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Probability Theory and Related Fields

Adaptive Tests in Statistical Problems with Finite Nuisance Parameter*

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1. Introduction and Summary

Suppose $x_1, \ldots, x_n$ are n independent identically distributed observations on a random variable X which has one of two possible distributions P or Q. Assume that a simple hypothesis P is to be tested against a simple alternative Q.

If for a given number β, $0<\beta<1$, which does not depend on n, a test $\varphi = \varphi(x_1, \ldots, x_n)$ has the guaranteed power β, $E^Q \varphi \geqq \beta$, then

$$\liminf_{n \to \infty} [E^P \varphi]^{1/n} \geqq \exp(-K(Q, P)), \tag{1.1}$$

where $K(Q, P) = E^Q \log(dQ/dP)$ is the information number (see Chernoff 1956 or Bahadur 1971). The equality sign in (1.1) is attained by the most powerful likelihood ratio test of P versus Q.

Suppose now that the distributions P and Q are not known exactly but only up to a finite-valued nuisance parameter α, $\alpha = 1, \ldots, l$. For instance, there are l measurement types and for each fixed (but unknown to the statistician) type α the measurements have one of two alternative distributions P_α or Q_α. Another example is the transmission of a message in one of l possible languages which use the same alphabet. Assume that the message in unknown language is sent n times over a noisy channel and the choice has to be made between two possible messages or rather between two probability distributions which correspond to them. Thus, one has the hypothesis P_α to be tested against Q_α for each value of α.

We call a test φ_a, such that $E^Q_\alpha \varphi_a \geqq \beta$ for all α, to be adaptive if for any α

$$\lim_{n \to \infty} [E^P_\alpha \varphi_a]^{1/n} = \exp(-K(Q_\alpha, P_\alpha)) = \exp(-K_\alpha). \tag{1.2}$$

In other terms an adaptive test is asymptotically optimal for any value of the nuisance parameter in the following sense: within the class of tests which

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have the guaranteed power it asymptotically minimizes the probability of the first kind error.

The existence of adaptive tests has been investigated by the author (Rukhin 1982). A necessary condition and a sufficient condition for the existence of such test were obtained. In this paper in Sect. 3, we show (Theorem 5) that an adaptive test exists if and only if the information numbers for members of one family do not exceed the information numbers for distributions in any two different families. In other terms an adaptive test exists if and only if the testing problem for any value of the nuisance parameter is "at least as difficult" as the testing problems for distributions corresponding to different values of this parameter. This condition is deduced from a study of tests of a hypothesis $\sum w_\alpha P_\alpha$ against an alternative $\sum u_\alpha Q_\alpha$ for some positive weights u_α and w_α which is performed in Sect. 2. In Sect. 4 we give an example which illustrates the main result in the case of an exponential family.

Notice that the existence of adaptive test is related to the structure of finite hypothesis $\{\theta_1, \ldots, \theta_l\}$ and $\{\eta_1, \ldots, \eta_l\}$ for which there exists a test φ_0 such that for all $k=1, \ldots, l$ $E_{\eta_k} \varphi_0 \geqq \beta$ and

$$\lim_{n\to\infty} [E_{\theta_k} \varphi_0]^{1/n} = \max_{1 \leqq i \leqq l} \exp\{-K(P_{\eta_i}, P_{\theta_k})\} = \exp\{-K(P_{\eta_k}, P_{\theta_k})\}.$$

In other words φ_0, which is a test of composite hypothesis $\{\theta_1, \ldots, \theta_l\}$ versus $\{\eta_1, \ldots, \eta_l\}$, is asymptotically as good as the most powerful test of a simple hypothesis θ_k against η_k for any k. It is easy to see that φ_0 is an adaptive test in the testing problem of θ_α versus η_α. In this setting for any k

$$K(P_{\eta_k}, P_{\theta_k}) = \min_i K(P_{\eta_i}, P_{\theta_i}),$$

so that according to Theorem 5 such test φ_0 always exists.

These notions of optimality have "non-local" character, i.e., exponential convergence to zero of the significance level is examined. Somewhat different but related concepts for composite hypotheses have been considered by Bahadur (1960), Brown (1971), Hoeffding (1965) and Tusnady (1977).

