Probab. Th. Rel. Fields 73, 585–595 (1986)

Probability Theory and Related Fields

Functional Limit Theorems for Linear Statistics from Sequential Ranks

E.V. Khmaladze[1] and A.M. Parjanadze[2]

[1] Matematičeskiĭ Institut im. V.A. Steklova AN SSSR, 117966 GSP-1, Moskva, ul. Vavilova, 42, USSR

[2] Matematičeskiĭ Institut im. A.M. Razmadze AN GSSR, 380012 Tbilisi, Plekhanov ave., 150A, USSR

Summary. Let $X_1, \dots, X_n$ be a sequence of continuously distributed independent random variables. The normalized ranks R_{kn} and sequential ranks S_k, $k=1, \dots, n$, are defined by

$$R_{kn}=\frac{1}{n}\sum_{j=1}^{n} I\{X_j<X_k\}, \quad S_k=\frac{1}{k}\sum_{j=1}^{k} I\{X_j<X_k\}.$$

The subject of the present paper is the asymptotic behavior, as $n\to\infty$, of the process

$$\frac{1}{\sqrt{n}}\sum_{k\leqq nt} a(S_k), \quad 0\leqq t\leqq 1,$$

for $a\in L^2(0,1)$, $\int_0^1 a\,dn=0$. For suitable a, the limiting law of that process is expressed as solution of a stochastic equation under the hypothesis of identically distributed $X_1, \dots, X_n$ as well as under a class of contiguous alternatives, which contains the occurrence of a change point in the series of measurements.

1. Introduction

Let $X_1, \dots, X_n$ be a sequence of continuously distributed random variables (r.v.). Consider the corresponding vectors of normalized ranks $\mathbb{R}_n=(R_{1n}, \dots, R_{nn})$ and normalized sequential ranks $\mathbb{S}_n=(S_1, \dots, S_n)$ which are defined in the following way:

$$R_{kn}=\frac{1}{n}\sum_{j=1}^{n} I\{X_j\leqq X_k\}, \quad S_k=R_{kk}=\frac{1}{k}\sum_{j=1}^{k} I\{X_j\leqq X_k\}, \qquad k=1, \dots, n,$$

where $I\{A\}$ is an indicator of an event A. There is a one-to-one correspondence between the vector of sequential ranks $\mathbb{S}_n$ and the vector of the "ordinary" ranks $\mathbb{R}_n$

$$S_k = \frac{1}{k} \sum_{j=1}^{k} I\{R_{jn} \leqq R_{kn}\}, \quad k=1, \ldots, n,$$

$$\frac{n}{n-k} R_{n-k,n} = S_{n-k} + \frac{1}{n-k} \sum_{j=1}^{k} I\{S_{n-k+j} \leqq S_{n-k}\}, \quad k=0, \ldots, n-1.$$

But at the same time the properties of the vectors $\mathbb{R}_n$ and $\mathbb{S}_n$ are very different. In particular, for any symmetrical function φ the r.v. $\varphi(\mathbb{R}_n)$ degenerates into a constant whereas $\varphi(\mathbb{S}_n)$ remains a non-degenerate r.v. For instance, the empirical distribution function of the vector $\mathbb{R}_n$ is a deterministic function $F_{R_n}(t) = [nt]/n$ where $[x]$ is the integer part of x and, hence, it is of no use for the aims of statistics. At the same time the empirical distribution function F_{S_n} of the vector $\mathbb{S}_n$ is a random function which is quite useful for testing of the hypotheses about the distribution of the r.v.'s $X_1, \ldots, X_n$ (see, e.g., [1] and the Corollary 2 below). Furthermore the coordinates of the vector $\mathbb{R}_n$ are dependent r.v.'s, while the following statement holds for the sequential ranks (see, e.g., Theorem 1.1 in [2]).

Lemma 1.1. *If the r.v.'s $X_1, \ldots, X_n$ are independent and identically continuously distributed then the r.v.'s $S_1, \ldots, S_n$ are independent and*

$$\mathbb{P}\left\{S_k = \frac{i}{k}\right\} = \frac{1}{n}, \quad i=1, \ldots, k.$$

Note one more difference between $\mathbb{R}_n$ and $\mathbb{S}_n$: if the sequence $X'_1, \ldots, X'_n$ is a permutation of the sequence $X_1, \ldots, X_n$ then the vector $\mathbb{R}_n$ is a similar permutation of the vector whereas this does not hold for the vector $\mathbb{S}_n$. However in problems when according to any of the hypotheses to be tested the r.v.'s $X_1, \ldots, X_n$ are not identically distributed, e.g., in change point problems (see below), it does not seem natural to be interested in the permutations of the sequence $X_1, \ldots, X_n$.

