i \geq 1}$ is a decreasing sequence of closed subsets of $\mathfrak{D}$ and set $\mathfrak{A}_\infty = \bigcap_{i=1}^{\infty} \mathfrak{A}_i$. Then $\Omega((\mathfrak{A}_i)_{i \geq 1})$ is represented at infinity in the Markov diagram.*

Proof. This follows from the fact that $\bigcap_{i=0}^{\infty} (\mathfrak{A}_i \setminus \mathfrak{A}_\infty) = \emptyset$ and Lemma 7 applied to $\mathfrak{A} = \mathfrak{A}' = \mathfrak{A}_i$ and $\mathfrak{B} = \mathfrak{A}_\infty$ for every i, since $x \in V_k(x) \cap \Psi(\mathfrak{A}_i, \mathfrak{A}_\infty)$ for every k and every $x \in \Omega((\mathfrak{A}_i)_{i \geq 1})$. □

We want to improve Theorem 2 in the case, when $\mathfrak{Z}$ is not a generator. This gives then a satisfactory result for piecewise monotonic transformations (cf. Theorem 4 below). To this end set $\mathfrak{Z}_\infty = \left\{ \bigcap_{i=0}^{\infty} T^{-i}(Z_i) \neq \emptyset \colon Z_i \in \mathfrak{Z} \right\}$. The elements of $\mathfrak{Z}_\infty$ are closed and called $\mathfrak{Z}$-atoms. If $I \in \mathfrak{Z}_\infty$ consists only of a single point, we call I a trivial $\mathfrak{Z}$-atom. $\mathfrak{Z}$ is a generator, if and only if all $\mathfrak{Z}$-atoms are trivial.

Lemma 9. (i) *X is the disjoint union of all $\mathfrak{Z}$-atoms and $T^k | I$ is invertible for all $k \geq 1$ and all $I \in \mathfrak{Z}_\infty$.*

(ii) *The image of a $\mathfrak{Z}$-atom is contained in a $\mathfrak{Z}$-atom.*

(iii) *For $I \in \mathfrak{Z}_\infty$ and $k \geq 1$, either $T^k(I) \cap I = \emptyset$ or $T^k(I) \subset I$.*

(iv) *Suppose (X, T) is piecewise monotonic and $I \in \mathfrak{Z}_\infty$. Then I is an interval and $T^k | I$ is monotone for all k.*

Proof. $\mathfrak{Z}_\infty$ is the refinement of the partitions $\mathfrak{Z}_k$ for $k \geq 0$. Hence (i) follows. Since the elements of $\mathfrak{Z}_k$ are intervals, on which T^k is monotone, if (X, T) is piecewise monotonic, we get also (iv). Every $I \in \mathfrak{Z}_\infty$ can be written as $I = Z \cap T^{-1}(J)$ with $Z \in \mathfrak{Z}$ and $J \in \mathfrak{Z}_\infty$. This gives (ii). Now (iii) follows from (i) and (ii). □

For $x \in X$ let $\tilde{\mathfrak{U}}(x)$ be a neighbourhood system of x. We define $\mathfrak{U}(x)$ as follows. By (i) of Lemma 9 there is a unique $I \in \mathfrak{Z}_\infty$ with $x \in I$. If $I = \{x\}$, set $\mathfrak{U}(x) = \tilde{\mathfrak{U}}(x)$. Otherwise, if I is nontrivial, set $\mathfrak{U}(x) = \{U \setminus I \colon U \in \tilde{\mathfrak{U}}(x)\}$. Since I is closed, the elements of $\mathfrak{U}(x)$ are open. If $x \in \operatorname{int} I$, we can set $\mathfrak{U}(x) = \{\emptyset\}$.

Lemma 10. *Suppose (X, T) is piecewise monotonic. For every $x \in X$ one can choose $\mathfrak{U}(x)$ such that every nonempty element of $\mathfrak{U}(x)$ is an interval and a union of elements of $\bigcup_{m=0}^{\infty} \mathfrak{Z}_m$.*

Proof. Let $I \in \mathfrak{Z}_\infty$ be such that $x \in I$. If $I = \{x\}$, set $\mathfrak{U}(x) = \{V_k(x) \colon k \geq 0\}$. Now let I be nontrivial. If $x \in \operatorname{int} I$, set $\mathfrak{U}(x) = \{\emptyset\}$. If $x \in bd\, I$, say the left endpoint of the interval I (cf. (iv) of Lemma 9), set $\mathfrak{U}(x) = \{(V_k(x) \setminus I) \cap \{y \in X \colon y < x\} \colon k \geq 0\}$. As I

is a subinterval of the interval $V_k(x)$, the set $V_k(x)\setminus I$ is the union of two intervals (the right one can be empty), the left one of which is an element of $\mathfrak{U}(x)$. It is the union over m of the sets $(V_m(x)\setminus V_{m+1}(x))\cap\{y: y<x\}$, each of which is a union of elements of $\mathfrak{Z}_{m+1}$ or empty. Furthermore $\bigcap_{k=0}^{\infty}(V_k(x)\setminus I)\cap\{y: y<x\}=\emptyset$. $\square$

For closed subsets $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathfrak{D}$ with $\mathfrak{B}\subset\mathfrak{A}$ define

$$L(\mathfrak{A}, \mathfrak{B})=\{x\in X: Q\cap\Psi(\mathfrak{A}, \mathfrak{B})\neq\emptyset \text{ for all } Q\in\mathfrak{U}(x)\}.$$

We have $L(\mathfrak{A}, \mathfrak{B})\subset\Psi(\mathfrak{A}, \mathfrak{B})$, since $\Psi(\mathfrak{A}, \mathfrak{B})$ is closed, and $L(\mathfrak{A}, \mathfrak{B})=\Psi(\mathfrak{A}, \mathfrak{B})$, if $\mathfrak{Z}$ is a generator. One deduces from (ii) of Lemma 6, that $L(\mathfrak{A}, \mathfrak{B})$ is T-invariant.

Let Γ be the set of all irreducible subsets of $\mathfrak{D}$. For $\mathfrak{C}\in\Gamma$ set $L(\mathfrak{C})=L(\bar{\mathfrak{C}}, \tilde{\mathfrak{C}})$. If $(\mathfrak{A}_i)_{i\geq 1}$ is a decreasing sequence of closed subsets of $\mathfrak{D}$, we set $L((\mathfrak{A}_i)_{i\geq 1}) =\bigcap_{i=1}^{\infty} L(\mathfrak{A}_i, \mathfrak{A}_\infty)$, where $\mathfrak{A}_\infty=\bigcap_{i=1}^{\infty}\mathfrak{A}_i$. Let L_∞ be the union of all such sets $L((\mathfrak{A}_i)_{i\geq 1})$.

Theorem 4. *Suppose (X, T) is piecewise monotonic.*

(i) *If $\mathfrak{C}\in\Gamma$, then $L(\mathfrak{C})=\emptyset$ or $L(\mathfrak{C})=\omega(z)$ where z is as in Theorem 2.*

(ii) *If $x\in L_\infty$, then $x\in\omega(c)$ for some c in the finite set K (cf. Lemma 3).*

Proof. (i) Let z be as in Theorem 2. If such a z does not exist, then $H(\bar{\mathfrak{C}})\setminus H(\tilde{\mathfrak{C}}) =\emptyset$ and $L(\mathfrak{C})\subset\Omega(\mathfrak{C})=\emptyset$. Fix some $x\in L(\mathfrak{C})$. Then $Q\cap\Omega(\mathfrak{C})=Q\cap\Psi(\bar{\mathfrak{C}}, \tilde{\mathfrak{C}})\neq\emptyset$ for all $Q\in\mathfrak{U}(x)$. By Lemma 10, for all $Q\in\mathfrak{U}(x)$, there is a $V\in\bigcup_{m=0}^{\infty}\mathfrak{Z}_m$ with $V\subset Q$ and $V\cap\Omega(\mathfrak{C})\neq\emptyset$. By Theorem 2, $V\cap\Omega(\mathfrak{C})\neq\emptyset$ implies $T^i(z)\in V$ for some i. Hence $x\in\omega(z)$ and $L(\mathfrak{C})\subset\omega(z)$.

Now suppose $x\in\omega(z)$. Let $I\in\mathfrak{Z}_\infty$ be such that $x\in I$. If $T^i(z)\in I$ for $i=m$ and n, then $T^p(I)\subset I$ for $p=n-m$, by (iii) of Lemma 9. Let J be the union of the $\mathfrak{Z}$-atoms which contain $T^j(I)$ for $0\leq j<p$. Then $T^i(z)\in J$ for $i\geq m$, every $\mathfrak{Z}$-limit point of the orbit of z is in J, and $\Omega(\mathfrak{C})\subset J$ by Theorem 2. This implies $L(\mathfrak{C}) =\emptyset$, if I is nontrivial. If $T^i(z)\in I$ for at most one i or if I is trivial, then for every $Q\in\mathfrak{U}(x)$ there is a j with $T^j(z)\in Q$, as $x\in\omega(z)$. Since $T^j(z)\in\Omega(\mathfrak{C})$ for all j by Theorem 2 and (ii) of Lemma 6, we get $x\in L(\mathfrak{C})$. Hence $\omega(z)\subset L(\mathfrak{C})$.

(ii) Suppose x is in some $L((\mathfrak{A}_i)_{i\geq 1})$ and $x\notin\omega(c)$ for all $c\in K$. In particular, there is a $Q\in\mathfrak{U}(x)$ with $T^j(c)\notin Q$ for all $c\in K$ and all $j\geq 0$, as K is finite. We choose $\mathfrak{U}(x)$ as in Lemma 10. For every $i\geq 1$, we find a $D\in\mathfrak{A}_i$ with $Q\cap D\neq\emptyset$, because otherwise $Q\cap H(\mathfrak{A}_i)=\emptyset$, which implies $Q\cap\Psi(\mathfrak{A}_i, \mathfrak{A}_\infty)=\emptyset$ and $x\notin L((\mathfrak{A}_i)_{i\geq 1})$. By Lemma 3, the interval Q cannot contain an endpoint of the interval D, hence $Q\subset D$. Let V be one of the elements of $\bigcup_{m=0}^{\infty}\mathfrak{Z}_m$, whose union is Q (cf. Lemma 10). If $V=\bigcap_{k=0}^{m} T^{-k}(Z_k)$, set $D_0=D$ and $D_k=T(D_{k-1})\cap Z_k$ for $1\leq k\leq m$. Since $V\subset Q\subset D$, we get $D_m=T^m(V)$, by Lemma 1. Hence D_m depends only on V and not on D. Thus we get the same D_m for every i. As $\mathfrak{A}_i$ is closed, we have $D_m\in\mathfrak{A}_i$ for all i and hence $D_m\in\mathfrak{A}_\infty=\bigcap_{i=1}^{\infty}\mathfrak{A}_i$. This implies $T^m(V)=D_m$

$\subset H(\mathfrak{A}_\infty)$ and $V \cap \Psi(\mathfrak{A}_i, \mathfrak{A}_\infty) = \emptyset$ for all i by (i) of Lemma 6. Therefore $Q \cap \Psi(\mathfrak{A}_i, \mathfrak{A}_\infty) = \emptyset$ for all i and $x \notin L((\mathfrak{A}_i)_{i \geqq 1})$, a contradiction. This proves (ii). □

Now we state the main result about $\Omega(X, T)$. If I is a nontrivial $\mathfrak{Z}$-atom and $T^k(I) \not\subset I$ for all $k \geqq 1$, then int $I \cap \Omega(X, T) = \emptyset$ by (iii) of Lemma 9. If $p \geqq 1$ and $T^p(I) \subset I$, set $P(I) = \Omega(I, T^p | I)$ and let P be the union of all such $P(I)$. If (X, T) is piecewise monotone, then P consists only of periodic points by (iv) of Lemma 9. Furthermore, set $\Omega^1(X, T) = \Omega(X, T)$. If $\Omega^k(X, T)$ is defined, let $\Omega^{k+1}(X, T)$ be the set of nonwandering points of $T | \Omega^k(X, T)$. Define W as the set of all $x \in \Omega(X, T)$ with $x \notin \Omega^m(X, T)$ for some m.

Theorem 5. *Let (X, T) be a piecewise invertible dynamical system. Suppose that every perfect subset of $\mathfrak{D}$ is a finite union of sets $\bar{\mathfrak{C}}$ with $\mathfrak{C} \in \Gamma$. Then*

$$\Omega(X, T) \subset \bigcup_{\mathfrak{C} \in \Gamma} L(\mathfrak{C}) \cup L_\infty \cup P \cup W.$$

If $\mathfrak{Z}$ is a generator or if (X, T) is piecewise monotonic, then $L(\mathfrak{C}) = \omega(z)$ for some $z \in X$. L_∞ can be represented at infinity in the Markov diagram. If (X, T) is piecewise monotonic, then $L_\infty \subset \bigcup_{c \in K} \omega(c)$, where K is finite.

Proof. Fix $x \in \Omega(X, T)$ and suppose $x \notin L(\mathfrak{C})$ for all $\mathfrak{C} \in \Gamma$ and $x \notin L_\infty$. We have to show $x \in W \cup P$. To this end we construct a sequence Δ of closed subsets of $\mathfrak{D}$ with the following two properties. If $\mathfrak{A}$ and $\mathfrak{B}$ are two successive elements of the sequence Δ, then

(a) $\mathfrak{B} \subsetneqq \mathfrak{A}$,
(b) $Q \cap \Psi(\mathfrak{A}, \mathfrak{B}) = \emptyset$ for some $Q \in \mathfrak{U}(x)$.