2. Asymptotic Behavior of Tests for Mixtures

We start with the following result which is proved with the help of a multivariate version of Chernoff's Theorem (Groeneboom et al. 1979).

Lemma. *Let c_n, $n=1,2,\ldots$ be a sequence of positive numbers such that $n^{-1}\log c_n$ converges to a finite limit L. Assume that p_i, q_i, $i=1,\ldots,l$ are strictly positive measurable functions, $w_i = \exp(nb_i)/[\sum_k \exp(nb_k)]$, where b_i are real constants, u_i are positive probabilities, $i=1,\ldots,l$, which do not depend on n, and for all positive probabilites v_i, $i=1,\ldots,l$,*

$$\Pr\{\sum_i v_i[\log(p_k(X)/q_i(X)) - b_i] > L\} > 0$$

for all $k=1,\ldots,l$. *Then*

$$\lim_{n\to\infty}\left[\Pr\left\{\sum_k u_k\prod_1^n p_k(x_j)\geqq c_n\sum_k w_k\prod_1^n q_k(x_j)\right\}\right]^{1/n}$$
$$=\lim_{n\to\infty}\left[\Pr\left\{\sum_k u_k\prod_1^n p_k(x_j)>c_n\sum_k w_k\prod_1^n q_k(x_j)\right\}\right]^{1/n}$$
$$=\max_{1\leqq k\leqq l}\ \inf_{s_1,\ldots,s_l\geqq 0}\ \exp\{-\sum_i s_i(b_i+L-\max b_k)\}E\prod_i[p_k(X)/q_i(X)]^{s_i}.$$

The proof of this lemma is essentially contained in Rukhin (1982) (with functions q_i being replaced by $q_i e^{c_i}$).

We introduce now the following notation. Let f_k, $k=1,\ldots,l$, denote the density of Q_k and g_k denote the density of P_k. We assume throughout the paper that these densities (with respect to some σ-finite measure) exist and are strictly positive. Also let

$$\rho_\alpha(b_1,\ldots,b_l,L)=\max_{1\leqq k\leqq l}\ \inf_{s_1,\ldots,s_l\geqq 0}\ \exp\{-\sum_i s_i(b_i+L)\}\,E_\alpha^P\prod_i[f_k(X)/g_i(X)]^{s_i}.$$

Now let φ be the most powerful test of the simple hypothesis $\sum_k w_k\prod_1^n g_k(x_j)$ against the simple alternative $\sum_k u_k\prod_1^n f_k(x_j)$.

Theorem 1. *For fixed positive probabilities* u_k, $k=1,\ldots,l$ *and positive probabilities* w_i *of the form* $w_i=\exp(nb_i)/[\sum_k\exp(nb_k)]$ *assume that the test* φ_1 *has a fixed power* β, $0<\beta<1$, *and* $u_m\geqq\max[\beta,1-\beta]$ *where* m *is defined by* (2.1). *Then for any* α, $\alpha=1,\ldots,l$

$$\lim_{n\to\infty}[E_\alpha^P\varphi_1]^{1/n}=\rho_\alpha(b_1,\ldots,b_l,L),$$

where

$$L=\min_{k,i}[K(Q_k,P_i)-b_i]=\min_i[K(Q_m,P_i)-b_i]. \tag{2.1}$$

Proof. It is well known that for some constants c_n and γ_n, $0\leqq\gamma_n<1$ test φ_1 has the form

$$\varphi_1=\begin{cases}1, & \sum_k u_k\prod_1^n f_k(x_j)>c_n\sum_k w_k\prod_1^n g_k(x_j),\\ \gamma_n, & \sum u_k\prod_1^n f_k(x_j)=c_n\sum_k w_k\prod_1^n g_k(x_j),\\ 0, & \sum u_k\prod_1^n f_k(x_j)<c_n\sum_k w_k\prod_1^n g_k(x_j).\end{cases} \tag{2.2}$$

It follows that

$$\sum_k u_k Q_k\left(\sum_i u_i\prod_1^n f_i(x_j)>c_n\sum_i w_i\prod_1^n g_i(x_j)\right)$$
$$\leqq\beta\leqq\sum_k u_k Q_k\left(\sum u_i\prod_1^n f_i(x_j)\geqq c_n\sum_i w_i\prod_1^n g_i(x_j)\right). \tag{2.3}$$