In the present work we obtain limit theorems for processes formed by partial sums $\sum_{k \leqq nt} a(S_k)$. Such partial sums play an important role when the hypothesis about independence and identical continuous distribution of the r.v.'s $X_1, \ldots, X_n$ is tested against various alternatives about different distribution of these r.v.'s. According to Lemma 1.1 under the hypothesis the functional limit theorem for these partial sums immediately follows from Donsker's theorem (see, e.g., [3], Chap. 3, §6) – this is what makes the application of sequential ranks so attractive. However, when the alternatives hold the situation is quite different.

In particular, let according to the hypothesis (the alternative) the distribution function (d.f.) of the r.v. X_i be $F(F_{in})$ and let $\mathscr{L}_n = \sum_{i \leqq n} \ln(dF_{in}/dF)(X_i)$. Suppose that the sequence of the direct products $F_{1n} \times \ldots \times F_{nn}$, $n=1, 2, \ldots$, is contiguous with respect to the sequence $F \times \ldots \times F$. Introduce σ=algebras $\mathscr{F}^X = \sigma\{X_1, \ldots, X_n\}$ and $\mathscr{F}^S = \sigma\{S_1, \ldots, S_n\}$. According to Le Cam's third lemma (see, e.g., [4], Chap. VI, §1) the limit distribution for $\sum_{k \leqq nt} a(S_k)$ under the alternatives will follow from the limit theorem for the pair $(\sum_{k \leqq nt} a(S_k), \mathscr{L}_n)$ under the hypothesis.

However it is not so easy to obtain this limit theorem: the distribution of $\sum_{k \leqq nt} a(S_k)$ with respect to $\mathscr{F}^X$ is not so "simple" as it is with respect to $\mathscr{F}^S$, but the r.v. $\mathscr{L}_n$ is not measurable with respect to $\mathscr{F}^S$. At the same time the classical method to obtain the joint limit theorem for linear rank statistics $\sum c_k a(R_{kn})$ and $\mathscr{L}_n$ cannot be applied here: as it is well-known, the essence of this method is that the relation

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} c_k a(R_{kn}) - \frac{1}{\sqrt{n}} \sum_{k=1}^{n} c_k \varphi(U_k) = o_P(1), \qquad n \to \infty$$

is established where $\sum c_k = 0$, $U_k = F(X_k)$ and, hence, it is sufficient to prove the joint limit theorem for two sums of independent r.v.'s $(1/\sqrt{n}) \sum_{k \leqq n} c_k \varphi(F(X_k))$ and $\mathscr{L}_n$ which is quite easy. In the case of sequential ranks such a way cannot be applied, as it is stated in

Lemma 2.1. *Let $U_1, \ldots, U_n$ be independent and uniformly distributed on $[0, 1]$ r.v.'s and $S_1, \ldots, S_k$ be their sequential ranks. Then for any $t \in (0, 1]$*

$$\frac{1}{\sqrt{n}} \sum_{k \leqq nt} a(S_k) - \frac{1}{\sqrt{n}} \sum_{k \leqq nt} \varphi(U_k) \neq o_P(1)$$

for any functions a and φ such that

$$\int_0^1 a(u)\,du = \int_0^1 \varphi(u)\,du = 0, \qquad 0 < \int_0^1 a^2(u)\,du < \infty$$

and a is left continuous and has finite right-hand limits.

Describe our alternatives more precisely: assume that all d.f. F_{in}, $i = 1, \ldots, n$, $n = 1, 2, \ldots$, are absolutely continuous with respect to the d.f. F and

$$\left[\frac{dF_{in}}{dF}(x)\right]^{1/2} = 1 + \frac{1}{2\sqrt{n}} h_n(t, u), \qquad \frac{i-1}{n} < t \leqq \frac{i}{n}, \quad u = F(x)$$

where

$$\int_0^1 \int_0^1 \left[h_n(t, u) - h\left(\frac{[nt]}{n}, u\right)\right]^2 dt\,du \to 0, \qquad \int_0^1 \int_0^1 h^2(t, u)\,dt\,du < \infty,$$

$$\int_0^1 h(t, u)\,du = 0 \qquad \text{for a.a. } t \in [0, 1]. \tag{1}$$

In particular, for the so-called change point alternatives there exists $t_0 \in (0, 1)$ such that $h(t, u) = I\{t \geqq t_0\}\, h(u)$ for some square integrable function h.

