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Probability Theory and Related Fields

Fermion Martingales

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Summary. We show that strictly quasi-free Fermion martingales may be expressed as a sum of quantum stochastic integrals with respect to the Fermi creation and annihilation processes and a multiple of the identity.

§ 0. Introduction

Fermion stochastic integrals have been constructed and studied in [1, 3]. In particular, the notion of a martingale has been made precise by means of a family of conditional expectation maps for a quasi-free ('Gaussian') state of the C^*-algebra of the canonical anti-commutation relations (CAR) (see (1.1)) which is determined by a multiplication operator on $L^2(\mathbb{R}^+)$, and, for the Fock state, an analogue of Itô's product formula has been proved for such stochastic integrals.

In the present note it is shown, by means of an orthogonal decomposition of the underlying Hilbert space, that for a strictly quasi-free filtration, any Fermion martingale M may be represented uniquely in the form

$$M(t)=M(0)+\int_0^t dA^* F+\int_0^t G\,dA \tag{0.1}$$

where (A, A^*) is the quasi-free creation, annihilation process, F and G are L^2-adapted processes and $M(0)$ is a multiple of the identity.

The methods used here also yield an elementary proof of the corresponding result for Boson martingales [6] (see also [9]). In that case an Itô formula [1, 7, 8] is not needed in view of the existence of 'Weyl martingales' with known stochastic integral representation. A unified treatment is given in [8].

Finally, I should mention the stochastic calculus of the Clifford process – the CAR analogue of the Wiener process [2]. Martingales with respect to the Clifford filtration are expressible as stochastic integrals, giving a third category of non-commutative martingale representation.

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The plan of the paper is as follows. Section 1 introduces the stochastic calculus and some elementary properties of the stochastic integrals and the quasi-free representations are proved there. The purpose of this section is to fix notation and facilitate the later arguments - the material is more or less covered by [3]. In the second section a Fermion Itô formula is established, which is then used to give stochastic integral representation for a large class of martingales. In the final section it is shown that all Fermion martingales have an integral representation (0.1), moreover those having an adjoint process $M^\dagger$ have stochastic derivatives which consist of closeable operators, and

$$M^\dagger(t)=M^\dagger(0)+\int_0^t dA^* G^\dagger+\int_0^t F^\dagger dA \tag{0.2}$$

($X^\dagger$ denoting the restriction of X^* to the appropriate domain - see Sect. 3). Both are proved by decomposing the relevant Hilbert spaces into three orthogonal components - cf. the proof of the classical Kunita-Watanabe theorem in [10].

N. Marshall [12] has also obtained a representation of quasi-free Fermion martingales in terms of integrals with respect to the creation and annihilation processes, in which he gives an explicit form for the 'stochastic derivatives'.

§ 1. Quasi-free Stochastic Calculus: Basics

If $\mathscr{h}$ is a complex Hilbert space, then the CAR algebra over $\mathscr{h}$, $\mathfrak{A}(h)$, is the unital C^*-algebra generated by elements $\{a(f), a^*(f): f\in h\}$ satisfying:

$$\begin{gathered} a^*(f+\lambda g)=a^*(f)+\lambda a^*(g); \quad a^*(f)=a(f)^* \\ a(f)\,a(g)+a(g)\,a(f)=0 \\ a(f)\,a^*(g)+a^*(g)\,a(f)=\langle f,g\rangle I \\ \|a(f)\|=\|a^*(f)\|=\|f\| \quad \forall f,g\in\mathscr{h},\ \lambda\in\mathbb{C}. \end{gathered} \tag{1.1}$$

The Fock representation of the CAR algebra is defined by

$$\Gamma_0(\mathscr{h})=\bigoplus_{n\in\mathbb{N}} \Lambda^n\mathscr{h}$$
$$a^*(f)\,g^1\wedge\ldots\wedge g^n=f\wedge g^1\wedge\ldots\wedge g^n, \qquad n\in\mathbb{N}, f, g^i\in\mathscr{h}\,^1$$

where $\Lambda^0\mathscr{h}=\mathbb{C}$ and, for $n\geq 1$ Λ^n denotes the n-fold anti-symmetric tensor product; it is irreducible and cyclic with cyclic vector $\Omega_0=(1,0,0,\ldots)$ satisfying

$$a(f)\,\Omega_0=0 \qquad \forall f\in h. \tag{1.2}$$

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[1] For $u\in\Lambda^j\mathscr{h}$, $v\in\Lambda^k\mathscr{h}$

$$u\wedge v:=\sqrt{\frac{(j+k)!}{j!\,k!}}\,A_{j+k}(u\otimes v)$$

where A_{j+k} is the orthogonal projection of $\overset{(j+k)}{\bigotimes}\mathscr{h}$ onto $\Lambda^{(j+k)}\mathscr{h}$ - *not* the usual normalisation for wedge products

Quasi-free representations may then be constructed as follows

$$\Gamma_R(\mathscr{h})=\Gamma_0(\mathscr{h})\otimes\Gamma_0(\mathscr{h})$$
$$a_R^*(f)=a_0^*(\sqrt{I-R}f)\otimes I+\theta\otimes a_0(J\sqrt{R}f)$$

where $R\in\mathfrak{B}(\mathscr{h})$ satisfies $0\leqq R\leqq I$, J is an involution on $\mathscr{h}$ and θ is the parity operator whose restriction to $\Lambda^n\mathscr{h}$ is $(-1)^n I$. The vector $\Omega=\Omega_0\otimes\Omega_0$ then determines the guage invariant quasi-free state ω_R of the CAR algebra for which

$$\omega_R(a^*(f)\,a(g))=\langle g, Rf\rangle. \tag{1.3}$$

That is to say the higher order 'cumulants' vanish and the state on $\mathfrak{A}(\mathscr{h})$ is thereby determined by its action on the products $\{a^*(f)\,a(g): f, g\in\mathscr{h}\}$. (1.3) gives the 'covariance' of the state – quasi-free states being the analogues of Gaussian distributions. For further details of the CAR algebra, its representations and (quasi-free) states we refer to [4].

