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Probability Theory and Related Fields

Excessive Measures and Markov Processes with Random Birth and Death

P.J. Fitzsimmons[1] and B. Maisonneuve[2]

[1] Department of Mathematical Sciences, The University of Akron, Akron, OH 44325, USA
[2] I.M.S.S., 47 X – F-38040 Grenoble Cedex, France

1. Introduction

Let $(X_s)_{s \geqq 0}$ be a right Markov process on a nice state space $(E, \mathscr{E})$, with semigroup $(P_s)_{s \geqq 0}$. Given a measure η on E which is excessive for (P_s) (i.e. η is σ-finite and $\eta \geqq \eta P_s$, $s>0$), one can construct a stationary right continuous Markov process (Y_t), defined on a random time interval $]\alpha, \beta[$ and admitting (P_s) as transition semigroup and η as one dimensional distribution. Similar constructions have been made by Kuznetsov [13] and by Mitro [15], the latter under duality hypotheses.

The process (Y_t) is a natural tool in the study of the class of excessive measures for the process (X_s). Our purpose in this paper is to use this tool in developing several aspects of the theory of excessive measures.

In Sect. 3, we obtain a characterization of the class of measure potentials (for (X_s)). This characterization leads to a natural Riesz decomposition of an excessive measure into potential and harmonic components. These results are obtained without transience hypotheses and so generalize recent work of Getoor and Glover ([8], [9]).

In Sect. 4, we show that Dynkin's [3] decomposition of an excessive measure into dissipative and conservative parts is the analog of the Riesz decomposition, appropriate to the *simple* ordering of excessive measures.

In Sect. 5, we develop a notion of balayage associated with a given terminal time R. This balayage operation is shown to coincide with that of Hunt when (X_s) is transient and R is the hitting time of a Borel set.

Inspired by recent work of Atkinson and Mitro [1], and Getoor and Sharpe [10], we consider characteristic measures in Sect. 6. The principal result of this section is a last exit decomposition of the balayage $L_R \eta$ of an excessive measure η. This decomposition resembles, in form, certain invariant measures constructed by Silverstein [18], Getoor [6], and Kaspi [11], [12], using invariant measures of a "process on the boundary".

2. Notation and Basic Results

Let E be a Borel subset of a compact metrizable space and let $\mathscr{E}$ denote the Borel sets in E. Let Δ be a point not in E and set $E_\Delta = E \cup \{\Delta\}$, $\mathscr{E}_\Delta = \mathscr{E} \vee \{\Delta\}$. We consider Δ as isolated in E_Δ. As usual, a function f on E is extended to E_Δ by setting $f(\Delta)=0$.

Denote by W the set of paths $w: \mathbb{R} \to E_\Delta$ which have the following property: there exists an open interval on which w is E-valued and right-continuous, and outside of which w is identically Δ. The coordinate process on W is denoted by $(Y_t)_{t\in\mathbb{R}}$. Let $(\mathscr{G}_t^0)_{t\in\mathbb{R}}$ denote the natural filtration of (Y_t) and set $\mathscr{G}^0 = \bigvee_t \mathscr{G}_t^0$. We define random variables

$$\alpha = \inf\{t\in\mathbb{R}: Y_t \in E\}, \qquad \beta = \sup\{t\in\mathbb{R}: Y_t\in E\}$$

with the conventions $\inf\emptyset = +\infty$, $\sup\emptyset = -\infty$. We also define two families of shift operators on W by

$$\begin{aligned} \sigma_t w(s) &= w(t+s) && s, t\in\mathbb{R}; \\ \tau_t w(s) &= w(t+s) && s>0, t\in\mathbb{R}, \\ &= \Delta && s\leqq 0, t\in\mathbb{R}. \end{aligned}$$

Clearly $\sigma_t \circ \sigma_s = \sigma_{t+s}$ and $\tau_t = \tau_0 \circ \sigma_t$.

Let $\Omega = \{\alpha = 0,\ Y_{0+}$ exists in $E\} \cup \{[\Delta]\}$, where $[\Delta]$ is the constant path $t\to\Delta$. Let X_s (resp. θ_s, resp. ζ) denote the restriction of Y_{s+} (resp. τ_s, resp. $\beta \vee 0$) to Ω, where $s\geqq 0$. Set $\mathscr{F}^0 = \mathscr{G}^0|_\Omega$, $\mathscr{F}_t^0 = \mathscr{G}_t^0|_\Omega$. Let $(P^x)_{x\in E}$ be an $\mathscr{E}$-measurable family of probabilities on $(\Omega, \mathscr{F}^0)$ such that $P^x(X_0 = x)=1$ and such that for s, $t\in\mathbb{R}_+$ and $f\in b\mathscr{E}$,

$$P^x(f\circ X_{t+s} \mid \mathscr{F}_t^0) = P^{X_t}(f\circ X_s).$$

Finally, we assume that the Markov process $X = (\Omega, \mathscr{F}^0, \mathscr{F}_t^0, \theta_t, X_t, P^x)$ satisfies the right hypothesis $(HD2)$ of Sharpe [17]. This is equivalent to the almost sure right continuity of $t\to P^{X_t}(f\circ X_s)$, whenever f is bounded and continuous on E. Thus $X=(X_s)$ is strong Markov with (Borel) semigroup (P_s) given by $P_s f(x) = P^x(f\circ X_s)$, $f\in b\mathscr{E}$. The resolvent family of X is denoted by $(U^r)_{r\geqq 0}$, with $U=U^0$.

The family $(\tau_t)_{t\in\mathbb{R}}$ plays an important role in the sequel. Owing to the identity $Y_{t+s} = X_s\circ\tau_t$ on $\{\alpha<t\}$, $t\in\mathbb{R}$, $s\geqq 0$, this family provides the link between W and Ω that is essential in many calculations.

Recall that a measure η on $(E,\mathscr{E})$ is *excessive* (for (P_s)) if η is σ-finite and if $\eta \geqq \eta P_s$ whenever $s>0$. As is well-known ([8]), if η is excessive then $\eta P_s \uparrow \eta$ as $s\downarrow 0$. We let Exc denote the class of excessive measures for (P_s).

(2.1) **Theorem.** *Fix $\eta\in$ Exc. There exists a unique σ-finite measure Q_η on $(W, \mathscr{G}^0)$, carried by $W\setminus\{[\Delta]\}$, and under which $(Y_t)_{t\in\mathbb{R}}$ is markovian with semigroup (P_s) and one-dimensional laws η: for $t\in\mathbb{R}$, $f\in\mathscr{E}_+$ and $F\in(\mathscr{F}^0)_+$,*

$$Q_\eta(F\circ\tau_t \mid \mathscr{G}_t^0) = P^{Y_t}(F) \quad on \quad \{\alpha < t < \beta\}, \tag{2.2}$$

$$Q_\eta(f\circ Y_t) = \eta(f). \tag{2.3}$$

Moreover, Q_η is stationary relative to $(\sigma_s)_{s\in\mathbb{R}}$.

This is essentially a right-continuous version of a result of Kuznetsov [13], and follows easily from our hypotheses on the process (X_s), together with [13].

Note that because of the uniqueness assertion in (2.1), if η and ξ are excessive, then $Q_{\eta+\xi}=Q_\eta+Q_\xi$.

For $\eta\in\mathrm{Exc}$ let $\mathscr{G}^\eta$ denote the Q_η-completion of $\mathscr{G}^0$ and let $\mathscr{G}_t^\eta$ denote $\mathscr{G}_{t+}^0\vee\mathscr{N}^\eta$, where $\mathscr{N}^\eta$ is the class of Q_η null sets in $\mathscr{G}^\eta$. As noted by Dynkin in [3], if $\{A_1, A_2\}$ is a partition of W into $\mathscr{G}_\alpha^\eta$-measurable, (σ_t)-invariant sets, then Q_η can be decomposed as $Q_{\eta_1}+Q_{\eta_2}$, where $\eta_i: f\to Q_\eta(f\circ Y_t; A_i)$ is excessive (and independent of $t\in\mathbb{R}$). Clearly $Q_{\eta_i}=Q_\eta(\cdot\,; A_i)$ and $\eta=\eta_1+\eta_2$.