Finally, consider the empirical process y_n which is based on the normalized sequential ranks:

$$y_n(s) = \sqrt{n}\,[F_{Sn}(s) - s].$$

In [1] the convergence in distribution is formulated for y_n under the hypothesis. Corollary 2 in Sect. 2 of the given paper establishes the convergence in distribution of y_n under the alternatives.

2. Main Results

Let a be a square integrable function on $[0,1]$ such that $\int_0^1 a(u)\,du=0$ and $\int_0^1 a^2(u)\,du=1$. Denote $b_n(t)=\frac{1}{\sqrt{n}}\sum_{i\leq nt} a(S_i)$ and introduce the σ-algebras

$$\mathscr{F}_k^X=\sigma\{X_1,\dots,X_k\} \quad \text{and} \quad \mathscr{F}_k^S=\sigma\{S_1,\dots,S_k\},$$

$k=1,\dots,n$, $\mathscr{F}_0^X=\mathscr{F}_0^S=\sigma\{\emptyset\}$.

The process $\{b_n(t),\mathscr{F}_{[nt]}^S\}$ is a process with independent increments, but we have to consider the process $\{b_n(t),\mathscr{F}_{[nt]}^X\}$. Consider Doob's decomposition for this process

$$b_n=A_n+M_n$$

$$A_n(t)=\frac{1}{\sqrt{n}}\sum_{k\leq nt} E[a(S_k)|\mathscr{F}_{k-1}^X] \tag{2}$$

$$M_n(t)=\frac{1}{\sqrt{n}}\sum_{k\leq nt} (a(S_k)-E[a(S_k)|\mathscr{F}_{k-1}^X])$$

and introduce the field V_n and the process C_n by means of the relations: $U_k=F(X_k)$ and

$$V_n(t,u)=0,\quad 0\leq t<\frac{1}{n},\qquad V_n(t,u)=\frac{1}{\sqrt{n}}\sum_{j\leq nt}(I\{U_j\leq u\}-u),\quad \frac{1}{n}\leq t\leq 1,$$

$$C_n(t)=-\int_0^t w_n\left(\frac{[n\tau]-1}{n}\right)\frac{n}{[n\tau]-1}\,d\tau$$

where

$$w_n(t)=\int_0^1 a(u)\,V_n(t,du)=\frac{1}{\sqrt{n}}\sum_{k\leq nt} a(U_k)$$

and suppose $w_n\left(\frac{[nt]-1}{n}\right)\Big/([nt]-1)=0$ for $t\in[0,1/n]$. Unless the opposite is stated, we assume the r.v.'s $X_1,\dots,X_n$ to be independent and have a continuous d.f. F, i.e. $U_1,\dots,U_n$ are independent and uniformly distributed.

Obviously, the process $\{w_n(t),\mathscr{F}_{[nt]}^X\}$ is a square integrable martingale and, hence, the process C_n has a "suitable" construction. According to Theorem 1, the process C_n approximates the compensator A_n.

Theorem 1. *Suppose that the derivative a' of the function a is bounded and continuous. Then for any $\varepsilon>0$*

$$\sup_{n\varepsilon\leq k\leq n}\left|\sqrt{n}\,E[a(S_k)|\mathscr{F}_{k-1}^X]+w_n\left(\frac{k-1}{n}\right)\frac{n}{k-1}\right|=o_P(1),\qquad n\to\infty.$$

Consequently,

$$\sup_{0 \leq t \leq 1} |A_n(t) - C_n(t)| = o_P(1), \qquad n \to \infty.$$

Remark. The statement of Theorem 1 holds for somewhat more general conditions: it is sufficient, for instance, to assume that $a' = \varphi - \psi$ where φ, ψ are increasing square integrable and continuous in the square mean. The proof for this case is given in Sect. 3. But such extension does not lead to the generalization of the basic Theorem 3.