We are interested in the case where $\mathscr{h}=L^2(\mathbb{R}^+)$, R is a multiplication operator M_ρ where ρ is bounded away from 0 and 1 locally (i.e. on each finite interval) and J is ordinary complex conjugation. The vector Ω is then cyclic and separating for $\mathscr{N}:=\{a_R(f),\ a_R^*(f)\colon f\in\mathscr{h}\}''$ – the von Neumann algebra generated by this representation – which allows us to establish results about operator-valued processes while working at the Hilbert space level. (All our quantum stochastic processes will be operator-valued.) From now on we shall drop the subscript R, denoting $\Gamma_R(L^2(\mathbb{R}^+))$, $a_R(\cdot)$ and $a_R^*(\cdot)$ respectively Γ, $a(\cdot)$ and $a^*(\cdot)$. Let σ denote the *-automorphism of $\mathscr{N}$ given by

$$\sigma\colon a^*(f)\to a^*(-f) \tag{1.4}$$

– this is implemented by the unitary operator $\theta\otimes\theta$.

Now let $\beta=\rho^{\frac{1}{2}}$, $\alpha=(1-\rho)^{\frac{1}{2}}$, $\mathscr{N}_t=\{a(f), a^*(f)\colon \operatorname{supp} f\subset[0,t]\}''$ and $\Gamma_t=\overline{\mathscr{N}_t\Omega}$ for each $t>0$. The following straight forward result will be repeatedly used.

Lemma 1.1. *Let* $t>0$, $\psi\in\Gamma_t$ *and* $f\in\mathscr{h}$ *with* $\operatorname{supp} f\subset[t,\infty)$. *Then*

$$\langle\psi, a(f)\,\Omega\rangle=\langle\psi, a^*(f)\,\Omega\rangle=0, \tag{1.5}$$

$$\begin{aligned}\langle\psi, a(f)\,a^*(f)\,\Omega\rangle&=\langle\psi,\Omega\rangle\int_t^\infty|\alpha f|^2,\\ \langle\psi, a^*(f)\,a(f)\,\Omega\rangle&=\langle\psi,\Omega\rangle\int_t^\infty|\beta f|^2.\end{aligned} \tag{1.6}$$

Proof. Applying the CAR's (1.1) and (1.2) we have

$$\forall f\in\mathscr{h},\quad a(f)\,\Omega=\Omega_0\otimes a_0^*(\beta\bar f)\,\Omega_0=[I\otimes a_0^*(\beta\bar f)]\,\Omega$$
$$\begin{aligned}a(f)\,a^*(f)\,\Omega&=[a_0(\alpha f)\otimes I+\theta\otimes a_0^*(\beta\bar f)](a_0^*(\alpha f)\,\Omega_0\otimes\Omega_0)\\&=[\|\alpha f\|^2 I-a_0^*(\alpha f)\otimes a_0^*(\beta\bar f)]\,\Omega\end{aligned}$$

so, for $\Psi=\prod_{i=1}^m a_0^*(f^i)\otimes\prod_{j=1}^n a_0^*(g^j)$ with $\operatorname{supp} f^i$, $g^j\subset[0,t]$ and h, k with support in $[t,\infty)$,

$$[I\otimes a_0(k)]\,\Psi\Omega=(-1)^n\,\Psi(\Omega_0\otimes a_0(k)\,\Omega_0)=0,$$
$$[a_0(h)\otimes I]\,\Psi\Omega=(-1)^m\,\Psi(a_0(h)\,\Omega_0\otimes\Omega_0)=0,$$
$$[a_0(h)\otimes a_0(k)]\,\Psi\Omega=(-1)^{n+m}\,\Psi(a_0(h)\,\Omega_0\otimes a_0(k)\,\Omega_0)=0.$$

Hence,

$$\langle\Psi\Omega, a(f)\,\Omega\rangle=\langle[I\otimes a_0(\beta\bar{f})]\,\Psi\Omega,\Omega\rangle=0,$$
$$\langle\Psi\Omega, a(f)\,a^*(f)\,\Omega\rangle=\left\langle\Psi\Omega,\int_t^\infty|\alpha f|^2\,\Omega\right\rangle$$

and the lemma follows by the totality of the vectors $\Psi\Omega$ in Γ_t and the CAR (1.1). $\square$