As an application of this observation, we deduce the "elementary" Riesz decomposition of an excessive measure η. Recall that $\eta\in\mathrm{Exc}$ is *invariant* if $\eta=\eta P_s$, $s>0$, and that η is *purely excessive* if $\eta P_s\downarrow 0$ as $s\uparrow\infty$. Let Inv and Pur denote the classes of invariant and purely excessive measures respectively. Then $\eta\in\mathrm{Exc}$ can be decomposed uniquely as $\eta_i+\eta_p$ when $\eta_i\in\mathrm{Inv}$, $\eta_p\in\mathrm{Pur}$. In fact, $\eta_i(f)=Q_\eta(f\circ Y_t; \alpha=-\infty)$ and $\eta_p(f)=Q_\eta(f\circ Y_t; \alpha>-\infty)$, for $f\in\mathscr{E}_+$. To see this, note that if $f\in\mathscr{E}_+$ with $\eta(f)<\infty$, then $\eta P_s(f)\downarrow\eta_i(f)$ as $s\uparrow\infty$, and so,

$$\begin{aligned}\eta_i(f)&=\lim_{s\to\infty} Q_\eta(P_s f\circ Y_t)\\&=\lim_{s\to\infty} Q_\eta(f\circ Y_{t+s}; \alpha<t)\\&=\lim_{s\to\infty} Q_\eta(f\circ Y_t; \alpha<t-s)\\&=Q_\eta(f\circ Y_t; \alpha=-\infty).\end{aligned}$$

We close this section with a result that is a key to later developments. In particular, it shows that if $\eta\in\mathrm{Pur}$, then $\eta=\int_0^\infty \mu_t\,dt$ where $(\mu_t)_{t>0}$ is an *entrance law* for (P_s): each μ_t is σ-finite and $\mu_{t+s}=\mu_t P_s$ for $t>0$, $s\geqq 0$. This representation, which is well-known (see [3]), will be easily obtained via Q_η.

Following Weil [19], a stopping time S of $(\mathscr{G}_t^\eta)$ will be called *intrinsic* provided $\alpha\leqq S<\beta$ on $\{S<\infty\}$ and $S=t+S\circ\sigma_t$, $\forall t\in\mathbb{R}$, except on a Q_η-null set. For the next result note that $\tau_S\in\{\alpha=0\}$ on $\{-\infty<S<\infty\}$.

(2.4) **Theorem.** *Let $\eta\in\mathrm{Exc}$ and let S be an intrinsic stopping time of $(\mathscr{G}_t^\eta)$. Define the measure Π_η^S on $(W,\mathscr{G}^0)$, carried by $\Omega_0=\{\alpha=0\}$, by*

$$\Pi_\eta^S(G)=Q_\eta(0<S\leqq 1; \tau_S\in G),\qquad G\in\mathscr{G}^0. \tag{2.5}$$

Then

(i) *Π_η^S is σ-finite and for $\phi\in(\mathscr{B}_{\mathbb{R}}\otimes\mathscr{G}^0)_+$,*

$$Q_\eta(\phi(S,\tau_S); S\in\mathbb{R})=(\lambda\otimes\Pi_\eta^S)(\phi), \tag{2.6}$$

where λ is Lebesque measure on $\mathbb{R}$;

(ii) *set $\mu_u^S=Y_u(\Pi_\eta^S)$ on E for $u>0$; then for $t\in\mathbb{R}$,*

$$Q_\eta(Y_t\in\cdot\,; -\infty<S<t)=\int_0^\infty \mu_u^S\,du; \tag{2.7}$$

(iii) *the measures* μ_u^S *are* σ*-finite and under* Π_η^S *the process* $(Y_u)_{u>0}$ *is markovian with semigroup* (P_s) *and entrance law* (μ_u^S);

(iv) *if* $\alpha<S$ a.e. - Q_η, *then under* Π_η^S, Y_{0+} *exists in* E a.e. *and the process* $(Y_{u+})_{u\geqq 0}$ *is markovian with semigroup* (P_s) *and initial measure* $\mu^S = Q_\eta(Y_S\in\cdot\,; 0<S\leqq 1)$; *in particular*

$$Q_\eta(Y_t\in\cdot\,; S<t)=\mu^S U. \tag{2.8}$$

The most important case is $S=\alpha$; we will then write simply Π_η, μ_u rather than $\Pi_\eta^\alpha, \mu_u^\alpha$. In this case, (2.7) yields $\eta_p=\int_0^\infty \mu_u\,du$; since (μ_u) is an entrance law for (P_s), one has

$$\mu_t U\uparrow\eta_p \quad \text{as} \quad t\downarrow 0. \tag{2.9}$$

Proof. (i) Fix $q\in\mathscr{E}_+$ such that $q>0$ and $\eta(q)<\infty$. Then $H\equiv\int_{\mathbb{R}} q\circ Y_t\,dt>0$ on Ω_0 since there $\alpha<\beta$; we have

$$\begin{aligned}\Pi_\eta^S(H)&=Q_\eta\left(\int_0^\infty q\circ Y_{S+u}\,du; 0<S\leqq 1\right)\\ &\leqq Q_\eta\Big(\int_{\mathbb{R}} q\circ Y_t\,dt; 0<S\leqq 1\Big)\\ &=\int_{\mathbb{R}} Q_\eta(q\circ Y_0; -t<S\leqq -t+1)\,dt\\ &=Q_\eta(q\circ Y_0)=\eta(q)<\infty,\end{aligned}$$

where the second equality is due to the (σ_t) invariance of Q_η. Thus Π_η^S is σ-finite. For (2.6) it now suffices to show that if $t\in\mathbb{R}$, $G\in\mathscr{G}^0$ with $\Pi_\eta^S(G)<\infty$, then

$$\phi(s)\equiv Q_\eta(t<S\leqq t+s, \tau_S\in G)=s\Pi_\eta^S(G), \qquad s>0.$$

But $\phi(s+s')=\phi(s)+\phi(s')$ by the σ_s-invariance of Q_η and the identity $\tau_S=\tau_S\circ\sigma_s$ on $\{S\in\mathbb{R}\}$. Since $\phi(1)=\Pi_\eta^S(G)$ and since $s\to\phi(s)$ is right continuous, it follows that $\phi(s)=s\Pi_\eta^S(G)$.

(ii) For $f\in\mathscr{E}_+$,

$$Q_\eta(f\circ Y_t; -\infty<S<t)=Q_\eta(f\circ Y_{t-S}(\tau_S); -\infty<S<t)=\Pi_\eta^S\left(\int_{-\infty}^t f\circ Y_{t-s}\,ds\right),$$

by (2.6) and Fubini's theorem. This is (2.7).

(iii) The strong Markov property of Y holds under Q_η at every stopping time of $(\mathscr{G}_t^\eta)$ (the argument is the same as in [15]; for a detailed proof see [2]); applied at time $S+t$ this property yields

$$\Pi_\eta^S(G_t F\circ\tau_t)=\Pi_\eta^S(G_t P^{Y_t}(F)) \tag{2.10}$$

for $t>0$ and $G_t\in(\mathscr{G}_t^0)_+$, $F\in\mathscr{F}_+^0$. With $G_t=1$ and $F=\int_0^\infty f\circ Y_u\,du$, $f\in\mathscr{E}_+$, we obtain

$$\Pi_\eta^S\left(\int_t^\infty f\circ Y_u\,du\right)=\mu_t^S U(f). \tag{2.11}$$

When $f=q$, we have $\mu_t^S U(q) \leqq \eta(q) < \infty$ and since $Uq > 0$, μ_t^S is σ-finite. The Markov property of $(Y_t)_{t>0}$ under Π_η^S follows from (2.10).

(iv) If $\alpha < S$ a.e. - Q_η, the strong Markov property of Y at S under Q_η implies the Markov property of $(Y_{u+})_{u \geqq 0}$ under Π_η^S and (2.8) follows. □

(2.12) *Remark.* Let $\eta \in \mathrm{Exc}$ and suppose that $(\mu_t)_{t>0}$ and $(v_t)_{t>0}$ are two entrance laws for (P_s) such that $\int_0^\infty \mu_t\,dt = \int_0^\infty v_t\,dt = \eta$. Then $\mu_t = v_t$ for all $t>0$. In fact, for $s>0$

$$\left(\int_0^\infty \mu_t\,dt\right) P_s = \int_s^\infty \mu_u\,du = \mu_s U;$$

hence $\mu_s U = v_s U$ and so $\mu_s = v_s$ by Proposition (1.1) of [8].

3. Measure Potentials and Riesz Decomposition

(3.1) *Definition.* An excessive measure η is a measure potential (or simply a potential) if $\eta = \mu U$ for some (necessarily σ-finite) measure μ on E. The class of potentials is denoted by Pot.

Note that $\mathrm{Pot} \subset \mathrm{Pur}$.