Define the process of the likelihood ratio L_n:

$$L_n(t) = \sum_{k \leq nt} \left[\ln \frac{dF_{kn}}{dF}(X_k) - E \ln \frac{dF_{kn}}{dF}(X_k) \right]$$

and the two-dimensional Gaussian process (w, L) where w is a standard Wiener process, L is a Gaussian process with mean 0 and correlation function $Q_L(t_1 \wedge t_2)$ where

$$Q_L(t) = \int_0^t \int_0^1 h^2(\tau, u)\, d\tau\, du$$

(i.e. L is a process with independent increments) and the mutual correlation function of w and L is $Q(t_1 \wedge t_2)$ where

$$Q(t) = \int_0^t \int_0^1 h(\tau, u)\, a(u)\, d\tau\, du.$$

Let $D^2[0, 1]$ denote the direct product of the spaces $D[0, 1]$.

Theorem 2. *Under the conditions of Theorem* 1

$$\sup_{0 \leq t \leq 1} |M_n(t) - w_n(t)| = o_P(1), \qquad n \to \infty.$$

Now we pass to the joint limit theorem for the processes $\{b_n, L_n\}$ and the resulting limit theorem for b_n under a sequence of alternatives. Let $b(t) = -\int_0^t (w(\tau)/\tau)\, d\tau + w(t)$. Then $\{b, L\}$ is a two-dimensional Gaussian process, b being a standard Wiener process and the mutual correlation function of the processes b and L is

$$E b(t_1) L(t_2) = R(t_1, t_2) = -\int_0^{t_1} \frac{Q(\tau \wedge t_2)}{\tau}\, d\tau + Q(t_1 \wedge t_2).$$

In particular, for the change point alternatives

$$R(t_1, t_2) = (h, a) \left[t_0 \left(\ln \frac{t_1 \wedge t_2}{t_0} \right)^+ - (t_2 - t_0) \ln \frac{t_2}{t_1 \wedge t_2} \right]$$

and

$$R(t, 1) = (h, a)\, t_0 \left(\ln \frac{t}{t_0} \right)^+$$

where

$$x^+ = xI\{x>0\} \quad \text{and} \quad (h, a) = \int_0^1 h(u)\, a(u)\, du.$$

Introduce a d.f. $\lambda_k(t) = [kt]/k$. Let $C_{1b}[0,1]$ denote a class of functions on $[0,1]$ with a continuous and bounded derivative and let $B[0,1]$ be a completion of the class $C_{1b}[0,1]$ in the norm

$$\rho(a) = \lim \left[\frac{1}{n} \sum_{k=1}^{n} \int_0^1 a^2(\tau)\, d\lambda_k(t)\right]^{1/2} + \left[\int_0^1 a^2(\tau)\, d\tau\right]^{1/2}.$$

It is evident that the right-(left-)continuous functions which have left (right) limits belong to $B[0,1]$ and that $B[0,1] \subset L_2[0,1]$. To explain the nature of the metric ρ we remind that λ_k is a d.f. of the (normalized) sequential rank S_k and that the predictable square characteristic (see [6], Chap. 5, § 1) of the difference of the two martingales $b_{nj}(t) = (1/\sqrt{n}) \sum_{k \leq nt} a_j(S_k)$, $j = 1, 2$ (with respect to the flow $\{\mathscr{F}_k^S\}$) is

$$\langle b_{n1} - b_{n2} \rangle_t = \frac{1}{n} \sum_{k \leq nt} \int_0^1 [a_1(\tau) - a_2(\tau)]^2\, d\lambda_k(\tau).$$

Theorem 3. *If the function* $a \in B[0,1]$ *then* $\{b_n, L_n\} \xrightarrow{\mathscr{D}} \{b, L\}$ *in* $D^2[0,1]$ *with* $n \to \infty$.

Corollary 1. *If the function* $a \in B[0,1]$ *then under the alternatives* (1) $b_n \xrightarrow{\mathscr{D}} b + R(\cdot, 1)$ *in* $D[0,1]$ *with* $n \to \infty$.