Let $\mathcal{N}_\eta=\{R\eta\mathcal{N}:\Omega\in\mathcal{D}(R)\}$ and $\mathcal{N}_\eta(t)=\{R\eta\mathcal{N}_t:\mathcal{D}(R)=\mathcal{N}_t'\Omega\}$ where η denotes affiliation (for not necessarily closed operators) and $\mathcal{D}(\cdot)$ denotes the domain of an operator. We extend σ to $\mathcal{N}_\eta(t)$ as follows: for $R\in\mathcal{N}_\eta(t)$, $\sigma(R)$ is the element of $\mathcal{N}_\eta(t)$ satisfying

$$\sigma(R)\,\Omega=\theta\otimes\theta R\Omega$$

– since $\theta\otimes\theta$ leaves Γ_t invariant, this is well defined. We define the following classes of processes:

$$\mathcal{A}=\{X:\mathbb{R}^+\to\mathcal{N}_\eta;\ X(t)\in\mathcal{N}_\eta(t)\ \forall\, t\},$$
$$\mathcal{S}=\{X\in\mathcal{A};\ \exists\{t_n\}\text{ with }0\leqq t_1<t_2<\ldots<t_n\to\infty\text{ and }$$
$$X(t)=X_n|_{\mathcal{N}_t'\Omega},\ X_n\in\mathcal{N}_{t_n}\text{ for }t\in[t_n,t_{n+1})\}$$
$$\mathcal{L}^2=\left\{X\in\mathcal{S};\ t\to X(t)\,\Omega\text{ is Lebesgue measurable and }\int_0^t\|X(s)\,\Omega\|^2\,ds<\infty\ \forall\, t>0\right\}$$

and L^2 the set $\mathcal{L}^2$ with processes, that agree on Ω almost every-where, identified. We call the elements of $\mathcal{A}$ adapted and elements of $\mathcal{S}$ simple. There are the following one to one correspondences:

$\mathcal{A}\leftrightarrow\{x:\mathbb{R}^+\to\Gamma;\ x(t)\in\Gamma_t\ \forall\, t\}$, denoted a.
$L^2\leftrightarrow\{x\in L^2_{\mathrm{loc}}(\mathbb{R}^+;\Gamma);\ x(t)\in\Gamma_t$ for almost all $t\}$, denoted l^2

given by

$$x(t)=X(t)\,\Omega;\qquad\text{for }\ R\in\mathcal{N}_t',\ X(t)\,R\Omega=R\,x(t).\tag{1.7}$$

The prototype processes are

$$A_f^*(t):=a^*(f\chi_{[0,t]})|_{\mathcal{N}_t'\Omega};\qquad A_f(t):=a(\bar{f}\chi_{[0,t]})|_{\mathcal{N}_t'\Omega},\qquad f\in L^2_{\mathrm{loc}}(\mathbb{R}^+)\tag{1.8}$$

and when f is the constant function equal to one, are called the creation and annihilation processes respectively and written A^*, A. Since these processes take bounded values, we use the same notation for the unrestricted operators.

For each $t\geqq 0$, the time-t conditional expectation $\mathbb{E}_t$ is defined by

$$\mathbb{E}_t:\mathcal{N}_\eta\to\mathcal{N}_\eta(t)\qquad\mathbb{E}_t[X]\,R\Omega=RP_tX\Omega\qquad\text{for }\ R\in\mathcal{N}_t'$$

where P_t is the orthogonal projection onto the subspace Γ_t – this extends the conditional expectations given in [3, 5] to deal with unbounded operators. An adapted process X is then called a martingale if it satisfies the martingale identity

$$\mathbb{E}_s[X(t)] = X(s) \qquad \forall\, s < t$$

and we denote the set of martingales by $\mathscr{M}$.

We next define stochastic integrals of simple processes. For $F \in \mathscr{S} = \sum_i F_i \chi_{[t_i, t_{i+1})}$ (slightly abusing notation),

$$\int_0^t dA^* F := \sum_{i < i(t)} a^*(\chi_{[t_i, t_{i+1})}) F_i + a^*(\chi_{[t_{i(t)}, t)}) F_{i(t)} \tag{1.9}$$

where $i(t) = \max_{t_i < t} i$.

By Lemma 1.1 we have

$$\langle a^*(\chi_{[t_i, t_{i+1})}) F_i \Omega, a^*(\chi_{[t_j, t_{j+1})}) F_j \Omega \rangle = \begin{cases} 0 & i \neq j \\ \int_{t_i}^{t_{i+1}} \|F_i \Omega\|^2 \{\alpha(s)\}^2 \, ds & i = j \end{cases}$$

from which we obtain the isometric relations

$$\left\| \int_0^t dA^* F \Omega \right\|^2 = \int_0^t \|F(s)\Omega\|^2 \{\alpha(s)\}^2 \, ds \tag{1.10}$$

$\int F \, dA$, $\int F \, dA^*$ and $\int dA F$ are defined in the same way and satisfy

$$\int_0^t dA F = \int_0^t \sigma(F) \, dA; \qquad \int_0^t F \, dA^* = \int_0^t dA^* \sigma(F), \tag{1.11}$$

$$\left\| \int_0^t F \, dA \Omega \right\|^2 = \int_0^t \|F(s)\Omega\|^2 \{\beta(s)\}^2 \, ds. \tag{1.12}$$

We next show that L^2-processes may be approximated by simple processes permitting an extension of the stochastic integral (1.9) to L^2-processes.