(3.2) **Proposition.** *Fix $\eta = \mu U \in \mathrm{Pot}$. Then Q_η is carried by $\{\alpha > -\infty,\ Y_{\alpha+}$ exists in $E\}$, $\Pi_\eta (= \Pi_\eta^\alpha)$ is carried by Ω, and $\Pi_\eta = P^\mu$ on Ω.*

Proof. Since η is purely excessive, $\eta = \int_0^\infty \mu_t\,dt$ with $\mu_t = Y_t(\Pi_\eta)$, using (2.7) with $S = \alpha$. But $\eta = \mu U = \int_0^\infty X_t(P^\mu)\,dt$. By Remark (2.12), the process $(Y_t)_{t>0}$ under Π_η and the process $(X_t)_{t>0}$ under P^μ have the same distribution; thus Π_η is carried by Ω and $\Pi_\eta = P^\mu$ on Ω.

The main result of this section, Theorem (3.3), is essentially the converse of (3.2). It may also be viewed as a time-reversed version of Theorem A-8 of [9]. However, unlike [9], we do not assume that X is transient. Our Riesz decomposition of $\eta \in \mathrm{Exc}$ into potential and "harmonic" parts is a simple corollary of (3.3).

To state (3.3), we introduce some notation. Let $C_u(E)$ denote the space of real bounded d-uniformly continuous functions on E, where d denotes a metric on E compatible with its topology. Then $C_u(E)$ is separable in the uniform norm; let D be a countable dense subset of $C_u(E)^+$. Given $\eta \in \mathrm{Pur}$, choose $q \in \mathscr{E}_+$ with $q>0$ and $\eta(q) < \infty$. Set $h = Uq$ and define

$$\Omega_q = \{\alpha = 0,\ Y_{0+} \text{ exists in } E,\ \phi(Y_{1/n}) \to \phi(Y_{0+}), \text{ as } n \to \infty, \text{ for } \phi = h \text{ and } \phi = U^r g,\ g \in D,\ r \in \mathbb{Q}_+^*\},$$

where $\mathbb{Q}_+^*$ denotes the set of strictly positive rationals.

(3.3) **Theorem.** *Fix $\eta \in \mathrm{Pur}$ and let q and Ω_q be as above. Then $\eta \in \mathrm{Pot}$ if and only if Π_η is carried by Ω_q. In this case, $v = Y_{0+}(\Pi_\eta)$ is σ-finite and under $\Pi_\eta, (Y_{t+})_{t \geqq 0}$ is markovian with semigroup (P_s) and initial law v. That is, $\Pi_\eta = P^v$ on Ω. In particular, $\eta = vU$.*

Proof. If $\eta=\mu U$, then Π_η is carried by Ω_q because of (3.2) and our right hypotheses (which imply that the mappings $t\to\phi(X_t)$ are right continuous a.e. $-P^\mu$). Conversely, suppose that Π_η is carried by Ω_q. Then by Fatou's lemma and (2.9),

$$\nu(h)=\Pi_\eta(h\circ Y_{0+})\leqq\liminf_{n\to\infty}\Pi_\eta(h\circ Y_{1/n})\leqq\eta(q).$$

Since $h=Uq>0$, ν is σ-finite. For $f\in b\mathscr{E}_+$ let Π^f denote the finite measure $(hf)\circ Y_{0+}\cdot\Pi_\eta$. In order to prove that $(Y_{t+})_{t\geqq 0}$ is markovian, it suffices, in view of (2.4), to prove that for every $g\in D$

$$\Pi^f(g\circ Y_t)=\Pi^f(P_t g(Y_{0+})),\qquad t>0.$$

Both sides of the above equality being right continuous in $t>0$, it suffices to show the equality of their Laplace transforms. But for $r\in\mathbb{Q}^*_+$

$$\int_0^\infty e^{-rt}\Pi^f(g\circ Y_t)\,dt=\lim_{n\to\infty}\mathrm{e}^{-r/n}\Pi^f(U^r g\circ Y_{1/n})=\Pi^f((U^r g\circ Y_{0+}),$$

where the second equality follows from the definition of Ω_q and dominated convergence. □

(3.4) *Definition.* Let ξ and η be excessive measures. We say that η *strongly* dominates ξ if there exists $\gamma\in\mathrm{Exc}$ such that $\eta=\xi+\gamma$.

Of course, if η strongly dominates ξ, then η dominates ξ (for the *simple* order of measures), that is $\eta(A)\geqq\xi(A)$ for all $A\in\mathscr{E}$.

(3.5) *Definition.* We say that $\eta\in\mathrm{Exc}$ is *harmonic* if 0 is the only potential strongly dominated by η. The class of harmonic measures is denoted by Har.

We can now formulate the *Riesz decomposition* of $\eta\in\mathrm{Exc}$ into potential and harmonic parts. Choose $q\in\mathscr{E}_+$ and define Ω_q as for Theorem (3.3). The set $W_q=\{\alpha>-\infty,\tau_\alpha\in\Omega_q\}$ is in $\mathscr{G}^0_{\alpha+}$ and is (σ_t)-invariant. By the discussion of Sect. 1, we can decompose η into (excessive) components μU and ρ as follows:

$$\text{(3.6)}\qquad \mu U(f)=Q_\eta(f\circ Y_t;\,W_q),\qquad \rho(f)=Q_\eta(f\circ Y_t;\,W_q^c).$$

By Theorem (3.3), the first equation in (3.6) actually defines a potential μU.

(3.7) **Theorem.** *Fix* $\eta\in\mathrm{Exc}$ *and let* $\eta=\mu U+\rho$ *be the decomposition of* η *described by* (3.6). *Then* ρ *is harmonic and the decomposition (into potential and harmonic parts) is unique. Moreover,*

(i) μU *is the largest element of* Pot *(in either order) which is strongly dominated by* η;

(ii) ρ *is the largest element of* Har *(in either order) which is strongly dominated by* η.

Proof. Let us first prove that $\rho\in\mathrm{Har}$. In fact, if $\rho=\nu U+\gamma$ with $\gamma\in\mathrm{Exc}$, then $Q_\rho=Q_{\nu U}+Q_\gamma\geqq Q_{\nu U}$. Thus $Q_{\nu U}$ is simultaneously carried by W_q^c and W_q ((3.3) applies to νU since $\nu U(q)\leqq\eta(q)<\infty$). Thus $\nu U=0$ and $\rho\in\mathrm{Har}$. As for the uniqueness, suppose that $\eta=\nu U+\gamma$ with $\nu U\in\mathrm{Pot}$, $\gamma\in\mathrm{Har}$. Then $Q_\eta=Q_{\nu U}+Q_\gamma$. But

$Q_{\nu U}$ is carried by W_q and Q_γ is carried by $W_q^c(Q_\gamma(Y_t\in\cdot\,;W_q)$ is a potential by (3.3) and is strongly dominated by γ). Thus $Q_{\nu U}\leqq Q_{\mu U}$ and $Q_\gamma\leqq Q_\rho$. These inequalities are in fact equalities since $Q_\eta=Q_{\mu U}+Q_\rho=Q_{\nu U}+Q_\gamma$ and so $\nu U=\mu U$ and $\gamma=\rho$.

To prove (i), let $\eta=\nu U+\eta'$ with $\eta'\in$Exc. Then $\eta'(q)<\infty$ and η' has the Riesz decomposition $\eta'=\mu' U+\rho'$. The uniqueness assertion proved earlier implies that $\mu U=\nu U+\nu' U$; thus μU strongly dominates νU and (i) follows. The proof of (ii) is similar. □

(3.8) *Remarks.* (a) It is clear from (3.7) that the Riesz decomposition does not depend on the choice of q used in defining Ω_q.

(b) A comparison with [(2.10), 9] shows that our decomposition (3.6) coincides with that of [9] when (as in that paper) X is assumed to be transient. The present decomposition, valid for an arbitrary Borel right process, appears to be new.

(c) The following properties of Exc are proved in [4]:

(i) Let $\xi,\eta_1,\eta_2\in$Exc and suppose $\xi\leqq\eta_1+\eta_2$. Then there exist $\xi_1,\xi_2\in$Exc such that $\xi_1\leqq\eta_1$, $\xi_2\leqq\eta_2$ and $\xi=\xi_1+\xi_2$.

(ii) If $\xi\in$Exc and $\mu U\in$Pot satisfy $\xi\leqq\mu U$, then $\xi=\nu U$ for some measure ν on E.