The construction of Δ is done by induction using the following three steps.

Step 1. Suppose Δ is finite and the perfect set $\mathfrak{A} \neq \emptyset$ is the last element of Δ. Let Γ' be a minimal subset of Γ with $\bigcup_{\mathfrak{C} \in \Gamma'} \bar{\mathfrak{C}} = \mathfrak{A}$. Fix some $\mathfrak{C} \in \Gamma'$. Set $\Gamma'' = \Gamma' \setminus \{\mathfrak{C}\}$, $\mathfrak{F} = \bigcup_{\mathfrak{H} \in \Gamma''} \bar{\mathfrak{H}}$ and $\mathfrak{B} = \mathfrak{F} \cup \tilde{\mathfrak{C}}$. A union of closed sets is closed, hence $\mathfrak{F}$ and $\mathfrak{B}$ are closed by Lemma 5. As Γ' was chosen minimal, we have $\mathfrak{C} \not\subset \mathfrak{F}$, because otherwise $\bar{\mathfrak{C}} \subset \mathfrak{F}$, as $\mathfrak{F}$ is closed, and $\mathfrak{F} = \mathfrak{A}$. Hence $\mathfrak{B} \subsetneqq \mathfrak{A}$, as $\mathfrak{C} \cap \tilde{\mathfrak{C}} = \emptyset$. Since $x \notin L(\mathfrak{C})$, we have for some $Q \in \mathfrak{U}(x)$ that $Q \cap \Psi(\bar{\mathfrak{C}}, \tilde{\mathfrak{C}}) = \emptyset$ and hence $Q \cap \Psi(\mathfrak{A}, \mathfrak{B}) = \emptyset$ by (iii) of Lemma 6, as $\mathfrak{A} = \bar{\mathfrak{C}} \cup \mathfrak{F}$ and $\mathfrak{B} = \tilde{\mathfrak{C}} \cup \mathfrak{F}$. We add $\mathfrak{B}$ to Δ as its last element and the two properties of Δ remain valid.

Step 2. Suppose Δ is finite and the closed set $\mathfrak{A} \neq \emptyset$ is the last element of Δ. Set $\mathfrak{A}_0 = \mathfrak{A}$ and $\mathfrak{A}_{i+1} = \nu(\mathfrak{A}_i)$ for $i \geqq 0$. By (ii) of Lemma 4, all $\mathfrak{A}_i$ are closed. Let $m \leqq \infty$ be maximal, such that $\mathfrak{A}_i$ for $i < m$ is not perfect. Add $\mathfrak{A}_i$ for $1 \leqq i \leqq m$ to Δ, if $m < \infty$, and add $\mathfrak{A}_i$ for $1 \leqq i < \infty$ to Δ, if $m = \infty$. For $0 \leqq i < m$, we have by (i) of Lemma 4, that $\mathfrak{A}_{i+1} \subsetneqq \mathfrak{A}_i$, and by (iii) of Lemma 4, that $T(H(\mathfrak{A}_i)) = H(\mathfrak{A}_{i+1})$, which implies $\Psi(\mathfrak{A}_i, \mathfrak{A}_{i+1}) = \emptyset$. Hence the two properties of Δ remain valid.

Step 3. Suppose Δ is infinite, say $\Delta = (\mathfrak{A}_i)_{i \geqq 1}$. Set $\mathfrak{A}_\infty = \bigcap_{i=1}^{\infty} \mathfrak{A}_i$. As $x \notin L_\infty$, we have $x \notin L(\mathfrak{A}_j, \mathfrak{A}_\infty)$ for some j, that is $Q \cap \Psi(\mathfrak{A}_j, \mathfrak{A}_\infty) = \emptyset$ for some $Q \in \mathfrak{U}(x)$. We cancel $\mathfrak{A}_i$ for $i>j$ from Δ and add $\mathfrak{A}_\infty$ to Δ. Since $\mathfrak{A}_\infty \subsetneqq \mathfrak{A}_i$ for all $i \geqq 1$, by (a), the two properties of Δ remain valid.

We begin the construction letting Δ be the sequence consisting only of $\mathfrak{D}$. If $\mathfrak{D}$ is perfect, apply Step 1, otherwise apply Step 2. After each Step 1 and 3 apply Step 1 or 2, depending on whether the last element of Δ is perfect or only closed. After each Step 2 apply Step 1, if $m<\infty$, and Step 3, if $m=\infty$. Step 3 also makes Δ again finite, if it has become infinite by repetition of the steps. If Δ is finite, its last element is a strict subset of the last element of Δ at any earlier step at which Δ was finite. If Δ is infinite, it is again finite after one step, and the last element of Δ goes through a totally ordered set in the set of all closed subsets of $\mathfrak{D}$ with inclusion as order relation. As the empty set is the minimum in this ordered set, the induction ends with a finite sequence Δ whose last element is the empty set. As Δ is finite, we can suppose that (b) holds for all successive $\mathfrak{A}$ and $\mathfrak{B}$ in Δ with the same $Q \in \mathfrak{U}(x)$.

Suppose $\Delta = (\mathfrak{A}_i)_{1 \leqq i \leqq n}$. Then $\mathfrak{A}_1 = \mathfrak{D}$ and $\mathfrak{A}_n = \emptyset$. Hence $Q \cap \Omega^1(X, T) \subset X = \overline{H(\mathfrak{A}_1)}$. Suppose we have shown that $Q \cap \Omega^i(X, T) \subset \overline{H(\mathfrak{A}_i)}$. We show $Q \cap \Omega^{i+1}(X, T) \subset \overline{H(\mathfrak{A}_{i+1})}$. To this end, suppose $y \in Q \setminus \overline{H(\mathfrak{A}_{i+1})}$. Since Q is open, we find a neighbourhood U of y with $U \subset Q$, $U \cap \overline{H(\mathfrak{A}_{i+1})} = \emptyset$, and $T^j(U \cap \overline{H(\mathfrak{A}_i)}) \subset \overline{H(\mathfrak{A}_{i+1})}$ for some j, by (b), (i) of Lemma 6, and the continuity of T. As $Q \cap \Omega^i(X, T) \subset \overline{H(\mathfrak{A}_i)}$, the set $U \cap \overline{H(\mathfrak{A}_i)}$ contains a neighbourhood of y in $\Omega^i(X, T)$. Since $U \cap \overline{H(\mathfrak{A}_{i+1})} = \emptyset$ and $\overline{H(\mathfrak{A}_{i+1})}$ is T-invariant, we get that y is wandering for $T \mid \Omega^i(X, T)$, that is $y \notin \Omega^{i+1}(X, T)$. This is the desired result.

We have shown by induction, that $Q \cap \Omega^n(X, T) \subset \overline{H(\mathfrak{A}_n)} = \emptyset$. Let $I \in \mathfrak{Z}_\infty$ be such that $x \in I$. If I is trivial, we have $x \in W$, since $x \in Q$. If I is nontrivial, then either $x \in P(I) \subset P$, or x has a neighbourhood U with $T^m(I \cap U) \cap (I \cap U) = \emptyset$ for $m \geqq 1$. As $Q \cap \Omega^n(X, T) = \emptyset$, the set $I \cap U$ contains a neighbourhood of x in $\Omega^n(X, T)$. Hence $x \notin \Omega^{n+1}(X, T)$, that is $x \in W$, if $x \notin P$.

We have shown that $\Omega(X, T) \subset \bigcup_{\mathfrak{C} \in \Gamma} L(\mathfrak{C}) \cup L_\infty \cup P \cup W$. The other assertions follow from Theorems 2, 3 and 4, as $L(\mathfrak{A}, \mathfrak{B}) \subset \Psi(\mathfrak{A}, \mathfrak{B})$ and equality holds, if $\mathfrak{Z}$ is a generator. □

We conclude §2 with a result which shows once more the analogy with Markov shifts.

Theorem 6. *Let (X, T) be a piecewise invertible dynamical system and suppose that $\mathfrak{Z}$ is a generator. If $\mathfrak{C} \in \Gamma$ and $\bar{\mathfrak{C}}$ has a finite subset $\mathfrak{C}'$ with $H(\mathfrak{C}') \supset H(\mathfrak{C})$, then $L(\mathfrak{C}) = \Omega(\mathfrak{C})$ is the set of all $x \in X$ represented in $\mathfrak{C}$.*

Proof. The assumptions of Lemma 7 are satisfied for $\mathfrak{A} = \bar{\mathfrak{C}}$, $\mathfrak{A}' = \mathfrak{C}'$ and $\mathfrak{B} = \tilde{\mathfrak{C}}$. If $x \in \Omega(\mathfrak{C})$, then $x \in V_k(x) \cap \Omega(\mathfrak{C})$ for all k and hence, by Lemma 7, for all k, there is a path $D_0 D_1 \dots D_k$ in $\mathfrak{C}$ with $D_0 \in \mathfrak{C}'$ and $\bigcap_{i=0}^{k} T^{-i}(D_i) \subset V_k(x)$. Since $\mathfrak{C}'$ is finite, infinitely many of these paths must begin at some fixed $C \in \mathfrak{C}'$. By Lemma 2, for two such paths the longer one is a continuation of the shorter one. Hence we can join these infinitely many paths beginning with C to an infinite path $C_0 C_1 C_2 \dots$ in $\mathfrak{C}$, which satisfies then $\bigcap_{i=0}^{\infty} T^{-i}(C_i) \subset \bigcap_{k=0}^{\infty} V_k(x) = \{x\}$, as $\mathfrak{Z}$ is a generator. By (i) of Theorem 1, every path in $\mathfrak{D}$ represents some element of X, hence $C_0 C_1 C_2 \ldots$ represents x.

On the other hand, every infinite path in $\mathfrak{C}$ represents a unique $x \in X$ by (i) of Theorem 1, as $\mathfrak{Z}$ is a generator. By Lemma 8, we get $x \in \Omega(\mathfrak{C})$. $\square$

§ 3. *Topological Entropy and Periodic Points*

Lemma 11. *Suppose (X, T) is piecewise monotonic and $R \subset X$ is closed and T-invariant.*

(i) *If μ is a T-invariant measure on R with $\mu(I) > 0$ for some $I \in \mathfrak{Z}_\infty$, then there is T-invariant measure μ' on R with $h(\mu') > h(\mu)$, where h denotes entropy.*

(ii) $h_{\mathrm{top}}(R) = \lim \frac{1}{k} \log \operatorname{card} \mathfrak{Z}'_k$, *where* $\mathfrak{Z}'_k = \{V \in \mathfrak{Z}_k : R \cap V \neq \emptyset\}$.

Proof. If $\mu(I) > 0$ for $I \in \mathfrak{Z}_\infty$, then, by (iii) of Lemma 9, $T^m(I) \subset I$ for some m and $\mu = q\mu' + (1-q)\mu''$, where $0 \leqq q < 1$ and μ'' is concentrated on $\bigcup_{i=0}^{m-1} T^i(I)$. By (iv) of Lemma 9, $h(\mu'') = 0$ and by Theorem 8.1 of [17] we get $h(\mu) = q h(\mu') < h(\mu')$. This proves (i). In order to show (ii) we use the variational principle (Theorem 8.6 of [17]). By (i) it suffices to take the supremum in this theorem only over those μ, which satisfy $\mu(I) = 0$ for all $I \in \mathfrak{Z}_\infty$. Hence $h_{\mathrm{top}}(R)$ does not change, if we consider the $\mathfrak{Z}$-atoms as single points, which is possible by (ii) of Lemma 9. But then $\mathfrak{Z}$ is a generator and (ii) follows. $\square$

We consider the Markov diagram as a $\mathfrak{D} \times \mathfrak{D}$-matrix M with entries 0 and 1. For $C, D \in \mathfrak{D}$ set $M_{CD} = 1$, if $C \to D$, and $M_{CD} = 0$ otherwise. As every $C \in \mathfrak{D}$ has at most $N := \operatorname{card} \mathfrak{Z}$ successors, $u \to uM$ is a positive $l^1(\mathfrak{D})$-operator with $\|M\|_1 \leqq N$. The same holds for $M \mid \mathfrak{A}$, where $\mathfrak{A} \subset \mathfrak{D}$. We denote the spectral radius by r. We need the complicated assumption of the following theorem in Chap. II. In particular, it is satisfied, if $\mathfrak{A}$ has a finite subset $\mathfrak{F}$ with $H(\mathfrak{F}) \supset H(\mathfrak{A}) \setminus H(\mathfrak{B})$.