Lemma 1.2. *Let $X \in L^2$. Then there is a sequence $X^{(n)}$, $n = 1, 2, \ldots$ of simple processes such that, for all $t > 0$*

$$\int_0^t \|(X(s) - X^{(n)}(s))\Omega\|^2 \, ds \to 0 \quad \text{as} \quad n \to \infty \tag{1.13}$$

and we shall write $X^{(n)} \to X$ when (1.13) *is satisfied.*

Proof. Let x be the element of l^2 corresponding to X (1.7) and, for each $n \in \mathbb{N}^+$, $u^{(n)} = x \chi_{[0, 2^n)}$. On $L^2([0, 2^n]; \Gamma)$ let S_n^k be the shift

$$(S_n^k f)(x) = \begin{cases} 0 & x < 2^{-k} \\ f(x - 2^{-k}) & x \geqq 2^{-k} \end{cases}$$

and Q_n^k the orthogonal projection onto the subspace $L^2([0, 2^n], \mathscr{F}_n^k; \Gamma)$ where $\mathscr{F}_n^k$ is the σ-algebra generated by the sub-intervals $\{[i2^{-k}, (i+1)2^{-k})$:

$0 \leqq i \leqq 2^{n+k}-1\}$. Now S_n^k and Q_n^k are contractions which converge strongly to the identity as $k \to \infty$, so that, considering $u^{(n)}$ as an element of $L^2([0, 2^n]; \Gamma)$, there is a number $k(n)$ such that

$$\|(I - Q_n^{k(n)} S_n^{k(n)}) u^{(n)}\|_{L^2} \leqq 1/2n.$$

Writing $v^{(n)}$ for $Q_n^{k(n)} S_n^{k(n)} u^{(n)}$, letting $w^{(n)} \in L^2([0, 2^n], \mathscr{F}_n^{k(n)}; \Gamma)$ be $\mathcal{N}\Omega$-valued and satisfy

$$\|v^{(n)} - w^{(n)}\|_{L^2} \leqq 1/2n$$

and $x^{(n)}$ be $w^{(n)}$ considered as an element of l^2, we have:

$$\left(\int_0^t \|(X(s) - X^{(n)}(s))\,\Omega\|^2\, ds\right)^{\frac{1}{2}} \leqq \|u^{(n)} - w^{(n)}\|_{L^2} \quad \text{as soon as } 2^n > t$$
$$\leqq 1/n$$

and $X^{(n)} \in \mathscr{S}$, completing the proof. □

Corollary 1.3. *Let $X \in L^2$, then there are adapted processes Y, Z, U and V such that if $X^{(n)} \to X$, then*

$$\int_0^t dA^* X^{(n)} \Omega \to Y(t)\,\Omega; \qquad \int_0^t X^{(n)}\, dA\Omega \to Z(t)\,\Omega,$$

$$\int_0^t X^{(n)}\, dA^* \Omega \to U(t)\,\Omega; \qquad \int_0^t dA X^{(n)}\, \Omega \to V(t)\,\Omega \qquad \forall\, t>0$$

moreover

$$\|Y(t)\,\Omega\|^2 = \int_0^t \|X(s)\,\Omega\|^2 \{\alpha(s)\}^2\, ds; \qquad \|Z(t)\,\Omega\|^2 = \int_0^t \|X(s)\,\Omega\|^2 \{\beta(s)\}^2\, ds. \tag{1.14}$$

Proof. This follows from (1.11), (1.12) and (1.13). □

Definition. For $X \in L^2$, $\int_0^{\cdot} dA^* X$, $\int_0^{\cdot} X\, dA$, $\int_0^{\cdot} X\, dA^*$ and $\int_0^{\cdot} dA X$ are the adapted processes Y, Z, U and V respectively of Corollary 1.3, and $\int_0^{\cdot} X\, ds$ is the adapted process corresponding to the collection of strong-sense integrals $\left\{\int_0^t X(s)\,\Omega ds \colon t>0\right\}$. We write $\int_r^t dA^* X$ for $\int_0^t dA^* X \chi_{[r, \infty)}$, etc.

Note. The integrals are linear and satisfy (1.11) for $F \in L^2$ in view of the identity $\|(\sigma(F^{(n)}) - \sigma(F))\,\Omega\| = \|(F^{(n)} - F)\,\Omega\|$, and (1.10), (1.12) in view of (1.14).

Lemma 1.4. *For $X \in L^2$, $\int_0^{\cdot} dA^* X$, $\int_0^{\cdot} X dA \in \mathscr{M}$ whereas $\int_0^{\cdot} X\, ds \notin \mathscr{M}$ unless $X = 0$.*

Proof. For $X = F\chi_{[u,v)}$, $F \in \mathcal{N}_u$, $s<t$ and $\phi \in \Gamma$,

$$\left\langle \phi, \mathbb{E}_s \int_s^t X\, dA\Omega \right\rangle = \langle F^* P_s \phi, a(\chi_{[u \vee s, v \wedge t)})\,\Omega\rangle$$
$$= 0 \quad \text{by (1.5)}$$

then, by linearity and taking strong limits on Ω, this holds for $X \in L^2$, i.e.

$$\mathbb{E}_s\left[\int_0^t X\,dA\right] = \int_0^s X\,dA, \quad \text{or} \quad \int_0^{\cdot} X\,dA \in \mathscr{M}$$

a similar argument shows that $\int_0^{\cdot} dA^* X \in \mathscr{M}$ also.