(See [8] for (ii) under transience hypotheses.) Using (i) and (ii), we have the following variant of (3.7, ii) (cf. [(2.10, i), 9]):

(3.9) ρ is the largest harmonic measure dominated by η.

To see this, let $\gamma\in$Har with $\gamma\leqq\eta$. Then $\gamma\leqq\mu U+\rho$ and by (3.8, c, i) there exist $\gamma_1\leqq\mu U$ and $\gamma_2\leqq\rho$, both in Exc, such that $\gamma=\gamma_1+\gamma_2$. By (3.8, c, ii), $\gamma_1=\nu U\in$Pot. But since $\gamma\in$Har, we must have $\nu U=0$. Thus $\gamma=\gamma_2\leqq\rho$.

4. Dissipative Measures, Conservative Measures

According to Dynkin [3], each $\eta\in$Exc has a unique decomposition $\eta=\eta_d+\eta_c$ where η_d is dissipative and η_c is conservative. Dynkin's proof is based on the Hopf decomposition of η with respect to each of the operators P_t, $t>0$ (see Neveu [16], Proposition V-5-2). In this section, we show that this decomposition is the analog of the Riesz decomposition $\eta=\mu U+\rho$, appropriate to the *simple* ordering on Exc. The following definition is different from Dynkin's, but it will turn out to be equivalent, due to Theorem (4.3).

(4.1) *Definition.* Let $\eta\in$Exc. We shall say that

(i) η is *dissipative* provided $\eta=\sup\{\pi\in\text{Exc}: \pi\in\text{Pot}, \pi\leqq\eta\}$ (here sup denotes the least upper bound in the simple order on Exc);

(ii) η is *conservative* provided 0 is the only potential (simply) dominated by η.

It is clear from (2.9) that Pur ⊂ Dis. Hence the following inclusions between the various classes of excessive measures:

$$\text{Pot}\subset\text{Pur}\subset\text{Dis},\quad \text{Har}\supset\text{Inv}\supset\text{Con}.$$

Let η be a fixed excessive measure and let $q \in \mathscr{E}_+$ be such that $q > 0$ and $\eta(q) < \infty$. Following [3] we define

$$\eta_d = \eta(\cdot\,; \{Uq < \infty\}), \qquad \eta_c = \eta(\cdot\,; \{Uq = \infty\}). \tag{4.2}$$

(4.3) **Theorem.** *The measure η_d (resp. η_c) as defined in* (4.2) *is a dissipative (resp. conservative) excessive measure. The decomposition $\eta = \eta_d + \eta_c$ into dissipative and conservative parts is unique and*

(i) $\eta_d = \sup\{\pi \in \mathrm{Pot}: \pi \leqq \eta\}$, *so that η_d is the largest element of* Dis *which is (simply) dominated by η.*

(ii) *η_c is the largest element of* Con *which is dominated by η.*

In particular, η_d and η_c do not depend on the particular choice of q in (4.2).

Proof. 1) Let us show that $\eta_d \in \mathrm{Exc}$. In fact, since $G = \{Uq < \infty\}$ is absorbing for $(X_s)_{s \geqq 0}$, we have $\eta_d P_s f = \eta(1_G P_s f) = \eta(1_G P_s(1_G f)) \leqq \eta P_s(1_G f) \leqq \eta_d(f)$ for any $f \in \mathscr{E}_+$.

2) We now prove that there exists a sequence (μ_n) of measures on E such that $\mu_n U \uparrow \eta_d$. In fact, let (S_n) be a decreasing sequence of intrinsic stopping times such that $S_n > \alpha$ and $S_n \downarrow \alpha$ a.e. $-Q_{\eta_d}$. Such a sequence exists by Lemma 6.3 of [3]; for the convenience of the reader, we provide a quick proof of this fact in (4.4) below. Then the measure $\pi_n = Q_{\eta_d}(Y_t \in \cdot, S_n < t)$ is a potential (apply (2.8) with $S = S_n$ to η_d), and π_n increases to η_d as $n \uparrow \infty$.

3) Let $\pi = \mu U \in \mathrm{Pot}$ with $\pi \leqq \eta$. Then $\mu(Uq) \leqq \eta(q) < \infty$, so μ is carried by $G = \{Uq < \infty\}$. Since G is absorbing, μU is carried by G as well and so $\pi \leqq \eta_d$. From this and 2), it follows that $\eta_d = \sup\{\pi \in \mathrm{Pot}: \pi \leqq \eta\}$ and that η_d is dissipative. Point (i) is proved.

4) Let us now prove that $\eta_c \in \mathrm{Con}$. First note that $\eta_c \in \mathrm{Exc}$, as proved by Dynkin in [3]. If π is a potential dominated by η_c, then π is simultaneously carried by G and G^c. Thus $\pi = 0$ and $\eta_c \in \mathrm{Con}$.

5) If $\eta'_c \in \mathrm{Con}$ and $\eta'_c \leqq \eta$, the measure $\rho = \eta'_c(\cdot\,; G)$ is dissipative by 3); thus ρ is a sup of potentials. But a potential dominated by ρ is dominated by η'_c and must be 0. Thus $\rho = 0$ and η'_c is carried by G^c, proving that $\eta'_c \leqq \eta_c$. Point (ii) is thereby established. The uniqueness of the decomposition is clear from (i) and (ii). □

(4.4) **Lemma.** *Let $\eta \in \mathrm{Exc}$ be such that there exists $q > 0$ with $\eta(q) < \infty$ and $Uq < \infty$ a.e. η (this amounts to saying that $\eta \in \mathrm{Dis}$, once* (4.3) *is proved...). Then there exists a sequence (S_n) of intrinsic stopping times of $(\mathscr{G}_t^0)$ such that $S_n > \alpha$, $S_n \downarrow \alpha$ a.e. Q_η.*

Proof. Let $k \in \mathscr{E}_+$ be such that $0 < k \leqq 1$ and $\eta(kUq) < \infty$ (note that the measure $Uq \cdot \eta$ is σ-finite since $Uq < \infty$ a.e.). We have

$$Q_\eta\left(q \circ Y_t \int_{-\infty}^{t} k \circ Y_s \, ds\right) = \int_{-\infty}^{t} \eta(kP_{t-s} q)\, ds = \eta(kUq) < \infty;$$

thus, $K_t \equiv \int_{-\infty}^{t} k \circ Y_s \, ds < \infty$ a.e. Q_η. The sequence $(S_n = \inf\{t < \beta: K_t \geqq 1/n\})_{n \geqq 1}$ has the desired properties (this argument is adapted from Weil [18], Proposition 1). □

(4.5) *Remark.* It is clear from Theorem (4.3) that $\eta \in \mathrm{Exc}$ is dissipative (resp. conservative) if and only if it is carried by $\{Uq<\infty\}$ (resp. $\{Uq=\infty\}$) for some (or every) $q \in \mathscr{E}_+$ with $q>0$ and $\eta(q)<\infty$. Hence our definitions are equivalent to Dynkin's. In addition, part 2) of the proof of (4.3) implies the following useful characterization of Dis.

(4.6) **Corollary.** *Let* $\eta \in \mathrm{Exc}$. *The following properties are equivalent:*

(i) $\eta \in \mathrm{Dis}$;

(ii) *there exists a sequence* (S_n) *of intrinsic stopping times of* $(\mathscr{G}_t^\eta)$ *such that* $S_n > \alpha$, $S_n \downarrow \alpha$ a.e. Q_η;

(iii) *there exists a sequence* (π_n) *of potentials such that* $\pi_n \uparrow \eta$.

In addition, if (ii) *is satisfied, then* $\mu^{S_n} U \uparrow \eta$, *where*

$$\mu^{S_n} = Q_\eta(Y_{S_n} \in \cdot\,;\, 0 \leqq S_n < 1). \tag{4.7}$$

Proof. (i)$\Rightarrow$(ii) by Lemma (4.4); (ii)$\Rightarrow$(iii) since $\mu^{S_n} U = Q_\eta(Y_t \in \cdot\,; S_n < t) \uparrow \eta$; (iii)$\Rightarrow$(i) by definition. □

(4.8) **Corollary.** *Let* (η_n) *be an increasing sequence from* Exc *with limit* η. *Then* $\eta_{nd} \uparrow \eta_d$ *and* $\eta_{nc} \uparrow \eta_c$.