Theorem 7. *Suppose either that (X, T) is piecewise invertible and $\mathfrak{Z}$ is a generator or that (X, T) is piecewise monotonic.*

(i) *Suppose $\mathfrak{B} \subset \mathfrak{A} \subset \mathfrak{D}$ and $\mathfrak{A}$ and $\mathfrak{B}$ are closed. If there are $q \in \mathbb{N}$ and $\mathfrak{F}_n \subset \mathfrak{A}$ with card $\mathfrak{F}_n \leqq q$ for $n \geqq 1$ and with $H\left(\bigcup_{n=1}^{\infty} \mathfrak{F}_n\right) \supset H(\mathfrak{A}) \setminus H(\mathfrak{B})$ and if for every n and $D \in \mathfrak{F}_n$ there is a $C \in \mathfrak{F}_{n+1}$ with $D \subset C$, then $h_{\mathrm{top}}(\Psi(\mathfrak{A}, \mathfrak{B})) \leqq \log r(M \mid \mathfrak{A} \setminus \mathfrak{B})$.*

(ii) *Suppose $\mathfrak{C} \in \Gamma$ and that there are $\mathfrak{F}_n$ with the same properties as in* (i) *for $\mathfrak{A} = \bar{\mathfrak{C}}$ and $\mathfrak{B} = \tilde{\mathfrak{C}}$. If $L(\mathfrak{C}) \neq \emptyset$, then*

$$h_{\mathrm{top}}(L(\mathfrak{C})) = h_{\mathrm{top}}(\Omega(\mathfrak{C})) = \log r(M \mid \mathfrak{C}).$$

(iii) $h_{\mathrm{top}}(X, T) = \log r(M)$.

Proof. (cf. [2]). Set $\tilde{M} = M \mid (\mathfrak{A} \setminus \mathfrak{B})$ and $\mathfrak{Z}'_k = \{V \in \mathfrak{Z}_k : V \cap \Psi(\mathfrak{A}, \mathfrak{B}) \neq \emptyset\}$. Fix $k \geqq 0$. If $V \in \mathfrak{Z}'_k$, we find by Lemma 7 a path $D_0 D_1 \ldots D_k$ in $\mathfrak{A} \setminus \mathfrak{B}$ with $D_0 \in \mathfrak{F}_n$ for some

n, $D_k \notin H(\mathfrak{B})$ and $\bigcap_{i=0}^{k} T^{-i}(D_i) \subset V$. Choose $Z_i \in \mathfrak{Z}$ such that $D_i \subset Z_i$. Then $V = \bigcap_{i=0}^{k} T^{-i}(Z_i)$. For all $m > n$ there is a $C_0 \in \mathfrak{F}_m$ with $D_0 \subset C_0$. Set $C_i = T(C_{i-1}) \cap Z_i$. Then $D_i \subset C_i$ for $0 \leqq i \leqq k$, as $D_i = T(D_{i-1}) \cap Z_i$, and $C_0\, C_1 \ldots C_k$ is a path in $\mathfrak{A}$, since $\mathfrak{A}$ is closed. We have $C_i \notin \mathfrak{B}$ for $0 \leqq i \leqq k$, because otherwise $C_k \in \mathfrak{B}$, as $\mathfrak{B}$ is closed, and then $D_k \subset C_k \subset H(\mathfrak{B})$, a contradiction. Hence for $V \in \mathfrak{Z}'_k$ there is an n such that for all $m \geqq n$ there is a path $C_0\, C_1 \ldots C_k$ in $\mathfrak{A} \setminus \mathfrak{B}$ with $C_0 \in \mathfrak{F}_m$ and $\bigcap_{i=0}^{k} T^{-i}(C_i) \subset V$. Since $\mathfrak{Z}'_k$ is finite we find such an m independent of V. The number of these paths is $\sum_{C \in \mathfrak{F}_m} \sum_{D \in \mathfrak{A}} \tilde{M}^{(k)}_{CD}$, where $\tilde{M}^{(k)}_{CD}$ denotes an entry of the matrix $\tilde{M}^k$. The elements of $\mathfrak{Z}'_k$ are pairwise disjoint. Hence different $V \in \mathfrak{Z}'_k$ give rise to different paths $C_0\, C_1 \ldots C_k$, as $\bigcap_{i=0}^{k} T^{-i}(C_i) \subset V$, and

$$\operatorname{card} \mathfrak{Z}'_k \leqq \sum_{C \in \mathfrak{F}_m} \sum_{D \in \mathfrak{A}} \tilde{M}^{(k)}_{CD} \leqq q \, \|\tilde{M}^k\|_1.$$

This implies $h_{\mathrm{top}}(\Psi(\mathfrak{A}, \mathfrak{B})) \leqq \log r(\tilde{M})$ proving (i). If $\mathfrak{Z}$ is not a generator, we use Lemma 11.

For (ii) we set $\mathfrak{A} = \bar{\mathfrak{C}}$, $\mathfrak{B} = \tilde{\mathfrak{C}}$ and for (iii) we set $\mathfrak{A} = \mathfrak{D}$, $\mathfrak{B} = \emptyset$, $\mathfrak{F}_n = \mathfrak{Z}$ for all n. We get then by (i) that

$$h_{\mathrm{top}}(L(\mathfrak{C})) \leqq h_{\mathrm{top}}(\Omega(\mathfrak{C})) \leqq \log r(M \mid \mathfrak{C}),$$

as $L(\mathfrak{C}) \subset \Omega(\mathfrak{C})$, and that $h_{\mathrm{top}}(X, T) \leqq \log r(M)$. On the other hand we have $\|\tilde{M}^k\|_1 = \sup_C \sum_D \tilde{M}^{(k)}_{CD}$. For fixed C, we get by $\bigcap_{i=0}^{k} T^{-i}(D_i) \subset V \in \mathfrak{Z}_k$ a map from the set of all paths $D_0\, D_1 \ldots D_k$ of length k in $\mathfrak{D}$ with $D_0 = C$ to the set $\mathfrak{Z}_k$, which is injective by Lemma 2. Hence $\|M^k\|_1 \leqq \operatorname{card} \mathfrak{Z}_k$. Furthermore, if $D_0\, D_1 \ldots D_k$ is in $\mathfrak{C}$, then it is an initial segment of infinitely many finite paths in $\mathfrak{C}$ and occurs therefore infinitely often in a path representing a z of Theorem 2. Hence $T^j(z) \in \bigcap_{i=0}^{k} T^{-i}(D_i) \subset V \in \mathfrak{Z}_k$ for infinitely many j and $\omega(z) \cap V \neq \emptyset$, as V is closed. Now $\omega(z) = L(\mathfrak{C})$ by Theorem 4, if (X, T) is piecewise monotonic, and by Theorem 2 and the fact that $L(\mathfrak{C}) = \Omega(\mathfrak{C})$, if $\mathfrak{Z}$ is a generator. This gives $V \in \mathfrak{Z}''_k := \{V \in \mathfrak{Z}_k : V \cap L(\mathfrak{C}) \neq \emptyset\}$. Hence $\|(M \mid \mathfrak{C})^k\|_1 \leqq \operatorname{card} \mathfrak{Z}''_k$. This and $\|M^k\|_1 \leqq \operatorname{card} \mathfrak{Z}_k$ imply (ii) and (iii) using Lemma 11. □

Now we consider periodic points of (X, T). We call a path $D_0\, D_1\, D_2 \ldots$ a closed path of length n, if $D_i = D_{i+n}$ for $i \geqq 0$.

Theorem 8. *Suppose $x \in X$ satisfies $T^n(x) = x$.*

(i) *If for all $D \in \mathfrak{D}$ with $x \in D$ there is a k with $V_k(x) \subset D$, then there is a unique closed path of length n in the Markov diagram which represents x.*

(ii) *If (X, T) is piecewise monotonic, $\bigcap_{k=0}^{\infty} V_k(x) = \{x\}$ and $x \in \operatorname{bd} D$ for some $D \in \mathfrak{D}$, then there are finitely many (at least one) closed paths in the Markov* *diagram of length n or 2n representing x. If (X, T) is piecewise increasing, then the length of these paths is n.*

Proof. Existence: By (i) of Theorem 1, we find a path $D_0 D_1 D_2 \ldots$ which represents x. Then $D_{rn} D_{rn+1} D_{rn+2} \ldots$ represents $T^{rn}(x) = x$ for all r. If there is a k with $D_0 \cap V_k(x) = D_n \cap V_k(x)$, then $D_i = D_{i+n}$ for $i \geqq k$ by (ii) of Theorem 1 and we get a closed path $D_{jn} D_{jn+1} \ldots$ of length n representing x, where j is such that $jn \geqq k$. This happens always in case of (i). In case of (ii) it may happen that $D_{rn} = \{x\}$ for some r. Then $D_{rn+n} = \{x\}$ and we get a closed path of length n as above. Otherwise suppose that x is the left endpoint of D_0 (cf. Lemma 3). Then $T^n(x) = x$ is also an endpoint of D_n by the definition of successor. If it is the left endpoint then we find a $V_k(x)$ with $D_0 \cap V_k(x) = D_n \cap V_k(x)$, since $D_n \neq \{x\}$ and $\bigcap_{k=0}^{\infty} V_k(x) = \{x\}$, and get a closed path of length n as above. This happens always, if (X, T) is piecewise increasing. If x is the right endpoint of D_n, then T^n is decreasing in a neighbourhood of x, and $T^{2n}(x) = x$ is the left endpoint of D_{2n}. As above, we find a $V_k(x)$ with $D_0 \cap V_k(x) = D_{2n} \cap V_k(x)$ and get a closed path of length $2n$.

Uniqueness: Suppose $D_0 D_1 D_2 \ldots$ and $C_0 C_1 C_2 \ldots$ are closed paths representing x. In case of (i) we find a k with $D_0 \cap V_k(x) = V_k(x) = C_0 \cap V_k(x)$. Hence, by (ii) of Theorem 1, $D_i = C_i$ for $i \geqq k$. As the paths are closed, we get $D_i = C_i$ for $i \geqq 0$. In case of (ii) there are four possibilities for $D_0 \cap V_k(x)$ with large enough k. It can be $V_k(x)$, $V_k(x) \cap \{y : y \leqq x\}$, $V_k(x) \cap \{y : y \geqq x\}$, or $\{x\}$. The same holds for $C_0 \cap V_k(x)$. We get as above, that there are at most four different closed paths representing x. □

Remark. (i) Suppose that (X, T) is piecewise monotonic and that $\mathfrak{Z}$ is a generator. By Lemma 3, the requirements of (i) of Theorem 8 hold for $x \notin \bigcup_{i=0}^{\infty} T^i(K)$, and the requirements of (ii) of Theorem 8 hold for $x \in \bigcup_{i=0}^{i=\infty} T^i(K)$. Hence every x of period n is represented by a closed path, which is unique and of length n, if x is not one of the finitely many periodic points in $\bigcup_{i=0}^{\infty} T^i(K)$.

(ii) The fixed point y of the example in §2 above is represented by the path $K_1 K_2 K_3 \ldots$, which is not closed.

§ 4. *A Two Dimensional Example*

It seems to be difficult to compute the Markov diagram for higher dimensional (X, T). One can compute it for the following simple class of transformations T_a in $[0, 1]^2$. Set $I = [0, \frac{2}{5}]$, $J = (\frac{2}{5}, 1]$ and $K = [0, 1]$ and define $T_a(x, y) = (1 - y, \frac{5}{2} x)$ for $(x, y) \in I \times K$ and $T_a(x, y) = (\frac{5}{3} a(x - \frac{2}{5}), y)$ for $(x, y) \in J \times K$, where $a \in (\frac{2}{5}, 1)$. The points in $\{\frac{2}{5}\} \times K$ and their inverse images are doubled and T_a is extended continuously (cf. the introduction), in order to get a piecewise invertible dynamical system. One can show that $\mathfrak{Z} = \{I \times K, J \times K\}$ is a generator. We compute the Markov diagram only for $a = 0.43\ldots$ and write T for T_a. The intervals I_1, I_2, J_1, J_2, K_1 and K_2 are indicated in Fig. 2 and defined such that

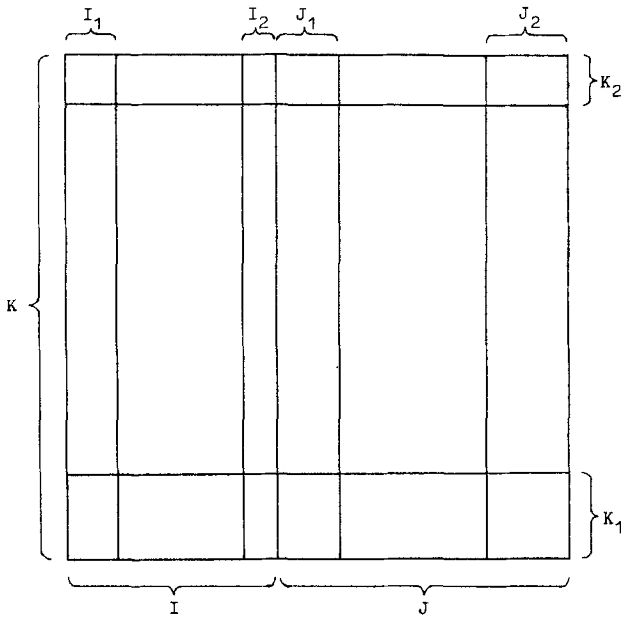

Fig. 2

the following hold:

$$J_1 \times K = (\tfrac{2}{5}, a] \times K = T(J \times K) \cap (J \times K),$$
$$T(J_1 \times K) = I_1 \times K, \qquad T(I_1 \times K) = (I \cup J) \times K_1,$$
$$T(I \times K_1) = J_2 \times K, \qquad T(J \times K_1) = (I \cup J_1) \times K_1, \qquad T(J_2 \times K) = (I_2 \cup J_1) \times K,$$
$$T(I_2 \times K) = (I \cup J) \times K_2, \qquad T(J \times K_2) = (I \cup J_1) \times K_2.$$

Finally we have $T(I \times K_2) = I_1 \times K$ because of the choice of a. Using these equations one gets the Markov diagram of T:

$$\begin{tikzcd}
I \times K \arrow[loop right] \arrow[d, shift left] & I \times K_1 \arrow[ddr] & J \times K_1 \arrow[l] \arrow[r] & J_1 \times K_1 \arrow[d] & J_2 \times K_1 \arrow[l] \arrow[d] \\
J \times K \arrow[u, shift left] \arrow[r] & J_1 \times K \arrow[r] & I_1 \times K \arrow[ul] \arrow[u] & I_1 \times K_1 \arrow[ur] & I_2 \times K_1 \arrow[d] \\
 & I \times K_2 \arrow[ur] & J_2 \times K \arrow[dl] \arrow[ul] & I_1 \times K_2 \arrow[u] & J_2 \times K_2 \arrow[dl] \arrow[d] \\
 & I_2 \times K \arrow[u] \arrow[r] & J \times K_2 \arrow[ul] \arrow[r] & J_1 \times K_2 \arrow[u] & I_2 \times K_2 \arrow[ul]
\end{tikzcd}$$

One easily sees that Γ consists of three irreducible subsets of $\mathfrak{D}$. Since $\mathfrak{D}$ is finite, we have $L_\infty = \emptyset$ (cf. Theorem 5). Hence all ω-limit sets are in some $L(\mathfrak{C}) = \Omega(\mathfrak{C})$ with $\mathfrak{C} \in \Gamma$. One gets $L(\mathfrak{C})$, if one takes away $T^{-k}(H(\tilde{\mathfrak{C}}))$ for $k = 0, 1, 2, \ldots$ from $H(\bar{\mathfrak{C}})$. In this way one gets that the three sets $L(\mathfrak{C})$ are a product of two Cantor sets, a finite union of products of an interval and a Cantor set, and a finite union of rectangles. A similar result holds for arbitrary $a \in (\frac{2}{5}, 1)$, but there can be a countable number of topologically transitive subsets $L(\mathfrak{C})$.