On the other hand,

$$\begin{aligned}
\int_0^{\cdot} X(s)\,ds \in \mathscr{M} &\Rightarrow \mathbb{E}_s\left[\int_0^t X(r)\,dr\right] = 0 \qquad \forall\, s<t \\
&\Rightarrow \left\langle \psi, \int_s^t X(r)\,dr\,\Omega \right\rangle = 0 \qquad \forall\, \psi \in \Gamma_s,\ s<t \\
&\Rightarrow \int_s^t \langle \psi, X(r)\,\Omega\rangle\,dr = 0 \qquad \forall\, \psi \in \Gamma_s,\ s<t \\
&\Rightarrow \langle \psi, X(t)\,\Omega\rangle = 0 \qquad \forall\, \psi \in \Gamma_s,\ s<t \\
&\Rightarrow X(t)\,\Omega \in \bigcap_{s<t} \Gamma_t \ominus \Gamma_s = \{0\} \\
&\Rightarrow X = 0. \quad \square
\end{aligned}$$

Lastly, a point of notation. A sentence or equation which contains terms of the form $A^\sharp$, $A^\flat$ should be read as two sentences or equations – one in which all the 's are deleted and each $\flat$ is replace by $*$, and one in which all the $\flat$'s are deleted and each $\sharp$ is replaced by $*$.

§ 2. Fermion Itô Formulae

In this section we establish a Fermion Itô formula [cf. 1]. Wick ordered products of processes of the form (1.8) which are normally ordered with respect to the state ω_R are martingales [5] and we show that these have a stochastic integral representation.

Lemma 2.1. *Let $r<t$ and write $A_r^\sharp$ for $A_f^\sharp$ where $f = \chi_{[r,\infty)}$, then*

$$A_r^*(t)\,A_r(t) = \int_r^t dA^* A_r(s) + \int_r^t A_r^*(s)\,dA + \int_r^t \{\beta(s)\}^2\,ds\,I, \tag{2.1a}$$

$$A_r(t)\,A_r^*(t) = \int_r^t dA\,A_r^*(s) + \int_r^t A_r(s)\,dA^* + \int_r^t \{\alpha(s)\}^2\,ds\,I \tag{2.1b}$$

$$(A_r^\sharp(t)^2 =) \int_r^t dA^\sharp A_r^\sharp(s) + \int_r^t A_r^\sharp(s)\,dA^\sharp = 0, \tag{2.1c}$$

$$A_r^\sharp(t)(t-r) = \int_r^t dA^\sharp (s-r) + \int_r^t A_r^\sharp(s)\,ds. \tag{2.1d}$$

Proof. For each $n \in \mathbb{N}^+$, let $A^{(n)\#}$ be the simple process defined by

$$A^{(n)\#}(s) = A_r^\#(s_n) \quad \text{where } s_n = \frac{t-r}{2^n}\left[\frac{s-r}{t-r} 2^n\right] + r$$

$[\cdot]$ denoting the integral part, then $A_r^\#(s) - A^{(n)\#}(s) = a^\#(\chi_{[s_n, s)})$, so that

$$\int_r^t \|A_r^\#(s) - A^{(n)\#}(s)\|^2 \, ds < 2^{-n}(t-r)^2 \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$

Writing $a_i^\#$ for $a^\#(\chi_{[s_i, s_{i+1})})$, $i = 0, 1, \ldots, 2^n - 1$ we have

$$\int_r^t dA^* A^{(n)}(s) = \sum_{i=1}^{2^n-1} a_i^* a(\chi_{[s_0, s_i)}) = \sum_{0 \leq j < i \leq 2^n - 1} a_i^* a_j$$

and $\int_r^t A^{(n)*}(s) \, dA = \sum_{0 \leq i < j \leq 2^n - 1} a_i^* a_j$, so that

$$\int_r^t dA^* A^{(n)}(s) + \int_r^t A^{(n)*}(s) \, dA = \sum_{i \neq j} a_i^* a_j = A_r^*(t) A_r(t) - \sum_i a_i^* a_i. \tag{2.3}$$

But

$$a^*(f) a(g) \Omega = [a_0^*(\alpha f) \otimes I + \theta \otimes a_0(\beta \bar{f})](\Omega_0 \otimes a_0^*(\beta \bar{g}) \Omega_0) = \int \beta^2 f \bar{g} \Omega + \alpha f \otimes \beta \bar{g}$$

and

$$\left\| \sum_{i=0}^{2^n-1} \alpha \chi_{[s_i, s_{i+1})} \otimes \beta \chi_{[s_i, s_{i+1})} \right\|^2 = \sum_{i=0}^{2^n-1} \|\alpha \chi_{[s_i, s_{i+1})}\|^2 \, \|\beta \chi_{[s_i, s_{i+1})}\|^2$$
$$\leq 2^{-n}(t-r)^2 \to 0 \quad \text{as } n \to \infty$$

so from (2.2) and (2.3)

$$\left(\int_r^t dA^* A_r(s) + \int_r^t A_r^*(s) \, dA\right) = \left(A_r^*(t) A_r(t) - \int_r^t \{\beta(s)\}^2 \, ds I\right) \Omega$$

and (2.1a) follows.

(2.1b, c) may be deduced from the identities

$$A_r^*(t) A_r(t) + A_r(t) A_r^*(t) = (t-r); \qquad \int_r^t dA^{\#1} A_r^{\#2}(s) = -\int_r^t A_r^{\#2}(s) \, dA^{\#1}$$

– the latter following from (1.11). (2.1d) is proved in a similar way to (2.1a). □

Next we prove the Fermion Itô formula for simple integrands.