Proof. Let q be as in (4.2). By Remark (4.5), we have $\eta_{nd} = \eta_n(\cdot\,; \{Uq<\infty\})$, $\eta_{nc} = \eta_n(\cdot\,; \{Uq=\infty\})$; the desired result is now immediate. □

(4.9) *Remarks.* a) Given $\eta \in \mathrm{Exc}$, if $\eta = \mu U + \rho = \eta_p + \eta_i = \eta_d + \eta_c$ are the associated Riesz-type decompositions, then $\mu U \leqq \eta_p \leqq \eta_d$ and $\rho \geqq \eta_i \geqq \eta_c$.

b) The classes Exc, Pot, Pur, Dis, Har, Inv, and Con are convex cones. In addition, Exc, Pot, Pur, Dis, and Con are $\wedge$-stable. The only properties not immediate from the definitions and Remark (4.5) are that Har is stable under $+$ and that Pot is $\wedge$-stable. But consider ξ, $\eta \in \mathrm{Har}$ and μU, $\nu U \in \mathrm{Pot}$. If $\pi \in \mathrm{Pot}$ is strongly dominated by $\xi + \eta$, then $Q_\pi \leqq Q_\xi + Q_\eta$ and so Q_π is carried by W_q^c (where $q \in \mathscr{E}_+$ is such that $q>0$, $(\xi+\eta)(q)<\infty$ and W_q is as in (3.6)). Thus $Q_\pi = 0$, $\pi = 0$ and $\xi + \eta \in \mathrm{Har}$. The measure $\mu U \wedge \nu U$ is excessive and dominated by μU; by Remark (3.8 ii) $\mu U \wedge \nu U \in \mathrm{Pot}$.

c) Suppose that X is transient, i.e. there exists $q \in \mathscr{E}_+$ such that $q>0$ and $Uq<\infty$ (see [7]). Given $\eta \in \mathrm{Exc}$, we can reduce q if necessary to insure $\eta(q)<\infty$ and then by Remark (4.5) $\eta \in \mathrm{Dis}$. Thus, under the transience hypothesis, Dis $=$ Exc and it now follows from Corollary (4.6) that every excessive measure is the limit of an increasing sequence of potentials. This is Hunt's approximation lemma (see [8], Theorem (1.5)).

5. Balayage

Our purpose in this section is to define and study a balayage operation on the class Exc which extends the original balayage of Hunt. Hunt's balayage (in the context of transient right processes) is discussed in [8] and [9]. Recall from [9] or Remark (4.8 c) that if X is transient and $\eta \in \mathrm{Exc}$, then there exists a sequence of measures (μ_n) on E such that $\mu_n U \uparrow \eta$ as $n \uparrow \infty$. One then defines

$L_B \eta$, the balayage of η on $B \in \mathscr{E}$, by the monotone limit

$$L_B \eta = \lim_n \mu_n P_B U.$$

Here P_B is the usual hitting operator for B: $P_B f(x) = P^x(f \circ X_{T_B})$, where $T_B = \inf\{t > 0 \colon X_t \in B\}$.

Our balayage operation will be defined by means of Q_η which allows us to proceed in slightly increased generality. To wit, let R be a perfect, exact terminal stopping time of $(\mathscr{F}_s)_{s \geqq 0}$ (the usual completion of $(\mathscr{F}_s^0)_{s \geqq 0}$ with respect to all the measures P^μ): $R = s + R \circ \theta_s$ on $\{R > s\}$ and $s + R \circ \theta_s \downarrow R$ as $s \downarrow 0$. The random time $\tilde{R}$ is then unambiguously defined on W by the condition

(5.1)
$$\tilde{R} = \inf_{\alpha < t < \beta} (t + R \circ \tau_t).$$

One checks easily that $\tilde{R}$ is an intrinsic stopping time of $(\mathscr{G}_t^\eta)$ and that $\tilde{R} = t + R \circ \tau_t$ on $\{R > t, \alpha < t < \beta\}$.

(5.2) *Definition.* Let η be an excessive measure. For every intrinsic stopping time S of $(\mathscr{G}_t^\eta)$ let $L^S \eta = Q_\eta(Y_t \in \cdot\,; S < t)$, where $t \in \mathbb{R}$ is arbitrary. For every perfect, exact terminal time R of $(\mathscr{F}_s)_{s \geqq 0}$ we define $L_R \eta$, the R-balayage of η, by $L_R \eta = L^{\tilde{R}} \eta$.

Since $Q_\eta(f \circ Y_{t+s};\ S < t + s) = Q_\eta(f \circ Y_t;\ S \circ \sigma_{-s} < t + s) = Q_\eta(f \circ Y_t;\ S < t)$, $L^S \eta$ and $L_R \eta$ do not depend on the choice of $t \in \mathbb{R}$. *In the sequel S denotes an intrinsic time*; *R and $\tilde{R}$ will be as described in the paragraph preceding* (5.2).

For the next result, define a family of "*birthing*" operators $(b^t)_{t \in \mathbb{R}}$ on W as follows:

$$b^t w(s) = w(s) \quad \text{if } s > t,$$
$$= \Delta \quad \text{if } s \leqq t.$$

The mapping $(t, w) \to b^t(w)$ is measurable. As is customary, b^S denotes the map $w \to b^{S(w)}(w)$.

(5.3) **Proposition.** *Let η and η' be excessive measures. Then $L^S \eta$ is excessive and $L^S \eta \leqq \eta$. Moreover*

(i) $L^S(\eta + \eta') = L^S(\eta) + L^S(\eta')$;

(ii) $Q_{L^S \eta} = b^S(Q_\eta(\cdot\,; S < \infty))$;

(iii) $(L^S \eta)_p = Q_\eta(Y_t \in \cdot\,; -\infty < S < t) = \int_0^\infty \mu_t^S\, dt$,
where (μ_t^S) *is as in* (2.4);

(iv) $(L^S \eta)_{\mathrm{Pot}} = Q_\eta(Y_t \in \cdot\,; -\infty < S < t, \tau_S \in \Omega_q) = \mu^S U$,
where $\mu^S = Q_\eta(Y_{S+} \in \cdot\,; 0 < S \leqq 1, \tau_S \in \Omega_q)$ *and* $q > 0$ *is such that* $\eta(q) < \infty$.

In particular, $L^S \eta$ is purely excessive (resp. a potential) if and only if $Q_\eta(\cdot\,; S < \infty)$ is carried by $\{S > -\infty\}$ (resp. $\{S > \infty, \tau_S \in \Omega_q\}$).

(5.4) *Remark.* The last assertion of this proposition implies $L^S\,\mathrm{Pur} \subset \mathrm{Pur}$ and $L^S\,\mathrm{Pot} \subset \mathrm{Pot}$ (since $S \geqq \alpha$ a.e. Q_η). However, L^S does not preserve the classes Inv and Har.

Proof. For $f \in \mathscr{E}_+$ and $s>0$,

$$\begin{aligned} L^S\eta(P_s f) &= Q_\eta(P_s f \circ Y_t; S<t) \\ &= Q_\eta(f \circ Y_{t+s}; S<t) \\ &= Q_\eta(f \circ Y_t; S<t-s) \leqq L^S\eta(f). \end{aligned}$$

The first assertion follows, the inequality $L^S\eta \leqq \eta$ being obvious. Point (i) is clear since $Q_{\eta+\eta'} = Q_\eta + Q_{\eta'}$. It is a simple matter to check that the finite dimensional distributions of the two measures in (ii) are identical; (ii) then follows from the uniqueness assertion in (2.1). Finally, (iii), (iv) follow from (ii), Theorem (2.4), and Theorem (3.7). □

In Proposition (5.3), the operation L^S can be replaced by L_R, since $L_R = L^{\tilde{R}}$. The next result shows that when restricted to Pur, the operation L_R can be described by means of the hitting operator P_R defined by

$$P_R f(x) = P^x(f \circ X_R), \qquad x \in E,\ f \in \mathscr{E}_+.$$

(5.5) **Proposition.** *Let* $\eta \in \mathrm{Pur}$ *and let* $(\mu_t)_{t>0}$ *be the entrance law for* (P_s) *such that* $\eta = \int_0^\infty \mu_t\, dt$. *Then* $\mu_t P_R U \uparrow L_R \eta$ *as* $t \downarrow 0$. *In particular, if* $\eta = \mu U$ *is a potential, then* $L_R(\mu U) = \mu P_R U$.