II. Piecewise Monotonic Transformations

Throughout Chap. II we suppose that (X, T) is piecewise monotonic.

§ 1. The Structure of the Markov Diagram

Our first goal is to show that the matrix M introduced in §3 of Chap. I behaves in some sense like a finite matrix. Recall Lemma 3, which says that all elements of $\mathfrak{D}$ are closed intervals. This is used permanently throughout this chapter. We call the endpoints of the intervals in $\mathfrak{Z}$ *critical points* and denote the set of critical points by K. The cardinality of K is at most twice the cardinality of $\mathfrak{Z}$. We call a subinterval of X *critical*, if it has an endpoint in K. For a subinterval I of some element of $\mathfrak{Z}$ set $\alpha(I) = \min\{i \geqq 1 \colon T^i(I) \cap K \neq \emptyset\}$ and $\alpha(I) = \infty$, if this set is empty.

Lemma 12. (i) *Suppose $C \in \mathfrak{D}$ has more than one successor. Then there are two successors of C, each of which has one common endpoint with $T(C)$ and has the other endpoint in K (its two endpoints may coincide). All other successors of C are in $\mathfrak{Z}$.*

(ii) *If $D \in \mathfrak{D}$, set $D_i = T^i(D)$. For $0 < i < \alpha(D)$, D_i is then in $\mathfrak{D}$, is not critical and is the only successor of D_{i-1}. If $\alpha(D) < \infty$, then the successors of $D_{\alpha(D)-1}$ are $T^{\alpha(D)}(D) \cap Z \neq \emptyset$ for $Z \in \mathfrak{Z}$, which are all critical. If $\alpha(D) = \infty$, then D is contained in some $\mathfrak{Z}$-atom.*

Proof. (i) Since C has more than one successor, the endpoints of the interval $T(C)$ are contained in two different elements Z and Z' of $\mathfrak{Z}$. Hence $T(C) \cap Z$ has a common endpoint with $T(C)$ and a common endpoint with Z. The same holds for $T(C) \cap Z'$. For all other $Y \in \mathfrak{Z}$, one has either $T(C) \cap Y = \emptyset$ or $T(C) \cap Y = Y$.

(ii) Suppose we have shown that $D_i \in \mathfrak{D}$. If $i < \alpha(D) - 1$, then $T(D_i) \cap K = \emptyset$. As $T(D_i)$ is an interval, we get $T(D_i) \subset Z$ for some $Z \in \mathfrak{Z}$. Hence $D_{i+1} = T(D_i)$ is not critical and the only successor of D_i. If $\alpha(D) < \infty$ and $i = \alpha(D) - 1$ then $T(D_i) \cap K \neq \emptyset$. If $T(D_i) \subset Z$ for some $Z \in \mathfrak{Z}$, then $T(D_i) = T^{\alpha(D)}(D) \cap Z$ is critical and the only successor of $D_{\alpha(D)-1}$. Otherwise, the successors of $D_{\alpha(D)-1}$ are $T(D_i) \cap Z = T^{\alpha(D)}(D) \cap Z \neq \emptyset$ for $Z \in \mathfrak{Z}$, which are all critical by (i). If $\alpha(D) = \infty$, choose $Z_i \in \mathfrak{Z}$, such that $D_i = T^i(D) \subset Z_i$. Then $D \subset \bigcap_{i=0}^{\infty} T^{-i}(Z_i)$, which is a $\mathfrak{Z}$-atom. $\square$

Now we consider a critical element $D \in \mathfrak{D}$. Set $\mathfrak{D}_0 = \mathfrak{Z}$ and $\mathfrak{D}_{i+1} = \mathfrak{D}_i \cup \nu(\mathfrak{D}_i)$, where ν is defined in § 2 of Chap. I.

Lemma 13. *Suppose that $D \in \mathfrak{D}$ has endpoints c and x with $c \in K$ and that $\alpha(D) < \infty$. Set $D_i = T^i(D) \in \mathfrak{D}$ for $0 \leqq i < \alpha(D)$ (cf. Lemma 12). Then all successors of $D_{\alpha(D)-1}$ are in $\mathfrak{Z}$, except at most two. At most one of these two successors, call it C, is not in $\mathfrak{D}_{\alpha(D)}$. If C exists, then C has endpoints $T^{\alpha(D)}(x)$ and c' with $c' \in K$ (its two endpoints may coincide).*

Proof. As $D_i \in \mathfrak{D}$ for $0 \leqq i < \alpha(D)$ we find $Z_i \in \mathfrak{Z}$ with $D_i = T^i(D) \subset Z_i$. Set $E_0 = Z_0$ and $E_i = T(E_{i-1}) \cap Z_i$ for $1 \leqq i < \alpha(D)$. We get by induction that $D_i \subset E_i$ and that D_i and E_i have the common endpoint $T^i(c)$ for $0 \leqq i < \alpha(D)$, since $T^i(c) \in D_i \subset Z_i$. For $i = 0$ this holds, since the endpoint c of D is in K and $D \subset Z_0$. Similarly, $T(D_{\alpha(D)-1}) \subset T(E_{\alpha(D)-1})$ and these two intervals have the common endpoint $T^{\alpha(D)}(c)$. The other endpoint of $T(D_{\alpha(D)-1}) = T^{\alpha(D)}(D)$ is $T^{\alpha(D)}(x)$. Let $Z, Z' \in \mathfrak{Z}$ be such that $T^{\alpha(D)}(c) \in Z$ and $T^{\alpha(D)}(x) \in Z'$. Set $E_{\alpha(D)} = T(E_{\alpha(D)-1}) \cap Z$. Then $T^{\alpha(D)}(c) \in E_{\alpha(D)}$, hence $E_{\alpha(D)} \neq \emptyset$. As $E_0 \in \mathfrak{Z}$ and $E_i \to E_{i+1}$ holds, we get $E_{\alpha(D)} \in \mathfrak{D}_{\alpha(D)}$. Suppose first that $Z = Z'$, i.e. $T^{\alpha(D)}(D) \subset Z$ is the only successor of $D_{\alpha(D)-1}$. Together with $T^{\alpha(D)}(D) \cap K \neq \emptyset$ this implies that one of the endpoints of $T^{\alpha(D)}(D)$ is in K. If $T^{\alpha(D)}(c) \in K$, then $T^{\alpha(D)}(D)$ satisfies the requirements of C. If $T^{\alpha(D)}(x) \in K$, then $T^{\alpha(D)}(D) = T(D_{\alpha(D)-1})$ has one endpoint in common with Z and one in common with $T(E_{\alpha(D)-1})$. This implies $T^{\alpha(D)}(D) = E_{\alpha(D)} \in \mathfrak{D}_{\alpha(D)}$ and C does not exist. Now suppose $Z \neq Z'$. For all $Y \in \mathfrak{Z}$ between (with respect to the order relation) Z and Z', one has $T^{\alpha(D)}(D) \cap Y = Y \in \mathfrak{Z}$. Furthermore, $T^{\alpha(D)}(D) \cap Z = E_{\alpha(D)} \in \mathfrak{D}_{\alpha(D)}$, since $T^{\alpha(D)}(D)$ and $T(E_{\alpha(D)-1})$ have the common endpoint $T^{\alpha(D)}(c)$. Finally $T^{\alpha(D)}(D) \cap Z'$ is C, as it has a common endpoint with Z' and its other endpoint is $T^{\alpha(D)}(x)$. □

Now we use Lemmas 12 and 13 to build up the Markovdiagram. For the sets $\mathfrak{D}_i$ defined above we have $\mathfrak{D} = \bigcup_{i=0}^{\infty} \mathfrak{D}_i$. We start with $\mathfrak{D}_0 = \mathfrak{Z}$. In a first step we add $\mathfrak{E}_1 := \mathfrak{D}_1 \setminus \mathfrak{D}_0$ to $\mathfrak{D}_0$. In the k-th step we add $\mathfrak{E}_k := \mathfrak{D}_k \setminus \mathfrak{D}_{k-1}$ to $\mathfrak{D}_{k-1}$. We fix some $D \in \mathfrak{D}_0 = \mathfrak{Z}$ and observe, what successors of D we get in $\mathfrak{E}_1$ in the first step. Then we observe what successors of these successors in $\mathfrak{E}_1$ we get in $\mathfrak{E}_2$ in the second step and so on.

By Lemma 12, $D_i = T^i(D)$ is the only successor of D_{i-1} for $1 \leqq i < \alpha(D)$. Hence in the i-th step we add the only successor D_i of D_{i-1} to $\mathfrak{D}_{i-1}$, if $D_i \in \mathfrak{E}_i$. If $D_i \notin \mathfrak{E}_i$, then $D_i \in \mathfrak{D}_{i-1}$, by definition of $\mathfrak{D}_i$, and D_i was added in an earlier step. All successors of $D_{\alpha(D)-1}$ are in $\mathfrak{Z} = \mathfrak{D}_0$ except at most two, say B and C, by Lemma 12. We add B and C in the $\alpha(D)$-th step, if they are in $\mathfrak{E}_{\alpha(D)}$. Furthermore B and C are critical.

Now consider a critical interval $C \in \mathfrak{E}_k$ with $k \geqq 1$. As above $C_i = T^i(C)$ is the only successor of C_{i-1} for $1 \leqq i < \alpha(C)$. Hence in the $(k+i)$-th step we add C_i to $\mathfrak{D}_{k+i-1}$, if $C_i \in \mathfrak{E}_{k+i}$. By Lemma 13, all successors of $C_{\alpha(C)-1}$ except at most one, call it C', are in $\mathfrak{D}_{\alpha(C)} \cup \mathfrak{Z} = \mathfrak{D}_{\alpha(C)}$. Since $k \geqq 1$, C' is the only successor of $C_{\alpha(C)-1}$, which need not be in $\mathfrak{D}_{k+\alpha(C)-1}$. Hence we add C' in the $(k+\alpha(C))$-th step, if $C' \in \mathfrak{E}_{k+\alpha(C)}$. Furthermore C' is critical by Lemma 12, so that one can iterate this procedure just described for $C \in \mathfrak{E}_k$.

$$
\begin{tikzcd}[column sep=small]
 & & \mathfrak{D}_{\alpha(C)} & & \mathfrak{D}_{\alpha(C')} & \\
 & C \to C_1 \to \dots \arrow[r] & C_{\alpha(C)-1} \arrow[r] \arrow[u] & C' \to C'_1 \to \dots \arrow[r] & C'_{\alpha(C')-1} \arrow[r] \arrow[u] & C'' \to \dots \\
D \to D_1 \to \dots \to D_{\alpha(D)-1} \arrow[ur] \arrow[dr] \arrow[d] & & & & & \\
\mathfrak{Z} & B \to B_1 \to \dots \arrow[r] & B_{\alpha(B)-1} \arrow[r] \arrow[d] & B' \to \dots & & \\
 & & \mathfrak{D}_{\alpha(B)} & & &
\end{tikzcd}
$$

Fig. 3

The part of the Markov diagram one gets is shown in Fig. 3. The elements of $\mathfrak{C}_j$ are placed in the j-th column. Since one has such a part for every $D \in \mathfrak{Z}$, we get card $\mathfrak{C}_k \leqq 2$ card $\mathfrak{Z}$ for all k. Figure 3 shows the case where all elements of $\mathfrak{D}$, which can be different, are different. It may happen, that a D shown to be in $\mathfrak{C}_j$ by Fig. 3 coincides with a $C \in \mathfrak{C}_i$ for $i<j$. Then D is not added in step j, but has already occured in the earlier step i. This makes the sets $\mathfrak{C}_k$ for $k \geqq j$ smaller. If $\mathfrak{C}_k = \emptyset$ for some k, then $\mathfrak{D}$ is finite. Otherwise, there is a $\gamma \in \mathbb{N}$ such that the sets $\mathfrak{C}_k$ for $k \geqq \gamma$ have all the same cardinality. Then it follows from the above results illustrated in Fig. 3, that the relation on $\mathfrak{D} \times \mathfrak{D}$ given by $D \to C$ is a bijective map from $\mathfrak{C}_k$ to $\mathfrak{C}_{k+1}$ for $k > \gamma$.