Proposition 2.2. *Let $F, G \in \mathscr{S}$ and $t > 0$, then the following product relations hold:*

$\underline{M}(t)$	$\underline{N}(t)$	$\underline{(MN)(t)}$	
$\int_0^t dA^* F$	$\int_0^t G\,dA$	$\int_0^t dA^* FN + \int_0^t MG\,dA + \int_0^t \sigma(FG)\,\beta^2\,ds,$	(2.4a)
$\int_0^t F\,dA$	$\int_0^t dA^* G$	$\int_0^t dA^*\,\sigma(M)\,G + \int_0^t F\,\sigma(N)\,dA + \int_0^t \alpha^2 FG\,ds,$	(2.4b)
$\int_0^t F\,dA^\#$	$\int_0^t G\,dA^\#$	$\int_0^t \{MG + F\,\sigma(N)\}\,dA^\#,$	(2.4c)
$\int_0^t dA^\# F$	$\int_0^t G\,ds$	$\int_0^t dA^\# FN + \int_0^t MG\,ds.$	(2.4d)

Proof. Without loss of generality we assume common intervals of constancy for F and G, and let t lie in the interval $[r, u)$, then in (2.4c),

$$M(t)\,N(t) = [M(r) + F(r)\,A_r(t)]\,[N(r) + G(r)\,A_r(t)].$$

Thus

$$M(t)\,N(t) - M(r)\,N(r) = M(r)\,G(r)\,A_r(t) + F(r)\,A_r(t)\,N(r)$$

and since

$$\begin{aligned}\int_r^t MG\,dA &= \int_r^t [M(s) - M(r)]\,G(r)\,dA + M(r)\,G(r)\,A_r(t)\\ &= \int_r^t F(r)\,\sigma(G(r))\,A_r(s)\,dA + M(r)\,G(r)\,A_r(t)\end{aligned}$$

while

$$\begin{aligned}\int_r^t dA\,\sigma(F)\,N &= \int_r^t dA\,\sigma(F(r))[N(s) - N(r)] + A_r(t)\,\sigma(F(r))\,N(r)\\ &= \int_r^t dA\,\sigma(F(r))\,G(r)\,A_r(s) + F(r)\,A_r(t)\,N(r)\\ &= F(r)\,\sigma(G(r))\int_r^t dA\,A_r(s) + F(r)\,A_r(t)\,N(r).\end{aligned}$$

(2.4c) follows from (2.1c) by summing over the intervals up to r. In (2.4a) we have

$$\begin{aligned}&M(t)\,N(t) - M(r)\,N(r)\\ &\quad = M(r)\,G(r)\,A_r(t) + A_r^*(t)\,F(r)\,N(r) + A_r^*(t)\,F(r)\,G(r)\,A_r(t)\end{aligned}$$

now

$$\begin{aligned}\int_r^t MG\,dA &= \int_r^t [M(s) - M(r)]\,G(r)\,dA + M(r)\,G(r)\,A_r(t)\\ &= \int_r^t A_r^*(s)\,F(r)\,G(r)\,dA + M(r)\,G(r)\,A_r(t)\\ &= \sigma(F(r)\,G(r))\int_r^t A_r^*(s)\,dA + M(r)\,G(r)\,A_r(t)\end{aligned}$$

while

$$\begin{aligned}\int_r^t dA^* FN &= \int_r^t dA^* F(r)[N(s)-N(r)] + A_r^*(t)\, F(r)\, N(r)\\ &= \int_r^t dA^* F(r)\, G(r)\, A_r(s) + A_r^*(t)\, F(r)\, N(r)\\ &= \sigma(F(r)\, G(r)) \int_r^t dA^* A_r(s) + A_r^*(t)\, F(r)\, N(r)\end{aligned}$$

and (2.4a) follows from (2.1a) by summation. (2.4d) is deduced from (2.1d) in a similar way and (2.4b) from (2.1b). □

Proposition 2.3. *Let $m, n \in \mathbb{N}$, $f^i, g^j \in L^2(\mathbb{R}^+)$ $(i=1, \ldots, m, j=1, \ldots, n)$ be simple (i.e. sums of elementary functions: $\lambda\chi_I$, $\lambda \in \mathbb{C}$, I sub-interval of $\mathbb{R}^+$) and let A_i^*, A_j denote the processes $A_{f^i}^*$, A_{g^j} respectively, then*

$$M(t) := \prod_{i=1}^{m} A_i^*(t) = \int_0^t dA^* F; \qquad N(t) := \prod_{j=n}^{1} A_j(t) = \int_0^t G\, dA \tag{2.5a, b}$$

and

$$M(t)\, N(t) = \int_0^t dA^* FN + \int_0^t MG\, dA + \int_0^t \sigma(FG)\, \beta^2\, ds \tag{2.5c}$$

where

$$F = \sum_{k=1}^{m} (-1)^{k+1} f^k \prod_{i \neq k} A_i^*; \qquad G = \sum_{l=1}^{n} (-1)^{n-l} g^l \prod_{j \neq l} A_j$$

and the ordering is respected, i.e. with subscript increasing in the former and decreasing in the latter case.