Proof. For $G \in \mathscr{G}^0_+$, set $\Phi(s,w) = G(\tau_{\tilde{R}} w) \cdot 1_{]0,1]}(s + \tilde{R}(w))$ and apply (2.6) with $S = \alpha$; we obtain

$$\Pi^{\tilde{R}}_\eta(G) = Q_\eta(\Phi(\alpha, \tau_\alpha)) = \Pi_\eta(G \circ \tau_{\tilde{R}}; \tilde{R} \in \mathbb{R}).$$

For $G = \int_0^\infty f \circ Y_t\, dt$, the above combined with (5.3 iii) yields

$$\begin{aligned} L_R \eta(f) &= \Pi_\eta\left(\int_{\tilde{R}}^\infty f \circ Y_u\, du\right) \\ &= \lim_{t \downarrow 0} \Pi_\eta\left(\int_{t + R \circ \tau_t}^\infty f \circ Y_u\, du\right) \\ &= \lim_{t \downarrow 0} \Pi_\eta(G \circ \tau_R \circ \tau_t) \\ &= \lim_{t \downarrow 0} \mu_t P_R U f. \end{aligned}$$

This proves the first assertion. If $\eta = \mu U$, one has $\mu_t = \mu P_t$ and the second assertion follows from the first. □

(5.6) *Remark.* As a consequence of (5.5) one has the following identity: for each measure μ on E such that μU is σ-finite

(5.7) $$\mu P_R = Q_{\mu U}(Y_{\tilde{R}+} \in \cdot\,; 0 < \tilde{R} \leqq 1).$$

This suggests the possibility of extending the notion of hitting operator by setting $P^S(x, \cdot) = Q_{\varepsilon_x U}(Y_{S+} \in \cdot\,; 0 < S \leqq 1)$, but we shall not pursue this idea.

The main result of this section is the next theorem, which provides a description of $(L_R \eta)_d$ and $(L_R \eta)_c$. As a consequence, Dis and Con are stable under L_R and L_R is seen to be a true extension of Hunt's balayage.

(5.8) **Theorem.** *Fix* $\eta \in \mathrm{Exc}$ *and let* $\eta = \eta_d + \eta_c$ *be Dynkin's decomposition of* η: $\eta_d \in \mathrm{Dis}$, $\eta_c \in \mathrm{Con}$. *Then*

(i) $(L^S \eta)_d = L^S(\eta_d)$ *and if* (μ_n) *is a sequence of measures such that* $\mu_n U \uparrow \eta_d$ *(see* (4.6))

$$\mu_n P_R U \uparrow (L_R \eta)_d; \tag{5.9}$$

(ii) $(L^S \eta)_c = L^S(\eta_c) = Q_{\eta_c}(Y_t \in \cdot\,; S = -\infty)$.

Proof. Note that $L^S(\eta_d) \leqq \eta_d$ and that $L^S(\eta_d)$ is dissipative by Remark (4.5). Similarly $L^S(\eta_c) \leqq \eta_c$ and $L^S(\eta_c)$ is conservative. Since $L^S(\eta) = L^S(\eta_d) + L^S(\eta_c)$ and since Dynkin's decomposition is unique, we must have $(L^S \eta)_d = L^S(\eta_d)$, $(L^S \eta)_c = L^S(\eta_c)$. The measure Q_{η_c} is carried by $\{\alpha = -\infty\}$ since η_c is invariant and so, by (5.3 iv), $Q_{\eta_c}(Y_t \in \cdot\,; -\infty < S < t)$ is a potential. This potential, being dominated by η_c, must be 0. Thus

$$L^S(\eta_c) = Q_{\eta_c}(Y_t \in \cdot\,; S < t) = Q_{\eta_c}(Y_t \in \cdot\,; S = -\infty).$$

For the proof of (5.9), we can (and do) assume that $\eta = \eta_d$. Because of Lemma (5.10) below, it suffices to prove (5.9) for one particular sequence (μ_n). We choose $\mu_n = \mu^{S_n}$ given by (4.7). By (5.5 ii)

$$\mu_n P_R U = L_R(\mu_n U) = Q_{\mu_n U}(Y_t \in \cdot\,; \tilde{R} < t).$$

But $\mu_n U = L^{S_n} \eta$ and so, by (5.3 ii),

$$\begin{aligned} \mu_n P_R U &= Q_\eta(Y_t \circ b^{S_n} \in \cdot\,; \tilde{R} \circ b^{S_n} < t, S_n < \infty) \\ &= Q_\eta(Y_t \in \cdot\,; \tilde{R} \circ b^{S_n} < t, S_n < t), \end{aligned}$$

which increases to $Q_\eta(Y_t \in \cdot\,; \tilde{R} < t)$, since $S_n \downarrow \alpha$ and $\tilde{R} \circ b^{S_n} \downarrow \tilde{R}$ a.e. Q_η. □

(5.10) **Lemma.** *If* $\eta \in \mathrm{Dis}$ *and if* (μ_n) *is a sequence of measures on E such that* $\mu_n U \uparrow \eta$, *then the sequence* $(\mu_n P_R U)_{n \geqq 1}$ *increases to a limit* ν *which does not depend on the particular sequence* (μ_n).

Proof. Fix $f \in \mathscr{E}_+$. If $Uf < \infty$ everywhere, then the excessive function $g = P_R Uf$ satisfies $P_t g \downarrow 0$ as $t \uparrow \infty$; by Lemma (3.1) of [7] we can deduce the existence of a sequence $(f_k) \subset \mathscr{E}_+$ such that $Uf_k \uparrow g$. In general, by restricting X to the absorbing set $\{Uf < \infty\}$, one can still find $(f_k) \subset \mathscr{E}_+$ such that $Uf_k \uparrow g$ on $\{Uf < \infty\}$.

Now if $\eta(f) < \infty$, the measures μ_n are carried by $\{Uf < \infty\}$, since $\mu_n Uf \leqq \eta(f)$. Thus $\mu_n(Uf_k) \uparrow \mu_n(g)$ as $k \uparrow \infty$. Since $\mu_n Uf_k \uparrow \eta(f_k)$ as $n \uparrow \infty$, the sequences $(\mu_n(g))$ and $(\eta(f_k))$ increase to the same limit, proving the lemma.

(5.11) *Remarks.* a) If $R = T_B$, where B is nearly Borel, and if X is transient, then

$$L_R \eta = L_B \eta \qquad \eta \in \mathrm{Exc},$$

owing to Remark (4.9 c) and Theorem (5.8 i). Here L_B is Hunt's balayage as described at the beginning of this section.

b) It follows from the proof of Lemma (5.10) that for given $f \in \mathscr{E}_+$ there exists a sequence $(f_k) \subset \mathscr{E}_+$ such that

(5.12) $$\eta(f_k)\uparrow L_R\eta(f),$$

provided $\eta\in\text{Dis}$ is such that $\eta(f)<\infty$.

(5.13) **Corollary.** *Let* (η_n) *be an increasing sequence from* Exc *with limit* $\eta\in\text{Exc}$. *Then*

$$L_R\eta_n\uparrow L_R\eta, \quad (L^S\eta_n)_c\uparrow L^S\eta_c.$$

Proof. $L_R\eta_n=L_R(\eta_{nd})+L_R(\eta_{nc})$ and $\eta_{nd}\uparrow\eta_d$, $\eta_{nc}\uparrow\eta_c$ by Corollary (4.8). Thus, we need only consider the special cases $\eta\in\text{Dis}$ and $\eta\in\text{Con}$.