Also notice that for any fixed m

$$Q_m\left(\sum_i u_i \prod_1^n f_i(x_j) > c_n \sum_i w_i \prod_1^n g_i(x_j)\right)$$
$$\geqq Q_m\left(\max_k \left[u_k \sum_1^n f_k(x_j)\right] > c_n l \max_i \left[w_i \prod_1^n g_i(x_j)\right]\right)$$
$$\geqq \max_k Q_m\left(u_k \prod_1^n f_k(x_j) > c_n l w_i \prod_1^n g_i(x_j),\ i=1, \ldots, l\right)$$

and

$$Q_m\left(\sum_i u_i \prod_1^n f_i(x_j) \geqq c_n \sum_i w_i \prod_1^n g_i(x_j)\right)$$
$$\leqq Q_m\left(l \max_k \left[u_k \prod_1^n f_k(x_j)\right] \geqq c_n \max_i \left[w_i \prod_1^n g_i(x_j)\right]\right)$$
$$\leqq \sum_k Q_m\left(l u_k \prod_1^n f_k(x_j) \geqq c_n \max_i \left[w_i \prod_1^n g_i(x_j)\right]\right)$$
$$\leqq l \max_k Q_m\left(u_k \prod_1^n f_k(x_j) \geqq c_n l^{-1} w_i \prod_1^n g_i(x_j),\ i=1, \ldots, l\right).$$

Since $u_m \geqq \beta$, formula (2.3) implies that

$$\limsup_{n\to\infty} \max_k Q_m\left(u_k \prod_1^n f_k(x_j) > c_n l w_i \prod_1^n g_i(x_j),\ i=1, \ldots, l\right) < 1, \tag{2.4}$$

and because of the inequality $u_m \geqq 1-\beta$ one has

$$\liminf_{n\to\infty} \max_k Q_m\left(u_k \prod_1^n f_k(x_j) > c_n l^{-1} w_i \prod_1^n g_i(x_j),\ i=1, \ldots, l\right) > 0. \tag{2.5}$$

For a fixed k let $Y_j^i = \log[f_k(x_j)/g_i(x_j)]$, $i=1, \ldots, l$, $j=1, 2, \ldots$,

$$y_n = n^{-1}(\log c_n + \log l - log \sum e^{nb_i}), \qquad v_n = n^{-1}(\log c_n - \log l - \log \sum e^{nb_i}).$$

Since $n^{-1} \sum_{j=1}^n Y_j^i$ converges in Q_m-probability to $E_m^Q \log[f_k(x)/g_i(X)] = e_{ik}$, one has

$$\limsup_{n\to\infty} Q_m\left(n^{-1} \sum_1^n Y_j^i > b_i + y_n,\ i=1, \ldots, l\right) = 1$$

if for all $i=1, \ldots, l$

$$b_i + \liminf_{n\to\infty} y_n < e_{ik}.$$

Also

$$\liminf_{n\to\infty} Q_m\left(n^{-1} \sum_1^n y_j^i \geqq b_i + v_n,\ i=1, \ldots, l\right) = 0$$

if for some i

$$b_i + \limsup_{n\to\infty} v_n > e_{ik}.$$

It follows now from (2.4) and (2.5) that for any k there exists i such that

$$b_i + \liminf_{n\to\infty} n^{-1}\log c_n \geqq e_{ik} + \max b_i,$$

and there exists k such that for all i

$$b_i + \limsup_{n\to\infty} n^{-1}\log c_n \leqq e_{ik} + \max b_i.$$

Therefore

$$\begin{aligned}\max_k \min_i [e_{ik} - b_i] + \max b_i &\leqq \liminf_{n\to\infty} n^{-1}\log c_n \leqq \limsup_{n\to\infty} n^{-1}\log c_n \\ &\leqq \max_k \min_i [e_{ik} - b_i] + \max_i b_i\end{aligned}$$