Proof. For $f = \lambda\chi_I$, $I = [u, v) \subset [0, t)$

$$\int_0^t dA^* f = \lambda a^*(\chi_I) = a^*(f) = a^*(f\chi_{[0,t)}) = A_f^*(t)$$

hence, by linearity (2.5a) is true for $m=1$. Suppose now that it is true for $m = 1, \ldots, p$ and $F = \sum_{k=1}^{p} (-1)^{k+1} f^k \prod_{i \neq k} A_i^*$. Then, letting $N^{(N)} = \int dA^* F^{(N)}$ where $F^{(N)} \in \mathscr{S}$, $N = 1, 2, \ldots$ and $F^{(N)} \to F$,

$$\begin{aligned}\prod_{i=1}^{p+1} A_i^*(t)\, \Omega &= A_1^*(t) \int_0^t dA^* F\Omega\\ &= \lim_{N\to\infty} \int_0^t dA^* f^1 \int_0^t dA^* F^{(N)}\, \Omega\\ &= \lim_{N\to\infty} \int_0^t dA^* \{f^1 N^{(N)} - A_1^* F^{(N)}\}\, \Omega\\ &= \int_0^t dA^* \left\{ f^1 \prod_{i=2}^{p+1} A_i^* - A_1^* \sum_{k=2}^{p+1} (-1)^{k+1} f^k \prod_{i \neq k} A_i^* \right\} \Omega\\ &= \int_0^t dA^* \sum_{k=1}^{p+1} (-1)^{k+1} f^k \prod_{i \neq k} A_i^*\, \Omega\end{aligned}$$

so that (2.5a) follows by induction. (2.5b) is proved similarly, or may be established by taking adjoints and ensuring that $F^{(N)*} \to F^*$ in the approximating sequence. (2.5c) holds for $n=0$ or $m=0$, so assume as induction hypothesis that it holds for all n and for $m=0, 1, \ldots, p$. Writing

$$\prod_{i=1}^{p} A_i^*(t) \prod_{j=n}^{1} A_j(t) = \int_0^t dA^* F + \int_0^t G\,dA + \int_0^t H\,ds$$

and letting $N^{(N)} = \int dA^* F^{(N)} + \int G^{(N)}\,dA + \int H^{(N)}\,ds$ where $F^{(N)}$, $G^{(N)}$, $H^{(N)} \to F, G, H$ we have

$$\begin{aligned}
\prod_{i=1}^{p+1} A_i^*(t) \prod_{j=n}^{1} A_j(t) &= \lim_{N\to\infty} \int_0^t dA^* f^1 N^{(N)}(t)\,\Omega \\
&= \lim_{N\to\infty} \left\{ \int_0^t dA^*(f^1 N^{(N)} - A_1^* F^{(N)}) + \int_0^t A_1^* G^{(N)}\,dA + \int_0^t (A_1^* H \right. \\
&\quad \left. + f^1 \sigma(G^{(N)})\,\beta^2)\,ds \right\} \Omega \\
&= \left\{ \int_0^t dA^* \sum_{k=1}^{p+1} (-1)^{k+1} f^k \prod_{i\neq k} A_i^* \prod_{j=n}^{1} A_j + \int_0^t \sum_{l=1}^{n} (-1)^{n-l} g^l \prod_{1}^{p+1} A_i^* \prod_{j\neq l} A_j\,dA \right. \\
&\quad \left. + \int_0^t \beta^2 \sum_{k=1}^{p+1} \sum_{l=1}^{n} (-1)^{k+p+l} f^k g^l \prod_{i\neq k} A_i^* \prod_{j\neq l} A_j\,ds \right\} \Omega
\end{aligned}$$

and (2.5c) follows by induction on p. □

If $P = {:}\prod_{i=1}^{m} a^*(f^i) \prod_{j=n}^{1} a(g^j){:}$ denotes the normal ordering of $\prod_{i=1}^{m} a^*(f^i) \prod_{j=n}^{1} a(g^j)$ with respect to the state ω_R, as defined in [5], then the time-t conditional expectation of P is: ${:}\prod_{i=1}^{m} A_i^*(t) \prod_{j=n}^{1} A_j(t){:}$ hence processes of the form

$$M: t \mapsto {:}\prod_{i=1}^{m} A_i^*(t) \prod_{j=n}^{1} A_j(t){:} \tag{2.6}$$

are martingales and we have

Corollary 2.4. *Let f^i, g^j be as in the previous proposition and M defined by* (2.6). *Then there are L^2-processes X and Y such that*

$$M(t) = \int_0^t dA^* X + \int_0^t Y\,dA \qquad \forall\, t > 0.$$

Proof. $M(t)$ is a polynomial in the A_i^*'s and A_j's. Wick ordering each product, applying the proposition and using the linearity of the integrals, the result follows from Lemma 1.4. □

§ 3. Martingale Representation

We are now in a position to establish the key lemma for the proof of the main result.

Lemma 3.1. *Let $t>0$. Then*

$$\Gamma_t = \mathbb{C}\Omega \oplus \left\{ \int_0^t dA^* F\Omega \colon F\in L^2 \right\} \oplus \left\{ \int_0^t G\,dA\,\Omega \colon G\in L^2 \right\}.$$

Proof. The mutual orthogonality of the vectors Ω, $\int_0^t dA^*F\Omega$, $\int_0^t G\,dA\,\Omega$ follows from (1.5) and the CAR's (1.2) when $F, G\in\mathscr{S}$ and for $F, G\in L^2$, it follows by taking limits. The density of the direct sum is a consequence of Corollary 2.4 and the totality in Γ_t of the set of vectors of the form $:\prod A_i^*(t)\prod A_j(t):\Omega$. It remains to show that these subspaces are closed. Let $\left\{ \int_0^t F^{(n)}\,dA\,\Omega \colon n=1,2,\dots \right\}$ be a Cauchy sequence with $F^{(n)}\in L^2$. Define $f\in l^2$ by $f(s)=\lim f^{n_i}(s)$ where $\{f^{n_i}\}$ is a point-wise (almost everywhere) convergent subsequence of $\{\chi_{[0,t]}F^{(n)}\Omega\}$, and F the corresponding L^2-process, then

$$\begin{aligned}\left\| \left(\int_0^t F\,dA - \int_0^t F^{(n)}\,dA \right)\Omega \right\|^2 &= \int_0^t \|(F-F^{(n)})\,\Omega\|^2\,\beta^2\,ds \\ &< \|f-f^{(n)}\|^2 \to 0 \quad \text{as} \quad n\to\infty\end{aligned}$$

so that $\left\{ \int_0^t F\,dA\,\Omega \colon F\in L^2 \right\}$ is closed. Similarly $\left\{ \int_0^t dA^*F\Omega \colon F\in L^2 \right\}$ is closed and the result follows. □