(i) Suppose $\eta\in\text{Dis}$ and consider $f\in\mathscr{E}_+$ such that $\eta(f)<\infty$. The sequence $(\eta_n(f_k))$, with (f_k) as in Remark (5.11 b), is increasing in both n and k; by (5.12) applied to η and η_n (note that $\eta_n(f)<\infty$)

$$\begin{aligned} L_R\eta(f)&=\lim_k\uparrow\lim_n\uparrow\eta_n(f_k)\\ &=\lim_n\uparrow L_R\eta_n(f). \end{aligned}$$

(ii) Suppose now that η is conservative. Then each measure η_n is conservative and hence invariant. For $n\geqq 1$ consider the measure γ_n such that $\eta_{n-1}+\gamma_n=\eta_n$ (γ_n is well-defined since each η_n is σ-finite). Then γ_n is clearly invariant and even conservative since $\gamma_n\leqq\eta_n$. If we put $\gamma_0=\eta_0$, then we have $\eta=\sum_{n\geqq 0}\gamma_n$ and $Q_\eta=\sum_n Q_{\gamma_n}$; thus

$$\sum_{k\leqq n}Q_{\gamma_k}(Y_t\in\cdot\,;S=-\infty)\uparrow Q_\eta(Y_t\in\cdot\,;S=-\infty).$$

By (5.8 ii) this is equivalent to $L^S\eta_n=\sum_{k\leqq n}L^S\gamma_k\uparrow L^S\eta$ as $n\uparrow\infty$. □

(5.14) *Remarks.* a) If ξ,η are excessive measures, then $\xi\leqq\eta\Rightarrow L_R\xi\leqq L_R\eta$: use (5.13) with $\eta_0=\xi$ and $\eta_n=\eta$ for $n\geqq 1$.

b) Let (R_n) be a decreasing sequence of perfect, exact terminal times with limit R. Then the conclusion of (5.13) can be strengthened to: $L_{R_n}\eta_n\uparrow L_R\eta$. In fact, $L_{R_n}\eta_k$ is increasing in n with limit $L_R\eta_k$ since $R_n\downarrow R$; $L_{R_n}\eta_k$ is increasing in k by (5.13). Thus $L_{R_n}\eta_n$ increases to $\lim_{n,k}L_{R_n}\eta_k=\lim_k L_R\eta_k=L_R\eta$.

c) If S_2 and S_1 are intrinsic times of $(\mathscr{G}_t^\eta)$, it follows from the definition of $L^S\eta$ that $L^{S_1\wedge S_2}\eta+L^{S_1\vee S_2}\eta=L^{S_1}\eta+L^{S_2}\eta$.

6. Characteristic Measures and a Last Exit Decomposition

Let $(B_s)_{s\geqq 0}$ be a perfect additive functional over $(X_s)_{s\geqq 0}$. More precisely, we require that (B_s) be an increasing (right continuous and $(\mathscr{F}_t)$ adapted) process such that

$$B_{t+s}=B_t+B_s\circ\theta_t \qquad t,s\geqq 0.$$

Define a homogeneous random measure $\tilde{B}(w,dt)$ ($w\in W$), carried by $]\alpha(w),\infty[$, by the formulae

(6.1) $$\tilde{B}(]t, t+s]) = B_s \circ \tau_t \quad \text{on} \quad \{\alpha < t\},$$

where $t \in \mathbb{R}$ and $s \geqq 0$.

(6.2) *Definition.* The characteristic measure of B, relative to $\eta \in \text{Exc}$, is the measure ν_η^B defined on $(E, \mathscr{E})$ by

(6.3) $$\nu_\eta^B(A) = Q_\eta\Big(\int_{]0,1]} 1_A \circ Y_s \tilde{B}(ds)\Big), \qquad A \in \mathscr{E}.$$

(6.4) *Remark.* The Revuz measure, as defined in [10], requires the existence of left limits for Y on $]\alpha, \beta[$ and then Y_s is replaced by Y_{s-} on the right hand side of (6.3). We follow Dynkin in calling ν_η^B the characteristic measure of B.

(6.5) **Theorem.** *Suppose that ν_η^B is σ-finite. Then for $f \geqq 0$ and $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{E}$-measurable,*

$$Q_\eta \int_{\mathbb{R}} f(s, Y_s) \tilde{B}(ds) = \int_{\mathbb{R} \times E} f(s,x)\, ds\, \nu_\eta^B(dx).$$

Proof. Fix $A \in \mathscr{E}$ with $\eta(A) < \infty$, and $t \in \mathbb{R}$. Define ϕ on $\mathbb{R}_+$ by

$$\phi(u) = Q_\eta \int 1_{]t, t+u]}(s)\, 1_A(Y_s) \tilde{B}(ds), \qquad u > 0.$$

Then $\phi(u+v) = \phi(u) + \phi(v)$, owing to the σ_u-invariance of Q_η and the fact that $\tilde{B}(u + \cdot) = \tilde{B} \circ \sigma_u$. Since $\phi(1) = \nu_\eta^B(A) < \infty$ and since $u \to \phi(u)$ is right continuous, it follows that $\phi(u) = u\phi(1) = u\nu_\eta^B(A)$. $\square$

We maintain in this section the notation R, $\tilde{R}$ introduced in Sect. 5. We shall also define

$$R_s = R \circ \theta_s, \qquad G = \{s \in]0, \zeta[: R_{s-} = 0, R_s > 0\},$$
$$\tilde{R}_t = R \circ \tau_t \quad \text{on} \quad \{\alpha < t\}, \qquad \tilde{G} = \{t \in]\alpha, \beta[: \tilde{R}_{t-} = 0, \tilde{R}_t > 0\}.$$

Note that G is the set of left hand endpoints of contiguous intervals contained in $]0, \zeta[$ of the homogeneous set $M = \{s + R_s : s \in \mathbb{R}_+\}$.

We assume that R is $\mathscr{F}^*$-measurable ($\mathscr{F}^*$ denotes the universal completion of $\mathscr{F}^\circ$). An obvious adaptation of the results of [14] yields the existence of a function $l \in (\mathscr{E}_\Delta^*)_+$ ($\mathscr{E}_\Delta^*$ denotes the universal completion of $\mathscr{E}_\Delta$) and an exit system $({}^*P^\cdot, B)$, where ${}^*P^\cdot$ is a kernel from $(E, \mathscr{E}^*)$ to $(\Omega, \mathscr{F}^*)$ and B is an additive functional of (X_s) with bounded 1-potential, such that

(6.6) (i) $\int_0^t 1_{\{R_s = 0\}} ds = \int_{[0,t]} l \circ X_s B(ds)$, $t \geqq 0$;

(ii) $P^\cdot(\sum_{s \in G} U_s f \circ \theta_s) = P^\cdot(\int_{\mathbb{R}_+} U_s {}^*P^{X_s}(f) B(ds))$;

(iii) $l + {}^*P^\cdot(1 - e^{-R}) = 1$ on $F = \operatorname{reg} R$;

(iv) $l = 0$ and ${}^*P^\cdot = P^\cdot / P^\cdot(1 - e^{-R})$ on $E_\Delta \setminus F$.

In (6.6 ii), $U \geqq 0$ is $(\mathscr{F}_s)$ optional and $f \geqq 0$ is $\mathscr{F}^*$-measurable. Recall also that $\operatorname{reg} R = \{x : P^x(R = 0) = 1\}$. Evidently one has $1_{\{\alpha < t, \tilde{R}_t = 0\}} dt = l(Y_t) \tilde{B}(dt)$ and

(6.7) $$Q_\eta\Big(\sum_{t \in \tilde{G}} V_t f \circ \tau_t\Big) = Q_\eta \int_{\mathbb{R}} V_t {}^*P^{Y_t}(f) \tilde{B}(dt),$$

where $\eta \in \mathrm{Exc}$, $V \geqq 0$ is $(\mathscr{G}_t^\eta)$ optional and $f \geqq 0$ is $\mathscr{F}^*$-measurable. For a detailed proof of (6.7) see [2].

Combining (6.7) with (6.5) we have

(6.8) **Theorem.** *Let $\eta \in \mathrm{Exc}$. With the above notation, ν_η^B is σ-finite and*

$$Q_\eta\Big(\sum_{t \in \tilde{G}} f(t, Y_t, \tau_t)\Big) = \int_{\mathbb{R} \times E} dt\, \nu_\eta^B(dx)\, {}^*P^x(f(t, x, \cdot)) \tag{6.9}$$

for $f \geqq 0$ universally measurable over $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{E} \otimes \mathscr{F}^0$. In particular, for $h \in \mathscr{E}_+$

$$L_R \eta(h) = \nu_\eta^B(lh + Wh), \tag{6.10}$$

where $Wh(x) = {}^*P^x\left(\int_0^R h \circ X_s\, ds\right)$.

Proof. Set $H = \int_0^\infty e^{-t} q \circ Y_t\, dt$, where $q \in \mathscr{E}_+$ is such that $0 < q \leqq 1$ and $\eta(q) < \infty$. One has

$$H \geqq \int_0^\infty e^{-t} q \circ Y_t 1_{\{\tilde{R}_t = 0\}}\, dt + \sum_{t \in \tilde{G}, t > 0} e^{-t} \left(\int_0^R e^{-s} q \circ X_s\, ds\right) \circ \tau_t.$$

Therefore, it follows from (6.6 i) and (6.7) that $\eta(q) = \mathbf{Q}_\eta(H) \geqq Q_\eta \int_{\mathbb{R}_+} e^{-t} r \circ Y_t \tilde{B}(dt) \geqq e^{-1} \nu_\eta^B(r)$ where $r = lq + {}^*P^\cdot \int_0^{R \wedge 1} q \circ X_s\, ds$. By (6.6 iii, iv), $r > 0$ and so ν_η^B is σ-finite.