Theorem 9 (cf. §1 of [11]).

(i) *For* $k \geqq 0$ *we have* card $\mathfrak{C}_k \leqq 2$ card $\mathfrak{Z}$.

(ii) *There is a* $\gamma \in \mathbb{N}$ *such that for* $k > \gamma$ *the relation in* $\mathfrak{D} \times \mathfrak{D}$ *given by* $C \to D$ *is a bijection between* $\mathfrak{C}_k$ *and* $\mathfrak{C}_{k+1}$, *i.e.* $\mathfrak{D} \setminus \mathfrak{D}_\gamma$ *is the disjoint union of finitely many sets* $\mathfrak{H} = \{C_i : i > \gamma\}$ *with* $C_i \to C_{i+1}$ *and* $C_i \in \mathfrak{C}_i$ *for* $i > \gamma$.

(iii) *Suppose that* $D_i \in \mathfrak{D}$ *for* $0 \leqq i < m$ *with* $D_{i-1} \to D_i$ *for* $1 \leqq i < m$, *that* D_0 *is critical and that* D_{m-1} *has more than one successor. Then all successors of* D_{m-1} *are in* $\mathfrak{Z}$ *except at most two, one of which is in* $\mathfrak{D}_m$.

Proof. We have shown (i) and (ii) above. In order to show (iii), choose $j < m$ maximal, such that D_j is critical. Then $j \geqq 0$, as D_0 is critical, and $\alpha(D_j) = m - j$ by Lemma 12, as $T(D_{m-1})$ is the union of the successors of D_{m-1} and hence $T(D_{m-1}) \cap K \neq \emptyset$. The result follows now from Lemma 13 with $D = D_j$. □

We prove two corollaries of Theorem 9. The first one shows, that there are not "many" paths at infinity in the Markov diagram, and a consequence of this fact. For this we use the matrix M introduced in §3 of Chap. I. The second corollary shows the assumption of Theorem 5.

Corollary 1. (i) *If* $n \to \infty$, *then* $r(M \mid \mathfrak{D} \setminus \mathfrak{D}_n) \to 1$.

(ii) *Let* $\mathfrak{A}$ *be a subset of* $\mathfrak{D}$ *with* $r(L) > 1$, *where* $L = M \mid \mathfrak{A}$. *Then the* $\mathfrak{A} \times \mathfrak{A}$-*matrix* L *has a nonnegative left eigenvector* $u \in l^1(\mathfrak{A})$ *and a nonnegative right eigenvector* $v \in l^\infty(\mathfrak{A})$ *for the eigenvalue* $\lambda = r(L)$.

Proof. (i) Suppose that $C \in \mathfrak{D} \setminus \mathfrak{D}_n$ is critical. Set $C_0 = C$. Suppose that C_i is the only successor of C_{i-1} in $\mathfrak{D} \setminus \mathfrak{D}_n$ for $1 \leqq i < m$ and that C_{m-1} has more than one successor in $\mathfrak{D} \setminus \mathfrak{D}_n$. Then $m \geqq n$ by (iii) of Theorem 9. Furthermore all

$D\in\mathfrak{D}\setminus\mathfrak{D}_n$, which have more than one successor in $\mathfrak{D}\setminus\mathfrak{D}_n$, have only two successors in $\mathfrak{D}\setminus\mathfrak{D}_n$ by (iii) of Theorem 9, which are both critical by (i) of Lemma 12. This implies that the number of paths of length nk in $\mathfrak{D}\setminus\mathfrak{D}_n$, which begin at some fixed $D\in\mathfrak{D}\setminus\mathfrak{D}_n$, is at most 2^k. This gives $r(M\,|\,\mathfrak{D}\setminus\mathfrak{D}_n)\leqq\sqrt[n]{2}$.

(ii) Set $\mathfrak{A}''=\mathfrak{A}\setminus\mathfrak{D}_n$, where n is such that $r(S)<r(L)$ and $S=L\,|\,\mathfrak{A}''$. This is possible by (i). Set $\mathfrak{A}'=\mathfrak{A}\setminus\mathfrak{A}''$. Let $L=\begin{pmatrix} P & Q \\ R & S\end{pmatrix}$ be the partition of the $\mathfrak{A}\times\mathfrak{A}$-matrix L according to the partition of $\mathfrak{A}$ into $\mathfrak{A}'$ and $\mathfrak{A}''$. Then $(I-xS)^{-1}=\sum_{k=0}^{\infty} x^k S^k$ exists for $|x|\leqq\lambda^{-1}$ and has nonnegative entries for $0\leqq x\leqq\lambda^{-1}$, as $\lambda:=r(L)>r(S)$. Here I is the unit matrix. For $E(x)=P+xQ(I-xS)^{-1}R$ one has the following matrix equation for $|x|\leqq\lambda^{-1}$ (cf. Lemma 2 of [11]).

$$\begin{bmatrix} I-xE(x) & -xQ(I-xS)^{-1} \\ 0 & I\end{bmatrix}\begin{bmatrix} I & 0 \\ -xR & I-xS\end{bmatrix}=I-xL.$$

Since $\lambda=r(L)$, we find an x with $|x|=\lambda^{-1}$, such that $I-xL$ is not invertible. By this matrix equation, we get that $I-xE(x)$ is not invertible, i.e. $r(E(x))\geqq\lambda$. Since the entries of $E(|x|)$ are greater than or equal to the absolute values of the entries of $E(x)$, we get $r(E(\lambda^{-1}))=r(E(|x|))\geqq r(E(x))\geqq\lambda$. Remark that the $\mathfrak{A}'\times\mathfrak{A}'$-matrix $E(x)$ is a finite matrix. For $t\in(0,\lambda^{-1}]$ the map $t\to r(E(t))$ is continuous and increasing, since the entries of $E(t)$ are continuous and increasing in t. Since $r(E(\lambda^{-1}))\geqq\lambda$, we find a $y\in(0,\lambda^{-1}]$ with $r(E(y))=y^{-1}$. Since $E(y)$ has nonnegative entries, this implies that $I-yE(y)$ is not invertible. Hence $I-yL$ is not invertible by the above matrix equation. As $\lambda=r(L)$, we get $y=\lambda^{-1}$. Since $E(y)$ is a finite matrix, we find a nonnegative vector u_1 with $u_1(I-yE(y))=0$. Set $u_2=yu_1Q(I-yS)^{-1}$, which is a nonnegative $l^1(\mathfrak{A}'')$-vector, since the rows of Q are in $l^1(\mathfrak{A}'')$. Hence $u=(u_1,u_2)$ is a nonnegative $l^1(\mathfrak{A})$-vector and $u(I-yL)=0$ by the above matrix equation.

Carrying out the same for the transpose of L, one gets a nonnegative $l^\infty(\mathfrak{A})$-vector v with $(I-\lambda^{-1}L)v=0$. □

Corollary 2. (i) *A closed subset $\mathfrak{A}$ of $\mathfrak{D}$ contains only finitely many elements, which have no predecessor in $\mathfrak{A}$. If $\mathfrak{H}$ is one of the sets in* (ii) *of Theorem* 9, *then $\mathfrak{H}\cap\mathfrak{A}=\emptyset$ or $\mathfrak{H}\setminus\mathfrak{A}$ is finite.*

(ii) *If $\mathfrak{B}\subset\mathfrak{D}$ is perfect, then there is a finite subset Γ' of Γ such that $\mathfrak{B}=\bigcup_{\mathfrak{C}\in\Gamma'}\bar{\mathfrak{C}}$.*

Proof. (i) Suppose $\mathfrak{H}=\{C_i\colon i>\gamma\}$. If $C_j\in\mathfrak{A}$, then $C_i\in\mathfrak{A}$ for $i\geqq j$, since $C_i\to C_{i+1}$ for $i>\gamma$ and $\mathfrak{A}$ is closed. Hence $\mathfrak{H}\setminus\mathfrak{A}$ is finite. Furthermore $\mathfrak{H}$ contains at most one $D\in\mathfrak{A}$, which has no predecessor in $\mathfrak{A}$. We get (i), since there are only finitely many $\mathfrak{H}$ by Theorem 9 and since $\mathfrak{D}_\gamma$ is finite.

(ii) Set $\Gamma'=\{\mathfrak{C}\in\Gamma\colon \mathfrak{C}\cap\mathfrak{B}\neq\emptyset\}$. Fix a $D\in\mathfrak{B}$. We show $D\in\bar{\mathfrak{C}}$ for some $\mathfrak{C}\in\Gamma'$. Set $D_0=D$. As $\mathfrak{B}$ is perfect, we find $D_i\in\mathfrak{B}$ for $i\geqq 1$ with $D_i\to D_{i-1}$. Let $\mathfrak{H}=\{C_i\colon i>\gamma\}$ be one of the sets in (ii) of Theorem 9. If $D_l=C_r$ and $D_m=C_s$, where $m>l\geqq 0$ and $s\geqq r>\gamma$, then we have a closed path $D_m\to D_{m-1}\to\ldots\to D_l=C_r\to C_{r+1}\to\ldots\to C_s=D_m$. It is contained in some $\mathfrak{C}\in\Gamma$. Hence $\mathfrak{C}\in\Gamma'$ and

$D\in\bar{\mathfrak{C}}$. If this holds for no $\mathfrak{H}$, then each of the finitely many sets $\mathfrak{H}$ contains only finitely many D_i. Hence there is an i_0 such that $D_i\in\mathfrak{D}_\gamma$ for all $i\geqq i_0$. As $\mathfrak{D}_\gamma$ is finite, we get $D_k=D_j$ for some j and k with $i_0\leqq j<k$. This gives again a closed path $D_k\to D_{k-1}\to\ldots\to D_j=D_k$ and we find a $\mathfrak{C}\in\Gamma$ with $\mathfrak{C}\in\Gamma'$ and $D\in\bar{\mathfrak{C}}$ as above. Hence $\mathfrak{B}\subset\bigcup_{\mathfrak{C}\in\Gamma'}\bar{\mathfrak{C}}$ follows. On the other hand, as $\mathfrak{B}$ is closed, $\mathfrak{B}\cap\mathfrak{C}\neq\emptyset$ implies $\bar{\mathfrak{C}}\subset\mathfrak{B}$ and we get equality. If $\mathfrak{H}\cap\bar{\mathfrak{C}}\neq\emptyset$ then $\mathfrak{H}\setminus\bar{\mathfrak{C}}$ is finite by (i). Hence by (ii) of Theorem 9 and as $\mathfrak{D}_\gamma$ is finite, we can make Γ' finite. □

§ 2. *The Nonwandering Set*

In Corollary 2 of Theorem 9 we have proved the requirement of Theorem 5. The next two results (Lemma 14 and Theorem 10) will be used to show the requirements of Theorems 6 and 7. The results, which follow from these theorems, are summarized in Theorem 11.

Lemma 14. (i) *There are only finitely many critical $D\in\mathfrak{D}$ with $\alpha(D)=\infty$.*

(ii) *There is a closed T-invariant subset Y of X, such that all ω-limit points in (X, T) are also ω-limit points in $(Y, T\mid Y)$, and such that $(Y, T\mid Y)$ is a piecewise monotonic dynamical system, whose Markov diagram contains no critical D with $\alpha(D)=\infty$ and $T^i(D)\not\subset D$ for all $i\geqq 1$.*

Proof. (i) By (ii) of Lemma 12 each set $\mathfrak{H}$ (cf. Theorem 9) contains at most one critical D with $\alpha(D)=\infty$. As there are only finitely many sets $\mathfrak{H}$ by Theorem 9 and as $\mathfrak{D}_\gamma$ is finite, we get (i).

(ii) Suppose $D\in\mathfrak{D}$ is critical with $\alpha(D)=\infty$. By definition of the Markov diagram, we find a path $C_0C_1\ldots C_k$ with $C_0\in\mathfrak{Z}$ and $C_k=D$. Choose $Z_i\in\mathfrak{Z}$ such that $C_i\subset Z_i$ and set $V=\bigcap_{i=0}^{k}T^{-i}(Z_i)$. By Lemma 1 we get $T^k(V)=D$. If V does not have a common endpoint with Z_0, then $V'=\bigcap_{i=0}^{k-1}T^{-i}(Z_{i+1})$ has no common endpoint with $T(Z_0)$, as $T\mid Z_0$ is monotone, and we have $T^{k-1}(V')=D$. Choosing k minimal, we can suppose that V has a common endpoint with Z_0. Remark that D itself has a common endpoint with Z_k. Furthermore $V\in\mathfrak{Z}_k$ is a $\mathfrak{Z}$-atom, since $T^k(V)=D$ and $\alpha(D)=\infty$ (cf. Lemma 12). As K is finite, (i) of Lemma 9 implies that there are only finitely many such V.