Theorem 3.2. *$M\in\mathscr{M} \Rightarrow \exists!\ \lambda\in\mathbb{C}$, $F, G\in L^2$ such that for all $t>0$*

$$M(t)=\lambda I + \int_0^t dA^* F + \int_0^t G\,dA.$$

Proof. Let $t>0$, then $M(t)\,\Omega\in\Gamma_t$ so by Lemma 3.1

$$M(t)=\lambda\Omega + \int_0^t dA^* F\Omega + \int_0^t G\,dA\,\Omega$$

for some $F, G\in L^2$ and $\lambda\in\mathbb{C}$. Now $\lambda I = M(0)$ and, by the orthogonality of the vectors and the isometric relations (1.14), F and G are uniquely determined on $[0, t]$, hence the vectors $\{M(t)\,\Omega \colon t>0\}$ together determine F and G uniquely on $\mathbb{R}^+$. □

We next consider the representation of martingales which consist of closeable operators. Sufficient conditions for this are $\Omega\in\mathscr{D}(M(t)^*)$ for each t [3]. The question is, do the stochastic derivatives then consist of closeable operators also? A positive answer to this is the content of the next theorem. First we introduce some notation. Let Σ denote the domain of the Tomita-Takesaki operator [11] $S=\bar{S}_0$, where $S_0\colon \Omega\to N$, $T\Omega\to T^*\Omega$, considered as a Hilbert space with the graph norm:

$$x\mapsto(\|x\|^2+\|Sx\|^2)^{\frac{1}{2}}.$$

We denote the set of processes $X\in\mathscr{A}$ for which $X(t)\Omega\in\mathscr{D}(S)$ for each t, (equivalently, $\Omega\in\mathscr{D}(X(t)^*)\,\forall\, t,$) by $\mathscr{A}_\Sigma$, and let $X^\dagger$ be the process defined by $X^\dagger(t)\Omega=SX(t)\Omega$ (i.e. $X^\dagger(t)=X^*(t)|_{\mathscr{N}_t'\Omega}$) and write $\mathscr{M}_\Sigma$, L^2_Σ for the sets $\{X\in\mathscr{M}\colon X^\dagger\in\mathscr{M}\}$, $\{X\in L^2\colon X^\dagger\in L^2\}$ respectively. We then have the following extension of Lemma 1.2.

Lemma 3.3. *Let $X\in L^2_\Sigma$. Then there is a sequence $X^{(n)}\in\mathscr{S}$ $n=1,2,\ldots$ such that*

$$X^{(n)}\to X \quad \textit{and} \quad X^{(n)\dagger}\to X^\dagger. \tag{3.1}$$

Moreover,

$$\int_0^{\cdot} dA^\sharp X\in\mathscr{A}_\Sigma \quad \text{and} \quad \left(\int_0^{\cdot} dA^\sharp X\right)^\dagger=\int_0^{\cdot} X^\dagger dA^\flat. \tag{3.2}$$

Proof. (3.1) is proved in the same way as Lemma 1.2 except that Γ should be replaced by the Hilbert space Σ throughout. Let $X^{(n)}$ satisfy (3.1), then for all $t>0$, $\int_0^t dA^\sharp X\Omega=\lim_{n\to\infty}\int_0^t dA^\sharp X^{(n)}\Omega$ and

$$\left(\int_0^t dA^\sharp X^{(n)}\right)^*\Omega=\int_0^t X^{(n)\dagger}dA^\flat\Omega\to\int_0^t X^\dagger dA^\flat\Omega$$

so that $\int_0^t dA^\sharp X\Omega\in\mathscr{D}(S)$ and $S\left(\int_0^t dA^\sharp X\Omega\right)=\int_0^t X dA^\flat\Omega$, i.e. (3.2) is satisfied. □

Theorem 3.4. *$M\in\mathscr{M}_\Sigma\Rightarrow\exists!\,\lambda\in\mathbb{C}$ and $F,G\in L^2_\Sigma$, such that*

$$\begin{aligned} M(t)&=\lambda I+\int_0^t dA^*F+\int_0^t G\,dA\\ M^\dagger(t)&=\bar{\lambda}I+\int_0^t dA^*G^\dagger+\int_0^t F^\dagger dA. \end{aligned}\tag{3.3}$$

Proof. Denoting the subspaces $\Gamma_t\cap\Sigma$ of Σ by Σ_t $(t>0)$ we have, for each $t>0$, the orthogonal decomposition

$$\Sigma_t=\mathbb{C}\Omega\oplus\left\{\int_0^t dA^*F\Omega\colon F\in L^2_\Sigma\right\}\oplus\left\{\int_0^t G\,dA\Omega\colon G\in L^2_\Sigma\right\}$$

which is proved in a similar way to Lemma 3.1 using the above lemma. The proof is completed in the same way as Theorem 3.2, with (3.3) following from (3.2). □

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