We have proved that the sequence $n^{-1}\log c_n$ converges and

$$\begin{aligned}L = \lim_{n\to\infty} n^{-1}\log c_n &= \max_k \min_i [e_{ik} - b_i] + \max_i b_i = \min_i [K(Q_m, P_i) - b_i] + \max_i b_i \\ &= \min_{k,i} [K(Q_k, P_i) - b_i] + \max_i b_i.\end{aligned}$$

For all positive probabilities q_i and any k

$$L \leqq \sum_i q_i (K(Q_k, P_i) - b_i),$$

so that

$$E_k^Q \sum_i q_i (\log [f_k(X)/g_i(X)] - b_i) \geqq L$$

and for any k

$$Q_k\Big(\sum_i q_i (\log [f_k(X)/g_i(X)] - b_i) > L\Big) > 0.$$

Since all measures P_k and Q_k are assumed to be mutually absolutely continuous,

$$P_k\Big(\sum_i q_i (\log [f_k(X)/g_i(X)] - b_i) > L\Big) > 0,$$

and our lemma is applicable.

This lemma entails

$$\begin{aligned}&\lim_{n\to\infty} \left[P_\alpha\left(\sum_k u_k \prod_1^n f_k(x_j) > c_n \sum w_k \prod_1^n g_k(x_j)\right)\right]^{1/n} \\ &= \lim_{n\to\infty} \max_{1\leqq k\leqq l} \left[P_\alpha\left(\prod_1^n f_k(x_j) > c_n w_i \prod_1^n g_k(x_j),\ i = 1, \ldots, l\right)\right]^{1/n} \\ &= \rho_\alpha(b_1, \ldots, b_l, L),\end{aligned}$$

and Theorem 1 is proven.

Corollary 1. *If φ is a test such that $E_k^Q \varphi \geqq \beta$ for all $k = 1, \ldots, l$, then for all real $b_1, \ldots, b_l$ and L defined by* (2.1)

$$\max_k \{e^{b_k} \liminf_{n\to\infty} [E_k^P \varphi]^{1/n}\} \geqq \max_k \{e^{b_k} \rho_k(b_1, \ldots, b_l, L)\}.$$

Indeed φ *as a test of* $\sum_k w_k \prod_1^n g_k(x_j)$ *versus* $\sum_k u_k \prod_1^n f_k(x_j)$ *has power* β *and therefore cannot have a significance level smaller than that of* φ_1.

Theorem 2. *For all real numbers* $b_1, \ldots, b_l$ *there exists a test* φ_2 *such that* $E_k^Q \varphi_2 \geqq \beta$ *for all* $k=1, \ldots, l$ *and*

$$\lim_{n\to\infty} [E_\alpha^P \varphi_2]^{1/n} = \rho_\alpha(b_1, \ldots, b_l, L) \tag{2.6}$$

where L is defined by (2.1).

Proof. For any α, $1 \leqq \alpha \leqq l$, define the constant $c_n(\alpha)$ so that for a test $\varphi^{(\alpha)}$ of form (2.2)

$$E_\alpha^Q \varphi^{(\alpha)} = \beta.$$

As in the proof of Theorem 1 we see that

$$\begin{aligned}\lim_{n\to\infty} n^{-1} \log c_n(\alpha) &= \max_k \min_i [E_\alpha^Q \log(f_k(X)/g_i(X)) - b_i] \\ &= \min_i [K(Q_\alpha, P_i) - b_i].\end{aligned}$$

Now let the test φ_2 of the form (2.2) be determined by $c_n = \min_\alpha c_n(\alpha)$. Then

$$\lim_{n\to\infty} n^{-1} \log c_n = L$$

and for all α

$$E_\alpha^Q \varphi_2 \geqq \beta.$$

The conclusion of Theorem 2 follows now from lemma.

Corollary 2. *For any* α *and all real* $b_1, \ldots, b_l$

$$\rho_\alpha(b_1, \ldots, b_l, L) \geqq \exp\{-K_\alpha\}. \tag{2.7}$$

This corollary follows directly from (1.1) *and* (2.6).

3. Conditions for the Existence of Adaptive Tests

We prove in this section our main results.