Now fix a critical $D\in\mathfrak{D}$ with $\alpha(D)=\infty$ and $T^i(D)\not\subset D$ for $i\geqq 1$. Let $\mathfrak{E}$ be the set of all critical intervals V, such that $T^k(V)=D$ for some $k\geqq 0$ and that $V\in\mathfrak{Z}_m$ for some $m\geqq k$. By the above, we get that $\mathfrak{E}\neq\emptyset$, that $\mathfrak{E}\subset\mathfrak{Z}_\infty$ and that $\mathfrak{E}$ is finite. The set $R=\bigcup_{V\in\mathfrak{E}}\bigcup_{i=0}^{\infty}T^{-i}(V)$ is open and T^{-1}-invariant. Hence $Y:=X\setminus R$ is closed and T-invariant. Suppose there is a $z\in X$ with $T^i(z)\in R$ for all $i\geqq 0$. Then $T^i(z)$ is in some $V\in\mathfrak{E}$ for infinitely many i, as $\mathfrak{E}$ is finite. Hence we find r and s with $r<s$ such that $T^r(z)\in V$ and $T^s(z)\in V$. By (iii) of Lemma 9, we get $T^{s-r}(V)\subset V$, since $T^s(z)\in T^{s-r}(V)\cap V$. Because $T^k(V)=D$, we get $T^{s-r}(D)\subset D$, a contradiction. Hence for all $z\in R$ there is an i with $T^i(z)\in Y$. This implies that the ω-limit set of z in (X, T) is the ω-limit set of $T^i(z)$ in $(Y, T\mid Y)$, since Y is closed.

Let J be a closed subinterval of X. We call I a boundary interval of J, if I contains an endpoint of J and if $I \cap J = U \cap J$ for some $U \in \bigcup_{m=0}^{\infty} \mathfrak{Z}_m$. Define $g(J)$ as follows: For $V \in \mathfrak{E}$ and $i \geqq 0$, cancel $T^i(V)$ from J, if $T^i(V)$ is a boundary interval of J and then cancel all elements of R from J. Call the remaining set $g(J)$. For $Z \in \mathfrak{Z}$, we show $g(Z) = Z \setminus R$. To this end suppose that $T^i(V)$ is a boundary interval of Z for some $i \geqq 0$ and some $V \in \mathfrak{E}$ and that $T^k(V) = D$. Then $T^i(V) \subset Z$, by (ii) of Lemma 9 and $\mathfrak{E} \subset \mathfrak{Z}_\infty$, and $T^i(V)$ is a critical interval. For $k < i$, $T^i(V) = T^{i-k}(D)$ is not critical by Lemma 12, hence $k \geqq i$. Now $T^i(V) \cap Z = U \cap Z$ implies $T^i(V) = U$, where $U \in \bigcup_{m=0}^{\infty} \mathfrak{Z}_m$. Hence $T^{k-i}(U) = D$ and $T^i(V) = U \in \mathfrak{E}$. Therefore $T^i(V) \subset R$ and $g(Z) = Z \setminus R$. This gives that $\tilde{\mathfrak{Z}} = \{g(Z) \colon Z \in \mathfrak{Z}\}$ is a partition of Y into intervals. Next we show for $C \in \mathfrak{D}$ that $g(T(C)) = T(g(C))$. Let $Z \in \mathfrak{Z}$ be such that $C \subset Z$. If $U \in \mathfrak{Z}_m$ for some m and if $U \cap Z \neq \emptyset$, then $U = Z \cap T^{-1}(U')$ for some $U' \in \mathfrak{Z}_{m-1}$. For $V \in \mathfrak{E}$ and $i \geqq 0$, $T^i(V) \cap C = U \cap C$ is equivalent with $T^{i+1}(V) \cap T(C) = U' \cap T(C)$, as $T \colon C \to T(C)$ is a homeomorphism. Hence $T^i(V)$ is a boundary interval of C, if and only if $T^{i+1}(V)$ is a boundary interval of $T(C)$, since $T \mid C$ is also monotone. Furthermore, if $V \cap C \neq \emptyset$ for $V \in \mathfrak{E}$, then V is a boundary interval of C, since V is critical and $C \subset Z$. These two facts imply that $g(T(C)) = T(g(C))$. Now, for $Z \in \mathfrak{Z}$, $T(g(Z)) = g(T(Z))$ gives that $(Y, T \mid Y)$ is piecewise monotonic with partition $\tilde{\mathfrak{Z}}$, since $g(J)$ is a subinterval of Y for every subinterval J of X. Finally we prove that $g(J \cap Z) = g(J) \cap g(Z)$, where $Z \in \mathfrak{Z}$ and J is a closed subinterval of X. To this end suppose, that $V \in \mathfrak{E}$, that $i \geqq 0$, and that $U \in \bigcup_{m=0}^{\infty} \mathfrak{Z}_m$. If $T^i(V) \cap Z \neq \emptyset$, then $T^i(V) \subset Z$ by (ii) of Lemma 9 and $\mathfrak{E} \subset \mathfrak{Z}_\infty$, and then $T^i(V) \cap (J \cap Z) = U \cap (J \cap Z)$ is equivalent with $T^i(V) \cap J = U \cap J$ provided that $T^i(V) \cap J \neq \emptyset$. Hence, if $T^i(V) \cap Z \neq \emptyset$, then $T^i(V)$ is a boundary interval of $J \cap Z$ if and only if it is a boundary interval of J or of Z. This gives $g(J \cap Z) = g(J) \cap g(Z)$. Together with $T(g(C)) = g(T(C))$ this implies that the successors of $g(C)$ are the sets $g(E) \neq \emptyset$, where E is a successor of $C \in \mathfrak{D}$. As $\tilde{\mathfrak{Z}} = \{g(Z) \colon Z \in \mathfrak{Z}\}$, the Markov diagram $\tilde{\mathfrak{D}}$ of $(Y, T \mid Y)$ is $\tilde{\mathfrak{D}} = \{g(C) \colon C \in \mathfrak{D}\} \setminus \{\emptyset\}$ with an arrow $g(C) \to g(E)$ if $C \to E$ holds in $\mathfrak{D}$. One checks that $g(C)$ is a critical interval in Y or empty, if $C \in \mathfrak{D}$ is critical. Hence $\alpha(g(C)) \leqq \alpha(C)$ by Lemma 12 and $\tilde{\mathfrak{D}}$ contains less critical C with $\alpha(C) = \infty$ than $\mathfrak{D}$, since $g(D) = \emptyset$. Repeating this procedure finitely many times, we can get rid of all critical intervals $D \in \mathfrak{D}$ with $\alpha(D) = \infty$ and $T^i(D) \not\subset D$ for all $i \geqq 1$, by (i). This proves (ii). □

Theorem 10. *Suppose $\mathfrak{A} \subset \mathfrak{D}$ is closed.*

(i) *$H(\mathfrak{A}) = H(\mathfrak{A}_1) \cup H(\mathfrak{A}_2) \cup H(\mathfrak{A}_3)$, where $\mathfrak{A}_1$ is a finite subset of $\mathfrak{A}$, $\mathfrak{A}_2$ is a finite union of sets $\{C_i \in \mathfrak{A} \colon i \geqq 0\}$, where $\alpha(C_0) = \infty$, $C_i = T^i(C_0)$ and $C_i \not\subset C_0$ for $i \geqq 1$, and $\mathfrak{A}_3$ is a finite union of sets $\{D_i \in \mathfrak{A} \colon i \geqq 0\}$, where $D_i \subset D_{i+1}$ for $i \geqq 0$ and all D_i have a common left or right endpoint d.*

(ii) *For every set $\{D_i \in \mathfrak{A} \colon i \geqq 0\}$ of which $\mathfrak{A}_3$ consists the union of the D_i is an interval with endpoints d and x, say. Then $T^p(x) = x$ for some p and x is contained in a nontrivial $\mathfrak{Z}$-atom I with $T^p(I) \subset I$ and I contains the other endpoint besides d of every D_i.*

Proof. For $c \in K$ set $\mathfrak{A}_c = \{D \in \mathfrak{A} \colon c \in D\}$. If Z_c is that element of $\mathfrak{Z}$, which contains c, then all $D \in \mathfrak{A}_c$ are subintervals of Z_c and have the common endpoint c. Hence $A_c = H(\mathfrak{A}_c)$ is a subinterval of Z_c with endpoint c. Let $K_1 \subset K$ be the set of those c, for which there is a $D \in \mathfrak{A}_c$ with $D = A_c$, i.e.

(1) $c \in K_1 \Leftrightarrow A_c \in \mathfrak{A}$.

Let K_2 be the set of those $c \in K \setminus K_1$ which have nonempty $\mathfrak{A}_c$. As $c \in A_c$, $E \in \mathfrak{A}$ and $A_c \subset E$ imply $E \in \mathfrak{A}_c$. Hence

(2) $c \in K_2 \Rightarrow A_c \not\subset E$, if $E \in \mathfrak{A}$, and there are $D_i \in \mathfrak{A}_c$ with $D_i \uparrow A_c$.

Set $\alpha(c) = \alpha(A_c)$. The endpoints of A_c are c and x_c, say. For $c \in K_2$ and D_i as in (2) we have $\alpha(c) = \min \alpha(D_i)$ and $T^{\alpha(c)}(D_i) \uparrow T^{\alpha(c)}(A_c)$. Hence for all large D_i we get $T^{\alpha(c)}(D_i) \cap Z_d = T^{\alpha(c)}(A_c) \cap Z_d$ for all $d \in K \cap T^{\alpha(c)}(A_c)$, except one, which we denote by $f(c)$ and which is uniquely determined by $T^{\alpha(c)}(x_c) \in Z_{f(c)}$. Set $\mathfrak{R} = \{D \in \mathfrak{A} : \nexists C \in \mathfrak{A}$ with $C \to D\} \cup \{A_c : c \in K_1 \cup K_2\}$. Together with (1) and (ii) of Lemma 12 we get for $B \in \mathfrak{R}$ and $Z \in \mathfrak{Z}$ with $T^{\alpha(B)}(B) \cap Z \neq \emptyset$ that

(3) $Z \neq Z_{f(c)}$, if $B = A_c$ for some $c \in K_2 \Rightarrow T^{\alpha(B)}(B) \cap Z \in \bigcup_{d \in K} \mathfrak{A}_d$.

Furthermore, the intervals $T^{\alpha(c)}(D_i) \cap Z_{f(c)}$, which are in $\mathfrak{A}_{f(c)}$ by Lemma 12 and definition of f, increase to $T^{\alpha(c)}(A_c) \cap Z_{f(c)}$. Hence

(4) $c \in K_2 \Rightarrow T^{\alpha(c)}(A_c) \cap Z_{f(c)} \subset A_{f(c)}$.

Now we can show

(5) $C \in \mathfrak{A} \Rightarrow C \subset T^i(B)$ where $B \in \mathfrak{R}$ and $0 \leqq i < \alpha(B)$.

If $C \in \mathfrak{R} \cup \bigcup_{d \in K} \mathfrak{A}_d$ then (5) holds. Suppose $C \subset T^i(B)$ is shown. Let $E = T(C) \cap Z$ be a successor of C, where $Z \in \mathfrak{Z}$. If $i < \alpha(B) - 1$, then $E \subset T^{i+1}(B)$ and (5) follows. If $i = \alpha(B) - 1$, then $E \subset T^{\alpha(B)}(B) \cap Z$ and E is a subset of some element of $\mathfrak{R}$ by (3) and (4). Hence (5) is proved by induction, since for every $C \in \mathfrak{A}$ there is a path from $\mathfrak{R} \cup \bigcup_{d \in K} \mathfrak{A}_d$ to C (cf. Fig. 3). Now (5) implies $H(\mathfrak{A}) \subset \bigcup_{B \in \mathfrak{R}} \bigcup_{0 \leqq i < \alpha(B)} T^i(B)$. Set $\mathfrak{R}_3 = \{A_c : c \in K_2\}$, $\mathfrak{R}_2 = \{B \in \mathfrak{R} \setminus \mathfrak{R}_3 :\ \alpha(B) = \infty,\ T^i(B) \not\subset B$ for $i \geqq 1\}$ and $\mathfrak{R}_1 = \mathfrak{R} \setminus (\mathfrak{R}_2 \cup \mathfrak{R}_3)$. If $B \in \mathfrak{R}_1$ and $\alpha(B) = \infty$, then $T^k(B) \subset B$ for some $k \geqq 1$. We redefine $\alpha(B) = k$ and get $\bigcup_{i=0}^{\infty} T^i(B) = \bigcup_{0 \leqq i < \alpha(B)} T^i(B)$. Set $\mathfrak{A}_1 = \{T^i(B) : B \in \mathfrak{R}_1,\ 0 \leqq i < \alpha(B)\}$ and $\mathfrak{A}_2 = \{T^i(C) : C \in \mathfrak{R}_2,\ 0 \leqq i < \infty\}$. By (i) of Corollary 2, we get that $\mathfrak{R}$ is finite. Hence $\mathfrak{A}_1$ and $\mathfrak{R}_2$ are finite. Let $\mathfrak{A}_3$ be the union over $c \in K_2$ and $0 \leqq j < \alpha(c)$ of the sets $\{T^j(D_i) : i \geqq 1\}$ where for every c the D_i are as in (2). This union is finite, since $\alpha(c) = \min \alpha(D_i) < \infty$, by (i) of Lemma 14. We have $H(\mathfrak{A}) \subset H(\mathfrak{A}_1) \cup H(\mathfrak{A}_2) \cup H(\mathfrak{A}_3)$. By (1), we have $\mathfrak{R}_1 \cup \mathfrak{R}_2 \subset \mathfrak{A}$. By (ii) of Lemma 12, we get then $\mathfrak{A}_1 \subset \mathfrak{A}$, $\mathfrak{A}_2 \subset \mathfrak{A}$ and $\mathfrak{A}_3 \subset \mathfrak{A}$. Hence (i) is shown.