Let

$$H(t,s) = \int_0^t \int_0^s h(\tau, u)\, d\tau\, du.$$

Corollary 2. *Under alternatives* (1)

$$y_n \xrightarrow{\mathscr{D}} v + c$$

in $D[0,1]$ *with* $n \to \infty$, *where* v *is a Brownian bridge and the function* c *is*

$$c(s) = -\int_0^1 \frac{H(\tau, s)}{\tau}\, d\tau + H(1, s).$$

In particular, for change point alternatives

$$c(s) = -\int_0^s h(u)\, du\, t_0 \ln t_0.$$

3. Proof of Theorems and Corollaries

Similar to Sect. 2 we assume everywhere that the r.v.'s $X_1, \ldots, X_n$ are independent and that the d.f. of X_k is F, i.e. the r.v. $U_k = F(X_k)$ are uniformly distributed on $[0,1]$ and $U_1, \ldots, U_n$ are independent. To prove Theorem 1 the following lemma is used.

Lemma 1.3. *For* $n \to \infty$

$$\sup_{0 \leqq t, u \leqq 1} |V_n(t,u)| = O_P(1), \qquad \sup_{0 \leqq t \leqq 1} \int_0^1 V_n^2(t,u)\, du = O_P(1).$$

Proof. Since

$$\sup_{0 \leqq t \leqq 1} \int_0^1 V_n^2(t,u)\, du \leqq \sup_{0 \leqq t, u \leqq 1} |V_n(t,u)|^2$$

it is sufficient to prove the first relation. But it is well-known ([5]) that $V_n \xrightarrow{\mathscr{D}} V$ in $D[0,1]^2$ where V is the so called Kiefer field, i.e. the Gaussian field with the mean 0 and the covariance function $(t_1 \wedge t_2)(s_1 \wedge s_2 - s_1 s_2)$. Therefore

$$\sup_{0 \leqq t, u \leqq 1} |V_n(t,u)| \xrightarrow{\mathscr{D}} \sup_{0 \leqq t, u \leqq 1} |V(t,u)| = O_P(1)$$

which implies the first relation of the lemma.

Proof of Theorem 1. The sequential rank S_k can be written as

$$S_k = \bar{F}_{k-1}(U_k) + \frac{1}{k}[1 - \bar{F}_{k-1}(U_k)]. \tag{3}$$

Hence,

$$\begin{aligned}
\sqrt{n}\, E[a(S_k) | \mathscr{F}_{k-1}^X] &= \sqrt{n} \int_0^1 a\left(\bar{F}_{k-1}(u) + \frac{1}{k}[1 - \bar{F}_{k-1}(u)]\right) du \\
&= \sqrt{n} \int_0^1 a'(u)[\bar{F}_{k-1}(u) - u]\, du + \frac{\sqrt{n}}{k} \int_0^1 a'(\bar{u})[1 - \bar{F}_{k-1}(u)]\, du \\
&\quad + \sqrt{n} \int_0^1 [a'(\bar{u}) - a'(u)][\bar{F}_{k-1}(u) - u]\, du,
\end{aligned} \tag{4}$$

where $\bar{u}$ lies between u and $\bar{F}_{k-1}(u) + \frac{1}{k}[1 - F_{k-1}(u)]$. But the first summand in the right-hand side can be written as $-w_n(k-1/n)\, n/(k-1)$ where, note $w_n(t) = \int_0^1 a(u)\, V_n(t, du)$. Let us show that the two remaining summands are small. We have (see the Remark after Theorem 1)

$$a'(\bar{u}) - a'(u) = \varphi(\bar{u}) - \varphi(u) - \psi(\bar{u}) + \psi(u)$$

and

$$0 \leqq |\bar{u} - u| \leqq \frac{1}{\sqrt{n\varepsilon}} \sup_{0 \leqq t, u \leqq 1} |V_n(t,u)| + \frac{1}{n\varepsilon} = \Delta_n. \tag{5}$$

Since φ is increasing

$$|\varphi(u - \Delta_n) - \varphi(u)| \leqq \sup_{n\varepsilon \leqq k \leqq n} |\varphi(\bar{u}) - \varphi(u)| \leqq \varphi(u + \Delta_n) - \varphi(u).$$

Lemma 1.3 and the continuity of φ in the square mean imply that

$$\int_0^1 [\varphi(u - \Delta_n) - \varphi(u)]^2\, du = o_P(1), \qquad n \to \infty$$

and the same relation holds for $\varphi(u+\Delta_n)-\varphi(u)$. Consequently,

$$\int_0^1 \sup_{n\varepsilon \leqq k \leqq n} [\varphi(\bar{u})-\varphi(u)]^2\, du = o_P(1), \qquad n\to\infty.$$