Formula (6.9) follows now from (6.5) by an easy extension of (6.7). To see (6.10), first note that $h \circ Y_t 1_{\{\tilde{R} < t\}} = h \circ Y_t 1_{\{\tilde{R}_t = 0\}} + h \circ Y_t 1_{\{\tilde{R} < t, \tilde{R}_t > 0\}}$ a.e. Q_η, since $Q_\eta(\tilde{R} = t) = 0$. By the (σ_s) invariance of Q_η, $Q_\eta(h \circ Y_t; \tilde{R}_t = 0)$ does not depend on $t \in \mathbb{R}$ and so is equal to

$$\int_0^1 Q_\eta(h \circ Y_t; \tilde{R}_t = 0)\, dt = Q_\eta \int_{]0,1]} h \circ Y_t\, l \circ Y_t \tilde{B}(dt) = \nu_\eta^B(lh). \tag{6.11}$$

On the other hand,

$$h \circ Y_t 1_{\{\tilde{R} < t, \tilde{R}_t > 0\}} = \sum_{s \in \tilde{G}, s \leqq t} (h \circ X_{t-s} 1_{\{R > t-s\}}) \circ \tau_s.$$

By an application of (6.9):

$$\begin{aligned} Q_\eta(h \circ Y_t 1_{\{\tilde{R} < t, \tilde{R}_t > 0\}}) &= Q_\eta\Big(\int_{]-\infty, t[\times E} ds\, \nu_\eta^B(dx)\, {}^*P^x(h \circ X_{t-s}; R > t-s)\Big) \\ &= \int_{\mathbb{R}_+ \times E} du\, \nu_\eta^B(dx)\, {}^*P^x(h \circ X_u; R > u) \\ &= \nu_\eta^B(Wh). \end{aligned} \tag{6.12}$$

Adding (6.11) to (6.12), we obtain (6.10). □

(6.13) *Remark.* Formula (6.10) is the "last exit" decomposition of $L_R \eta$ mentioned in the introduction.

The next result describes the characteristic measure ν_η^B for a *general* additive functional B and for $\eta \in \mathrm{Dis}$, by means of the potential operator U_B of B. Recall that

$$U_B f = P^{\cdot} \int_{\mathbb{R}_+} f \circ X_s \, B(ds), \qquad f \in \mathscr{E}_+ .$$

(6.14) **Theorem.** 1) *If $\eta = \mu U$ is a potential, then $\nu_\eta^B = \mu U_B$.*

2) *Let $\eta = \int_0^\infty \mu_t \, dt \in \mathrm{Pur}$, where (μ_t) is an entrance law for (P_s). Then $\mu_t U_B \uparrow \nu_\eta^B$ as $t \downarrow 0$.*

3) *Let $\eta \in \mathrm{Dis}$ and let $(\mu_n U)$ be an increasing sequence of potentials such that $\mu_n U \uparrow \eta$. Then for any $f \in \mathscr{E}_+$ such that $\nu_\eta^B(f) < \infty$ one has $\mu_n U_B(f) \uparrow \nu_\eta^B(f)$. In particular, $\mu_n U_B \uparrow \nu_\eta^B$ if ν_η^B is σ-finite.*

Proof. Fix $f \in \mathscr{E}_+$ and set $\Phi(u, w) = \int_{\mathbb{R}_+} 1_{]0,1]}(u+s) f \circ Y_s(w) \tilde{B}(w, ds)$. Then if η is purely excessive

$$\nu_\eta^B(f) = Q_\eta(\Phi(\alpha, \tau_\alpha)) = \Pi_\eta \int_{\mathbb{R}_+} f \circ Y_s \tilde{B}(ds), \tag{6.15}$$

by Theorem (2.4). Points 1) and 2) follow easily from (6.15).

For the proof of 3), consider a sequence (S_n) of intrinsic stopping times times such that $S_n > \alpha$, $S_n \downarrow \alpha$ a.e. Q_η. By (5.3 ii),

$$\nu_{\mu^{S_n} U}^B(f) = \nu_{L S_n \eta}^B(f) = Q_\eta \int_{]0,1]} f \circ Y_t 1_{\{S_n < t\}} \tilde{B}(dt).$$

Hence, $\mu^{S_n} U_B(f) \uparrow \nu_\eta^B(f)$. Since by assumption, $\nu_\eta^B(f) < \infty$, each of the measures $\mu^{S_n}, \mu^{S_n} U, \eta, \mu_n U$, and $\mu_n P_t$ is carried by the absorbing set $A = \{U_B f < \infty\}$. If we consider a sequence $(f_k) \subset \mathscr{E}_+$ such that $U f_k \uparrow U_B f$ on A (the existence of (f_k) is proved as in (5.10)), then $\mu_n P_t U f_k \uparrow \mu_n P_t U_B f$ as $k \uparrow \infty$ for all n and $t > 0$. But the family $(\mu_n P_t U f_k)$ is also increasing in n and decreasing in t. Thus $\mu_n U_B f = \lim_{t \downarrow 0} \mu_n P_t U_B f$ increases in n, while $\eta(f_k) = \lim_n \lim_{t \downarrow 0} \mu_n P_t U f_k$ increases in k, with a limit independent of (μ_n). With the choice $\mu_n = \mu^{S_n}$, we see that $\mu_n U_B f \uparrow \nu_\eta^B(f)$ as desired. For the last assertion, note that if ν_η^B is σ-finite, then we can find $f \in \mathscr{E}_+$ such that $0 < f < \infty$ and $\nu_\eta^B(f) < \infty$. By the previous argument, if $g \in b\mathscr{E}_+$ then $\mu_n U_B(gf) \uparrow \nu_\eta^B(gf)$, proving that $\mu_n U_B \uparrow \nu_\eta^B$. □

(6.16) *Remark.* Fix $\eta \in \mathrm{Exc}$. Easy calculations (for which the σ-finiteness of ν_η^B is not needed) show that for $r > 0$ and $f \in \mathscr{E}_+$,

$$r \eta_i U_B^r f = \nu_{\eta_i}^B(f),$$
$$r \eta_p U_B^r f = \Pi_\eta \int_{\mathbb{R}_+} (1 - e^{-ru}) f \circ Y_u \tilde{B}(du).$$

Thus $r \eta U_B^r \uparrow \nu_\eta^B$ as $r \uparrow \infty$ (owing to (6.15)) and we recover the original definition of Revuz.

Postcript. 1) Our Theorem (4.3 i) generalizes part of Theorem (1.5) of Getoor and Glover [8]. On the other hand this latter result, when combined with

Dynkin's decomposition of an excessive measure, yields a direct proof of (4.3 i) without using Q_η. One reason for the use of Q_η is to provide a new and simple proof of Getoor and Glover's result.

2) The following alternative proof of the sufficiency in Theorem (3.3) has been pointed out to us by H. Kaspi. Let $\eta \in \mathrm{Pur}$ be such that Π_η is carried by Ω_q (where q and Ω_q are as in (3.3), and $\eta(q)=1$). It follows from Theorem 0.1 of Dynkin [3] that $\eta = \int_{M_q} \sigma(d\xi)\,\xi$, where σ is a probability measure on the set M_q of *minimal* purely excessive measures ξ satisfying $\xi(q)=1$, and such that Π_ξ is carried by Ω_q. Using Theorem 7.2 of [3], one can show that each $\xi \in M_q$ has the form $Uq(x_\xi)^{-1}\,U(x_\xi, \cdot)$ for some (unique) $x_\xi \in E$. (In fact M_q is precisely $\{Uq(x)^{-1}\,U(x, \cdot)\colon x \in E,\ Uq(x) < \infty\}$.) One checks that $\xi \to x_\xi$ is measurable, thereby obtaining $\eta = \mu U$, where $\mu = \int_{M_q} \sigma(d\xi)\,Uq(x_\xi)^{-1}\varepsilon_{x_\xi}$. Of course this proof is less elementary than our own, but it serves to relate our result to the deep results of Dynkin. Moreover, in working out the details of the above argument, we found that an alternative characterization of Pot could be obtained by replacing Ω_q by the set

$$\Omega'_q = \{\alpha = 0,\ Y_{0+} \text{ exists in } E,\ \phi(Y_{1/n}) \to \phi(Y_{0+}) \text{ as } n \to \infty, \text{ for } \phi = Uq \text{ and } \phi = Ug,\ g \in D \text{ with } g \leqq q\}.$$

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