It remains to show (ii). For $d \in K_2$, by (i) of Corollary 2, all D_i in (2) except finitely many have a $C_i \in \mathfrak{A}$ with $C_i \to D_i$. By (5), $C_i \subset T^j(B)$ for some $B \in \mathfrak{R}$. As $d \in D_i \subset T(C_i)$, we get $j = \alpha(B) - 1 < \infty$, since $T^{j+1}(B) \cap K = \emptyset$ for $0 \leqq j < \alpha(B) - 1$. Hence $T^{\alpha(B)}(B) \supset D_i$. As $\mathfrak{R}$ is finite, we get $T^{\alpha(B)}(B) \supset A_d$ for some $B \in \mathfrak{R}$. Since $A_d \subset Z_d$, we have $T^{\alpha(B)}(B) \cap Z_d \supset A_d$. Hence (2) and (3) imply that $B = A_c$ for some $c \in K_2$ and $d = f(c)$. By (4) we get then $T^{\alpha(B)}(B) \cap Z_d = A_d$. We have shown

(6) $\forall d \in K_2\ \exists c \in K_2$ with $f(c) = d$ and $T^{\alpha(c)}(A_c) \cap Z_d = A_d$.

In particular $f: K_2 \to K_2$ is surjective and as K_2 is finite, we get

(7) $f: K_2 \to K_2$ is bijective.

For $c \in K_2$ define $h: \mathfrak{A}_c \to \mathfrak{A}_{f(c)}$ by $h(D) = T^{\alpha(c)}(D) \cap Z_{f(c)}$. If the D_i are as in (2), we get from (6) and (7)

(8) $D_i \uparrow A_c \Rightarrow h(D_i) \uparrow A_{f(c)}$.

Let c and x_c be the endpoints of A_c and let c and x_D be the endpoints of $D \in \mathfrak{A}_c$. For large $D \in \mathfrak{A}_c$, i.e. $\alpha(D) = \alpha(c)$, we have by definition of f and h and by (8)

(9) $T^{\alpha(c)}(x_D) = x_{h(D)}$ and $T^{\alpha(c)}(x_c) = x_{f(c)}$.

Now fix some $c \in K_2$. By (7) we find $c_0 = c, c_1, \ldots, c_n = c$ in K_2 with $f(c_i) = c_{i+1}$ for $0 \leqq i < n$. By (9) we get $T^p(x_c) = x_c$, where $p = \alpha(c_1) + \ldots + \alpha(c_n)$. We find a set $\mathfrak{H} = \{C_i : i > \gamma\}$ as in (ii) of Theorem 9, which contains infinitely many elements of $\mathfrak{A}_c$. If $C_j \in \mathfrak{A}_c$, choose $k < j$ maximal such that C_k is critical. By (ii) of Lemma 12, we have $T^{\alpha(C_k)}(C_k) \cap Z_c = C_j$. Furthermore $T^{\alpha(C_k)}(x_{C_k}) = x_{C_j}$ by Lemma 13 applied to $D = C_k$, since $C_j \in \mathfrak{G}_{i+\alpha(C_k)}$, if $C_k \in \mathfrak{G}_i$, and $i > \gamma \geqq 0$. As $x_{C_j} \in C_j \subset Z_c$, by definition of f we get $C_k \in \mathfrak{A}_{f^{-1}(c)} = \mathfrak{A}_{c_{n-1}}$ and $h(C_k) = C_j$. Furthermore $C_i \notin \mathfrak{A}_c$ for $k < i < j$, since C_i is not critical. We can iterate this step and find a $j' < j$ with $C_{j'} \in \mathfrak{A}_c$ and $h^n(C_{j'}) = C_j$. Furthermore, $C_i \notin \mathfrak{A}_c$ for $j' < i < j$, if $c_i \neq c$ for $0 < i < n$. Now fix some m with $C_m \in \mathfrak{H} \cap \mathfrak{A}_c$, such that $x_{C_j} \in V_p(x_c)$ (this is defined before Theorem 1) for all $j \geqq m$ with $C_j \in \mathfrak{A}_c \cap \mathfrak{H}$. As $\mathfrak{H} \cap \mathfrak{A}_c$ is infinite, there is a $j > m$ with $C_m \subsetneqq C_j$. We find $j' < j$ as above. If $C_{j'} \supset C_j$, we repeat the above argument and find $j'' < j'$ with $C_{j''} \in \mathfrak{A}_c$. If $C_{j''} \supset C_{j'}$, we iterate this, reach C_m, since $C_i \notin \mathfrak{A}_c$ for $i \neq j, j', \ldots$ and get a contradiction to $C_m \subsetneqq C_j$. Hence we can suppose $C_{j'} \subset C_j$. Let J be the interval with endpoints $x_{C_{j'}}$ and x_c. By the choice of m, we have $J \subset V_p(x_c)$, since $j' \geqq m$, and by (9) we get $T^p(x_{C_{j'}}) = x_{C_j}$. Hence $T^p(J) \subset J$, as $C_{j'} \subset C_j$, and there is a $\mathfrak{Z}$-atom I containing J with $T^p(I) \subset I$ (cf. Lemma 9). Hence I is the desired $\mathfrak{Z}$-atom for the set $\{D_i : i \geqq 1\} \subset \mathfrak{A}_c$. Using the $\mathfrak{Z}$-atom containing $T^j(I)$ for the set $\{T^j(D_i) : i \geqq 1\}$, where $j < \alpha(c)$, we get (ii). □

Remark. For the closed set $\mathfrak{C}_2$ in the example in §2 of Chap. I we have $H(\mathfrak{C}_2) = J_1 \cup J_2 \cup \bigcup_{i=1}^{\infty} K_i$ and $\mathfrak{A}_3 = \{K_i : i \geqq 1\}$.

Corollary. *Assume that $\mathfrak{D}$ contains no D with $\alpha(D) = \infty$ and $T^i(D) \not\subset D$ for all $i \geqq 1$. If $\mathfrak{A}$ and $\mathfrak{B}$ are closed subsets of $\mathfrak{D}$ with $\mathfrak{B} \subset \mathfrak{A}$, then there are subsets $\mathfrak{F}_n$ of $\mathfrak{A}$ for $n \geqq 1$ which satisfy the requirements of Theorem 7. Furthermore set $\mathfrak{C} = \mathfrak{A} \setminus \mathfrak{B}$. Then $H(\mathfrak{C}) \setminus H(\mathfrak{F}_n)$ is a subset of a finite union of nontrivial $\mathfrak{Z}$-atoms for all $n \geqq 1$. If $\mathfrak{Z}$ is a generator, then there is a finite subset $\mathfrak{C}'$ of $\mathfrak{A}$ with $H(\mathfrak{C}) \subset H(\mathfrak{C}')$.*

Proof. Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$ be the subsets of $\mathfrak{A}$ found in (i) of Theorem 10. As $\mathfrak{D}$ contains no D with $\alpha(D) = \infty$ and $T^i(D) \not\subset D$ for all $i \geqq 1$, we have $\mathfrak{A}_2 = \emptyset$. Hence $H(\mathfrak{A}) \subset H(\mathfrak{A}_1 \cup \mathfrak{A}_3)$. Let $\mathfrak{F}_n$ contain $\mathfrak{A}_1$ and the n-th element of each of the finitely many sets, of which $\mathfrak{A}_3$ consists. By (i) of Theorem 10, these $\mathfrak{F}_n$ satisfy the requirements of Theorem 7. That $H(\mathfrak{A}) \setminus H(\mathfrak{F}_n)$ and hence also $H(\mathfrak{C}) \setminus H(\mathfrak{F}_n)$ is contained in a finite union of nontrivial $\mathfrak{Z}$-atoms, follows from (ii) of Theorem 10. If $\mathfrak{Z}$ is a generator, all $\mathfrak{Z}$-atoms are trivial. Hence $\mathfrak{A}_3 = \emptyset$ by (ii) of Theorem 10. We get $H(\mathfrak{A}) \subset H(\mathfrak{A}_1)$ and hence $H(\mathfrak{C}) \subset H(\mathfrak{C}')$ for $\mathfrak{C}' = \mathfrak{A}_1$. □

Remark. The proof of (i) of Theorem 10 shows even, that for every $D \in \mathfrak{A}$ there is a $C \in \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \mathfrak{A}_3$ with $D \subset C$. Let $\mathfrak{C} \in \Gamma$, $\mathfrak{A} = \bar{\mathfrak{C}}$ and $\mathfrak{B} = \tilde{\mathfrak{C}}$. As $\mathfrak{C}$ is irreducible, by the properties of the elements of $\mathfrak{A}_2$ it is impossible, that $D \subset C$ for

$D \in \mathfrak{C}$ and $C \in \mathfrak{A}_2$ (cf. (ii) of Lemma 12). We get $H(\mathfrak{C}) \subset H(\mathfrak{A}_1 \cup \mathfrak{A}_3)$. The above proof shows now, that the corollary holds without the assumption.

We state now the main results about $\Omega(X, T)$ for piecewise monotonic (X, T). To this end we assume that the Markov diagram of (X, T) contains no D with $\alpha(D) = \infty$ and $T^i(D) \not\subset D$ for all $i \geq 1$. Otherwise we consider $(Y, T|Y)$ found in Lemma 14 instead of (X, T). We can do this without loss of generality, since moving to $(Y, T|Y)$ may decrease the set W, but all other parts of $\Omega(X, T)$ remain unchanged by Lemma 14 and Theorem 5. The proof of Lemma 14 gives a method, how one can reduce (X, T) to $(Y, T|Y)$ and how one can get the Markov diagram of $(Y, T|Y)$ from that of (X, T).

Theorem 11. *Suppose (X, T) is piecewise monotonic. Then*

$$\Omega(X, T) = \bigcup_{\mathfrak{C} \in \Gamma} L(\mathfrak{C}) \cup L_\infty \cup P \cup W.$$

(i) *For $\mathfrak{C} \in \Gamma$, $L(\mathfrak{C}) = \omega(z)$ for some $z \in X$ and $h_{\text{top}}(L(\mathfrak{C})) = \log r(M|\mathfrak{C})$. If $\mathfrak{Z}$ is a generator, then $L(\mathfrak{C})$ is the set of all x which are represented in $\mathfrak{C}$.*

(ii) *L_∞ is contained in a finite union of ω-limit sets and satisfies $h_{\text{top}}(L_\infty) = 0$.*

(iii) *If $x \in W$ then $x \notin \Omega^n(X, T)$ for some n.*

(iv) *P is the set of periodic points contained in nontrivial $\mathfrak{Z}$-atoms.*

Proof. By (ii) of Corollary 2 of Theorem 9 we can apply Theorem 5. As L_∞, P and every $L(\mathfrak{C})$ consist of ω-limit points and as $W \subset \Omega(X, T)$ by definition, we get the equality for $\Omega(X, T)$. If $\mathfrak{C} \in \Gamma$, the corollary of Theorem 10 shows the requirements of Theorems 6 and 7. Hence (i) follows from Theorems 5, 6 and 7. We have assumed that the Markov diagram contains no D with $\alpha(D) = \infty$ and $T^i(D) \not\subset D$ for all $i \geq 1$. By the corollary of Theorem 10, the requirements of Theorem 7 are satisfied. Hence $h_{\text{top}}(\Psi(\mathfrak{A}_i, \mathfrak{A}_\infty)) \leq \log r(M|\mathfrak{A}_i \setminus \mathfrak{A}_\infty)$ for every decreasing sequence of closed sets $\mathfrak{A}_i$. As $\bigcap_{i=0}^{\infty} (\mathfrak{A}_i \setminus \mathfrak{A}_\infty) = \emptyset$, we get by (i) of Corollary 1 of Theorem 9, that $h_{\text{top}}(\Omega((\mathfrak{A}_i)_{i \geq 1})) = 0$. We apply the variational principle in the version of Corollary 8.6.1 of [17]. Every ergodic T-invariant measure on L_∞ is concentrated on one of the T-invariant sets $\Omega((\mathfrak{A}_i)_{i \geq 1}) \supset L((\mathfrak{A}_i)_{i \geq 1})$ and has therefore entropy 0. This implies $h_{\text{top}}(L_\infty) = 0$. The other assertion of (ii) follows from Theorem 4. Finally (iii) is the definition of W and (iv) follows from the definition of P and (iv) of Lemma 9. □

Remark. One can show that every path in $\mathfrak{C}$ represents an $x \in L(\mathfrak{C})$, also if $\mathfrak{Z}$ is not a generator.

We conclude with a result about the growth rate of the number of inverse images of an $x \in L(\mathfrak{C})$ under $T^k|L(\mathfrak{C})$, where $\mathfrak{C} \in \Gamma$. We need the following consequence of Corollary 1 of Theorem 9.