A similar relation is true for $\psi(\bar{u})-\psi(u)$. Hence,

$$\int_0^1 \sup_{n\varepsilon \leqq k \leqq n} [a'(\bar{u})-a'(u)]^2\, du = o_P(1), \qquad n\to\infty. \tag{6}$$

Consequently, uniformly with respect to $k \geqq n\varepsilon$, $n\to\infty$ the third summand in the right-hand side of (4) is $o_P(1)$ and by virtue of (6) and the finiteness of the integral $\int_0^1 (a'(u))^2\, du$ we get $\int_0^1 (a'(\bar{u}))^2\, du = O(1)$. The latter relation implies that the second summand in the right-hand side of (4) is also $o_P(1)$ uniformly with respect to $k \geqq n\varepsilon$, $n\to\infty$. This proves the first assertion of the theorem.

In order to prove the second assertion note that

$$\sup_{0\leqq t\leqq 1} |A_n(t)-C_n(t)| \leqq \sup_{0\leqq t\leqq \varepsilon} |A_n(t)| + \sup_{0\leqq t\leqq \varepsilon} |C_n(t)| + |A_n(\varepsilon)| + |C_n(\varepsilon)|$$
$$+ \sup_{\varepsilon\leqq t\leqq 1} |A_n(t)-A_n(\varepsilon)-C_n(t)+C_n(\varepsilon)|.$$

It follows from the first assertion of the theorem that for any $\varepsilon>0$ the last summand in the right-hand side of this inequality is $o_P(1)$ with $n\to\infty$. Now for all $n\geqq 1$ and for $\varepsilon\to 0$

$$|C_n(\varepsilon)| \leqq \sup_{0\leqq t\leqq \varepsilon} |C_n(t)| \leqq \int_{1/n}^{\varepsilon} \left| w_n\left(\frac{[n\tau]-1}{n}\right) \frac{n}{[n\tau]-1}\right| d\tau = o_P(1),$$

since $E|w_n(\tau)| \leqq \sqrt{\tau}$ and, consequently, the expectation of the right-hand side of this inequality is small for small ε. Besides

$$|A_n(\varepsilon)| \leqq \sup_{0\leqq t\leqq \varepsilon} |A_n(t)| \leqq \sup_{0\leqq t\leqq \varepsilon} |b_n(t)| + \sup_{0\leqq t\leqq \varepsilon} |M_n(t)| = o_P(1)$$

as it can be easily seen from the fact that the processes $\{M_n(t), \mathscr{F}^X_{[nt]}\}$ and $\{b_n(t), \mathscr{F}^S_{[nt]}\}$ are martingales. These estimates lead to the second assertion of the theorem. Theorem 1 is proved.

Proof of Theorem 2. Using equality (3) write the expansion for $a(S_k)$:

$$a(S_k) = a(U_k) + a'(U_k)[F_{k-1}(U_k) - U_k] + \gamma_k$$

where we have for the remainder γ_k

$$|\gamma_k| \leqq \frac{1}{k} \sup_{0\leqq u\leqq 1} |a'(u)| + |a'(\bar{U}_k) - a'(U_k)| \cdot |S_k - U_k|.$$

Therefore

$$\frac{1}{\sqrt{n}} \sum_{k=[n\varepsilon]+1}^{[nt]} |\gamma_k| \leqq \sup_{0\leqq u\leqq 1} |a'(u)| \frac{1}{\varepsilon\sqrt{n}} + \sqrt{n}\,\Delta_n \max_{n\varepsilon\leqq k\leqq n} |a'(\bar{U}_k) - a'(U_k)| = o_P(1),$$

since it follows from inequality (5) that $\max_{n\varepsilon \leqq k \leqq n} |S_k - U_k| = O_P(1/\sqrt{n})$ and relation (6) implies $\max_{n\varepsilon \leqq k \leqq n} |a'(\bar{U}_k) - a'(U_k)| = o_P(1)$.