Lemma 15. *Suppose $\mathfrak{C} \in \Gamma$. Set $\tilde{M} = M|\mathfrak{C}$ and $\lambda = r(\tilde{M})$. Let $u \in l^1(\mathfrak{C})$ be the left and $v \in l^\infty(\mathfrak{C})$ be the right eigenvector of $\tilde{M}$ for the eigenvalue λ found in Corollary 1 of Theorem 9. We have $u_C > 0$ and $v_C > 0$ for all $C \in \mathfrak{C}$. Set $P_{CD} = \tilde{M}_{CD} v_D / \lambda v_C$ and $\pi_C = u_C v_C$, where $C, D \in \mathfrak{C}$. The $\mathfrak{C} \times \mathfrak{C}$-matrix P is then a stochastic matrix and $\pi \in l^1(\mathfrak{C})$ satisfies $\pi P = \pi$.*

Proof. Since $\mathfrak{C}$ is irreducible, we get from Corollary 1 of Theorem 9, that $u_C > 0$ and $v_C > 0$ for all $C \in \mathfrak{C}$. The results about π and P follow by a simple computation from $u\tilde{M} = \lambda u$ and $\tilde{M} v = \lambda v$. □

Theorem 12. *Suppose* $\mathfrak{C}\in\Gamma$ *and* $L:=L(\mathfrak{C})\neq\emptyset$. *Set* $\lambda=r(M|\mathfrak{C})$ *and* $n_k(x)=\operatorname{card}(T|L)^{-k}(\{x\})$ *for* $x\in L$. *Then there are* $c>0$ *and* $d<\infty$ *with*

$$c\leqq\liminf_{k\to\infty}(\lambda^{-k}\inf_{x\in L}n_k(x))\leqq\limsup_{k\to\infty}(\lambda^{-k}\sup_{x\in L}n_k(x))\leqq d.$$

Proof. We consider the finite sets $\mathfrak{F}_n\subset\bar{\mathfrak{C}}$ of the corollary of Theorem 10 with $\mathfrak{A}=\bar{\mathfrak{C}}$ and $\mathfrak{B}=\tilde{\mathfrak{C}}$. Fix $x\in L(\mathfrak{C})$, fix some $C\in\mathfrak{C}$ and fix some $\mathfrak{F}_n$. Set $\mathfrak{F}=\mathfrak{F}_n$. By the corollary, $H(\mathfrak{C})\setminus H(\mathfrak{F})$ is contained in a finite union of nontrivial $\mathfrak{Z}$-atoms. Hence, by the definition of $\mathfrak{U}(x)$, we can suppose that $Q\cap(H(\mathfrak{C})\setminus H(\mathfrak{F}))=\emptyset$ for all $Q\in\mathfrak{U}(x)$. We suppose that $\mathfrak{U}(x)$ is as in Lemma 10. For every $Q\in\mathfrak{U}(x)$ we find an element V of some $\mathfrak{Z}_m$ with $V\subset Q$ and $V\cap\Omega(\mathfrak{C})\neq\emptyset$. Then $H(\mathfrak{F})\supset V\cap H(\mathfrak{C})\supset(V\cap H(\bar{\mathfrak{C}}))\setminus H(\tilde{\mathfrak{C}})$ and hence, by Lemma 7, for every $Q\in\mathfrak{U}(x)$ there is a path $D_0D_1\dots D_m$ in $\mathfrak{C}$ with $D_0\in\mathfrak{F}$ and $\bigcap_{i=0}^{m}T^{-i}(D_i)\subset V\subset Q$. As $\mathfrak{F}$ is finite, there is a $D\in\mathfrak{F}$ with $D_0=D$ for infinitely many $Q\in\mathfrak{U}(x)$. Choosing a subsystem of $\mathfrak{U}(x)$, we can suppose $D_0=D$ for all $Q\in\mathfrak{U}(x)$. Let $C_0C_1\dots C_k$ be a path in $\mathfrak{C}$ with $C_0=C$, which was fixed above, and $C_k=D$. Then $C_0C_1\dots C_{k-1}D_0\dots D_m$ is a path in $\mathfrak{C}$. Let z be as in Theorem 2. Then z is represented by a path, which contains every finite path of $\mathfrak{C}$. Hence for every $Q\in\mathfrak{U}(x)$ there is an i with

$$T^i(z)\in\bigcap_{j=0}^{k-1}T^{-j}(C_j)\cap\bigcap_{j=0}^{m}T^{-j-k}(D_j)\subset\bigcap_{j=0}^{k-1}T^{-j}(C_j)\cap T^{-k}(Q).$$

Let y be a limit point of these points $T^i(z)$. Then $y\in\bigcap_{j=0}^{k-1}T^{-j}(C_j)$, which is a closed set, $y\in L(\mathfrak{C})$ by Theorem 4, and $T^k(y)=x$, since $y\in\overline{T^{-k}(Q)}$ for all $Q\in\mathfrak{U}(x)$.

For every $x\in L(\mathfrak{C})$ we have found a $D\in\mathfrak{F}$, such that for every path $C_0C_1\dots C_k$ in $\mathfrak{C}$ with $C_0=C$ and $C_k=D$ there is a $y\in\bigcap_{j=0}^{k-1}T^{-j}(C_j)\cap L(\mathfrak{C})$ with $T^k(y)=x$. By Lemma 2, the sets $\bigcap_{j=0}^{k-1}T^{-j}(C_j)$ are contained in different elements of $\mathfrak{Z}_k$, hence different paths $C_0C_1\dots C_{k-1}$ give rise to different y. Therefore $n_k(x)\geqq\inf_{D\in\mathfrak{F}}\tilde{M}_{CD}^{(k)}$, where $\tilde{M}=M|\mathfrak{C}$.

On the other hand $n_k(x)\leqq\operatorname{card}\mathfrak{Z}'_k$, as

$$\mathfrak{Z}'_k:=\{V\in\mathfrak{Z}_k\colon V\cap\Omega(\mathfrak{C})\neq\emptyset\}\supset\{V\in\mathfrak{Z}_k\colon V\cap L(\mathfrak{C})\neq\emptyset\}$$

and T^k is monotone on $V\in\mathfrak{Z}_k$. The assumption of (i) of Theorem 7 is shown to hold by the corollary of Theorem 10. Hence we get as in the proof of Theorem 7, that $n_k(x)\leqq\sum_{C\in\mathfrak{F}_m}\sum_{D\in\mathfrak{C}}\tilde{M}_{CD}^{(k)}$ for some m.

By Lemma 15 and the renewal theorem (cf. [1]), we have

$$\lambda^{-k}\tilde{M}_{CD}^{(k)}=P_{CD}^{(k)}v_C/v_D\to\pi_D v_C/v_D=u_D v_C>0.$$

Hence the result follows with $c=\inf_{D\in\mathfrak{F}}u_D v_C$, which is greater than zero, since $\mathfrak{F}$ is finite, and with $d=q\|u\|_1\|v\|_\infty$, where $q\geqq\operatorname{card}\mathfrak{F}_m$ for all m. $\square$

§ 3. *Further Results*

The Markov diagram has been used for the study of different properties of piecewise monotonic transformations. We give a description of these results:

The Nonwandering Set. A part of the results of Theorem 11 is shown in [2] and [4], in the case where $\mathfrak{Z}$ is a generator. Also some further results are proved there in this case: For $\mathfrak{C}, \mathfrak{C}' \in \Gamma$, it is shown that $L(\mathfrak{C}) \cap L(\mathfrak{C}')$ and $L(\mathfrak{C}) \cap L_\infty$ are finite or empty. The set W is finite and $W \cap \Omega^2(X, T) = \emptyset$. The geometric structure of $L(\mathfrak{C})$ for $\mathfrak{C} \in \Gamma$ is determined. The matrix $M \mid \mathfrak{C}$ has period p, if and only if $T^p \mid L(\mathfrak{C})$ consists of p ergodic components, on each of which T^p is weak mixing. The special case of monotonic mod 1 transformations is considered in [7]. The special structure of these transformations gives a special structure for the Markov diagram, which can be used to get further results about $\Omega(X, T)$. In [12], [13] and [14] other methods are used to investigate the nonwandering set of piecewise monotonic transformations on $[0, 1]$, which are additionally continuous. In [14] those parts of the nonwandering set are considered, to which most (in a topological sense) orbits converge. In the language of this paper, these parts are the sets $L(\mathfrak{C})$ with $\mathfrak{C} \in \Gamma$ and $\tilde{\mathfrak{C}} = \emptyset$, the sets $L((\mathfrak{A}_i)_{i \geq 1})$ for decreasing sequences of closed sets $\mathfrak{A}_i$ with $\bigcap_{i=0}^{\infty} \mathfrak{A}_i = \emptyset$, and the attracting periodic orbits in P.

Maximal Measures. By Lemma 11, one can assume without loss of generality, that $\mathfrak{Z}$ is a generator. For $\mathfrak{C} \in \Gamma$, it is shown in [3], that there is a $1-1$-correspondence between maximal measures for $T \mid L(\mathfrak{C})$ and maximal measures for the Markov shift S with transition matrix $M \mid \mathfrak{C}$. Furthermore it is shown that there is at most one maximal measure on S. Lemma 15 of this paper gives a maximal measure on S. Hence S and $T \mid L(\mathfrak{C})$ have unique maximal measure. This problem for the special case of the β-transformation is considered in [16], where a transition matrix is used, which is exactly the Markov diagram for this special case. In [5], it is shown that the transformations $x \to \beta x + \alpha (\text{mod } 1)$ with $\beta > 1$ have unique maximal measure. Its support is a finite union Y of intervals and the nonwandering set of $x \to \beta x + \alpha (\text{mod } 1)$ consists of Y and finitely many periodic orbits. In [6], the region of the (β, α)-plane is determined, where Y is the whole unit interval.

Periodic Points. If $\mathfrak{Z}$ is a generator, then all periodic points are represented as closed paths in the Markov diagram by Theorem 8. Consider the set $Z(X, T)$ of $p \in \mathbb{N}$ such that an $x \in X$ exists, which has minimal period p under T. In [8], the sets, which occur as $Z(X, T)$ for monotonic mod 1 transformations, are determined. The ζ-function of (X, T) is defined by $\zeta(x) = \exp \sum_{n=1}^{\infty} p_n \frac{x^n}{n}$, where p_n is the number of fixed points of T^n. In [8] it is shown that $\frac{1}{\zeta(x)}$ is a kind of characteristic power series of the matrix M. This interpretation is supported by results of [11], where it is shown that, for $1 < t \leq r(M)$, $1/\zeta(t^{-1}) = 0$ if and only if t is an eigenvalue of the $l^1(\mathfrak{D})$-operator M.

Transfer Operator. The transfer operator P on the set of functions of bounded variation is defined by $Pf(x) = \sum_{y \in T^{-1}\{x\}} g(y) f(y)$, where g is a given positive function of bounded variation. It is shown in [9] and [15], that a spectral theorem about P, which holds under certain conditions on g, implies existence and ergodic properties of equilibrium states for $\log g$. In [10], the requirements of this spectral theorem are proved to hold in different situations, using the Markov diagram (cf. Theorem 1 of [10], a special case of Theorem 12 above and Theorem 3 of [10]). For piecewise constant g, the spectrum of P is investigated further in [11] using the inverse of the ζ-function as a characteristic power series of M.

References

1. Feller, W.: Introduction to probability theory and its applications. New York: J. Wiley & Sons Inc. 1950
2. Hamedinger, W.: Die topologische Struktur stückweise monotoner Transformationen. Diplomarbeit. Wien 1984
3. Hofbauer, F.: On intrinsic ergodicity of piecewise monotonic transformations with positive entropy. Isr. J. Math. **34**, 213–237 (1979). Part II: Isr. J. Math. **38**, 107–115 (1981)
4. Hofbauer, F.: The structure of piecewise monotonic transformations. Ergodic Theory Dyn. Syst. **1**, 159–178 (1981)
5. Hofbauer, F.: Maximal measures for simple piecewise monotonic transformations. Z. Wahrscheinlichkeitstheor. Verw. Geb. **52**, 289–300 (1980)
6. Hofbauer, F.: The maximal measure for linear mod one transformations. J.L.M.S. **23**, 92–112 (1981)
7. Hofbauer, F.: Monotonic mod one transformations. Studia Math. **80**, 17–40 (1984)
8. Hofbauer, F.: Periodic points for piecewise monotonic transformations. Ergodic Theory Dyn. Syst. **5**, 237–256 (1985)
9. Hofbauer, F., Keller, G.: Ergodic properties of invariant measures for piecewise monotonic transformations. Math. Z. **180**, 119–140 (1982)
10. Hofbauer, F., Keller, G.: Equilibrium states for piecewise monotonic transformations. Ergodic Theory Dyn. Syst. **2**, 23–43 (1982)
11. Hofbauer, F., Keller, G.: Zeta-functions and transfer operators for piecewise linear transformations. J. Reine Angew. Math. **352**, 100–113 (1984)
12. Jonker, L., Rand, D.: Bifurcations in one dimension. I: The nonwandering set. Invent. Math. **62**, 347–365 (1981)
13. Nitecki, Z.: Topological dynamics on the interval. Ergodic Theory and Dynamical Systems, vol. II. Progress in Math. Boston: Birkhäuser 1981
14. Preston, C.: Iterates of piecewise monotone maps on an interval. Preprint. Bielefeld 1984
15. Rychlik, M.: Bounded variation and invariant measures. Studia Math. **76**, 69–80 (1983)
16. Takahashi, Y.: Isomorphism of β-automorphisms to Markov automorphisms. Osaka J. Math. **10**, 175–184 (1973)
17. Walters, P.: An introduction to ergodic theory. Berlin-Heidelberg-New York: Springer 1982

Received March 3, 1985