Further it immediately follows from the proof of Theorem 1 that

$$E[a(S_k) | \mathscr{F}_{k-1}^X] = \int_0^1 a'(u) [\bar{F}_{k-1}(u) - u] \, du + \delta_k$$

with $\max_{n\varepsilon \leqq k} |\delta_k| = o_P(1/\sqrt{n})$. So finally we have

$$\begin{aligned} M_n(t) - M_n(\varepsilon) &= \frac{1}{\sqrt{n}} \sum_{k=[n\varepsilon]+1}^{[nt]} (a(S_k) - E[a(S_k) | \mathscr{F}_{k-1}^X]) \\ &= w_n(t) - w_n(\varepsilon) + m_n(t) + \beta_n(t) \end{aligned}$$

where

$$\begin{aligned} m_n(t) &= \frac{1}{\sqrt{n}} \sum_{k=[n\varepsilon]+1}^{[nt]} (d_k - E[d_k | \mathscr{F}_{k-1}^X]), \\ d_k &= a'(U_k) [\bar{F}_{k-1}(U_k) - U_k], \\ &\sup_{\varepsilon \leqq t \leqq 1} |\beta_n(t)| = o_P(1). \end{aligned}$$

It can be easily seen that the martingale m_n converges in probability to 0. Indeed,

$$\begin{aligned} \langle m_n \rangle_1 &= \frac{1}{n} \sum_{k=[n\varepsilon]+1}^{n} E[d_k^2 | \mathscr{F}_{k-1}^X] \\ &\leqq \int_0^1 (a'(u))^2 \, du \max_{n\varepsilon \leqq k \leqq n} \sup_{0 \leqq u \leqq 1} |\bar{F}_{k-1}(u) - u| = o(1), \qquad n \to \infty \end{aligned}$$

by virtue of Glivenko's theorem, and from Kolmogorov's inequality we get

$$\sup_{0 \leqq t \leqq 1} |m_n(t)| = o_P(1).$$

So

$$\sup_{\varepsilon \leqq t \leqq 1} |M_n(t) - M_n(\varepsilon) - w_n(t) + w_n(\varepsilon)| = o_P(1), \qquad n \to \infty,$$

and besides it can be easily seen that

$$|M_n(\varepsilon)| \leqq \sup_{0 \leqq t \leqq \varepsilon} |M_n(t)| = o_P(1), \qquad \varepsilon \to 0,$$

and

$$|w_n(\varepsilon)| \leqq \sup_{0 \leqq t \leqq \varepsilon} |w_n(t)| = o_P(1), \qquad \varepsilon \to 0.$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Consider the process $C_n + w_n$. Assume first that the function a satisfies the conditions of Theorem 1. Theorems 1 and 2 imply that $\sup_{0 \leqq t \leqq 1} |b_n(t) - C_n(t) - w_n(t)| = o_P(1)$ with $n \to \infty$. Hence it is sufficient to prove the convergence of the processes $\{C_n+w_n, L_n\}$. But by means of the theorem of continuous mappings (see [3] Chap. 1, § 5) it can be easily proved that $\{b_n^\varepsilon, L_n\} \xrightarrow{\mathscr{D}} \{b^\varepsilon, L\}$ where

$$b_n^\varepsilon(t)=I\{t\geqq\varepsilon\}\left[-\int_\varepsilon^t \frac{n}{[n\tau]-1} w_n\left(\frac{[n\tau]-1}{n}\right)d\tau+w_n(t)\right]$$

and

$$b^\varepsilon(t)=I\{t\geqq\varepsilon\}\left[-\int_\varepsilon^t \frac{1}{\tau} w(\tau)\,d\tau+w(t)\right].$$

At the same time with $n\to\infty$ and when $\varepsilon\to 0$,

$$\sup_{0\leqq t\leqq 1}|b_n^\varepsilon(t)-C_n(t)-w_n(t)|=o_P(1) \quad\text{and}\quad \sup_{0\leqq t\leqq 1}|b_n^\varepsilon(t)-b(t)|=o_P(1).$$

So we can apply Theorem 4.2, Chap. 1 in [3] and show that

$$\{C_n+w_n, L_n\}\xrightarrow{\mathscr{D}}\{b, L\} \quad\text{and, hence,}\quad \{b_n, L_n\}\xrightarrow{\mathscr{D}}\{b, L\}$$

if $a\in C_{1b}[0,1]$. Suppose now that the function $\bar a\in B[0,1]$ is such that $\int_0^1 \bar a(\tau)\,d\tau=0$, $\int_0^1 \bar a^2(\tau)\,d\tau=1$. Consider the corresponding process $\bar b_n$. Let a^m be a sequence of functions from $C_{1b}[0,1]$ converging to $\bar a$ in the metric ρ and consider the corresponding processes b_n^m. By virtue of the just proved result $\{b_n^m, L_n\}\xrightarrow{\mathscr{D}}\{b^m, L\}$ for every m. Consider the difference $\{\bar b_n, L_n\}-\{b_n^m, L_n\}=\{\bar b_n-b_n^m, 0\}$ which with respect to the flow $\{\mathscr{F}_k^S\}$ is a martingale. Consequently, $\langle \bar b_n, b_n^m\rangle_1\to\rho(\bar a, a_m)$ with $n\to\infty$. Hence, using Kolmogorov's inequality we have

$$\lim_{m\to\infty}\lim_{n\to\infty} P\{\sup_{0\leqq t\leqq 1}|\bar b_n(t)-b_n^m(t)|>\varepsilon\}=0.$$

Applying Theorem 4.2, Chap. 1 from [3] once more, we complete the proof of Theorem 3.

Proof of Corollary 1 immediately follows from Le Cam's third lemma (see, e.g., [4], Chap. VI, § 1).

Proof of Corollary 2. For any fixed set $t_1, \ldots, t_m\in[0,1]$ consider the linear combination

$$\sum_{j=1}^m \alpha_j y_n(t_j)=\frac{1}{\sqrt{n}}\sum_{k=1}^n\sum_{j=1}^m \alpha_j(I\{S_k\leqq t_j\}-t_j).$$

It follows from Corollary 1 that

$$\sum_{j=1}^m \alpha_j y_n(t_j)\xrightarrow{\mathscr{D}}\sum_{j=1}^m \alpha_j[v(t_j)+c(t_j)].$$

It means that the finite-dimensional distributions of the process y_n converge to the finite-dimensional distributions of the process $v+c$. Besides under the alternatives (1) the family of probability measures of the processes y_n, $n=1,2,\ldots$, is tight since this family is tight under the hypothesis (see [1]). Corollary 2 is proved.

Proof of Lemma 2.1. Suppose the opposite, i.e. suppose one can find for the function a, satisfying the conditions of the lemma, a function φ from $L_2[0,1]$ such that $b_n(t)-g_n(t)=o_P(1)$ for $n\to\infty$ where $g_n(t)=(1/\sqrt{n})\sum_{k\leq nt}\varphi(U_k)$. But according to Theorem 3 $\{b_n, L_n\}\xrightarrow{\mathscr{D}}\{b, L\}$. Since the processes b and L have continuous trajectories $\{b_n(t), L_n(t)\}\xrightarrow{\mathscr{D}}\{b(t), L(t)\}$ where the correlation between $b(t)$ and $L(t)$ under the change point alternatives is $(h,a)t_0(\ln(t/t_0))^+$. Then by virtue of our assumption we must have $\{g_n(t), L_n(t)\}\xrightarrow{\mathscr{D}}\{b(t), L(t)\}$. But it is evident that the correlation between the r.v.'s $g_n(t)$ and $L_n(t)$ is $Eg_n(t)L_n(t)\to(h,\varphi)(t-t_0)^+$. Consequently, for any φ some t_0 can be found such that $(h,a)\,t_0(\ln(t/t_0))^+=(h,\varphi)(t-t_0)^+$ and this contradicts our assumption. Lemma 2.1 is proved.

Note added in proof

After we submitted our paper for publication we found in Z. Wahrscheinlichkeitstheor. Verw. Geb. **70**, 395–410 (1985) the paper by F. Lombard and D. Mason "Limit theorems for generalized sequential rank statistics". Though with different goals and different general approach and also different mathematical equipment this paper uses relation (3) above and therefore has distinct similarity with the present paper. We find it necessary to admit it here.

References

1. Parjanadze, A.M.: Application of sequential ranks for a change point detection (in Russian). Bulletin of the Acad. of Sci. of the Georgian SSR. **116**, 241–243 (1984)
2. Barndorff-Nielsen, O.: On limit behaviour of extreme order statistics. Ann. Math. Stat. **34**, 992–1002 (1963)
3. Billingsley, P.: Convergence of probability measures. New York-London-Sydney-Toronto. Wiley: 1968
4. Hájek, J., Šhidak, Z.: Theory of rank tests. Prague: Academia 1967
5. Bickel, P., Wichura, M.: Convergence criteria for multiparameter stochastic processes and some applications. Ann. Math. Stat. **42**, 1656–1670 (1971)
6. Liptser, R.Sh., Shiryayev, A.N.: Statistics of random processes. I. General theory. Berlin Heidelberg New York: Springer 1977

Received September 2, 1985