Theorem 3. *If an adaptive test exists then for all real* $b_1, \ldots, b_l$

$$\max_\alpha \exp(b_\alpha - K_\alpha) \geqq \max_\alpha [\exp(b_\alpha)\, \rho_\alpha(b_1, \ldots, b_l, L)]. \tag{3.1}$$

If for some $b_1, \ldots, b_l$

$$\exp(-K_\alpha) \geqq \rho_\alpha(b_1, \ldots, b_l, L) \tag{3.2}$$

for $\alpha = 1, \ldots, l$, *then an adaptive test exists.*

Proof. Assume that φ_a is an adaptive test. Then because of Corollary 1 one has

$$\max_k \{e^{b_k} \lim_{n\to\infty} [E_k^P \varphi_a]^{1/n}\} = \max_k \{\exp(b_k - K_k)\}$$
$$\geqq \max_k [\exp(b_k)\, \rho_k(b_1, \ldots, b_l, L)],$$

so that (3.1) is proven.

If (3.2) is met for some $b_1, \ldots, b_l$ then the test φ_2 of Theorem 2 is adaptive. Indeed (3.2) and (2.7) imply that for any α

$$\lim_{n\to\infty} [E^P \varphi_2]^{1/n} = \exp(-K_\alpha)$$

and

$$E_\alpha^Q \varphi_2 \geqq \beta.$$

Corollary 3. *If for some β, γ $f_\beta = g_\gamma$ then an adaptive test does not exist.*

Indeed put $b_1 = \ldots = b_l = 0$. Then

$$L = \min_{k,i} K(Q_k, P_i) = K(Q_\beta, P_\gamma) = 0$$

and

$$\rho_\gamma(b_1, \ldots, b_e, L) = \max_{1\leqq k\leqq l} \inf_{s_1,\ldots,s_l \geqq 0} E_\gamma^P \prod_{i=1}^{l} (f_k(X)/g_i(X))^{s_i}$$
$$\geqq \inf_{s_1,\ldots,s_l \geqq 0} E_\gamma^P \prod_{i=1}^{l} (g_\gamma(X)/g_i(X))^{s_i} = 1,$$

since all partial derivatives of the convex function $E_\gamma^P \prod_{i=1}^{l} (g_\gamma(X)/g_i(X))^{s_i}$ at the origin are nonnegative:

$$E_\gamma^P \log(g_\gamma(X)/g_i(X)) \geqq 0, \qquad i = 1, \ldots, l.$$

Thus

$$\rho_\gamma(b_1, \ldots, b_l, L) = 1,$$

and (3.1) cannot hold.

Theorem 4. *An adaptive test exists if and only if for any $\alpha = 1, \ldots, l$*

$$\rho_\alpha(K_1, \ldots, K_l, L_0) = \exp(-K_\alpha), \tag{3.3}$$

where

$$L_0 = \min_{k,i} [K(Q_k, P_i) - K_i]. \tag{3.4}$$

Proof. If an adaptive test exists then

$$1 = \max_\alpha \exp(K_\alpha - K_\alpha) \geqq \max_\alpha \exp(K_\alpha)\, \rho_\alpha(K_1, \ldots, K_l, L_0).$$

But because of (2.7) for any α

$$\exp(K_\alpha)\, \rho_\alpha(K_1, \ldots, K_l, L_0) \geqq 1.$$

Therefore (3.3) holds.

If condition (3.3) is met then test φ_2 of Theorem 2 is adaptive.

Theorem 5. *An adaptive test exists if and only if $L_0=0$, i.e., for all $i \neq k$*

$$K(Q_k, P_i) \geqq K_i = K(Q_i, P_i). \tag{3.5}$$

Proof. Assume first that $L_0=0$. Then

$$\rho_\alpha(K_1, \ldots, K_l, 0) \leqq \max_k \inf_{s>0} e^{-sK_\alpha} E_\alpha^P [f_k(X)/g_\alpha(X)]^s \leqq \exp(-K_\alpha).$$

But because of (2.7)

$$\rho_\alpha(K_1, \ldots, K_l, 0) \geqq \exp(-K_\alpha)$$

so that (3.3) is met and an adaptive test exists.

Because of Theorem 5.1 of Groeneboom et al. (1979) we have

$$\rho_\alpha(K_1, \ldots, K_l, L_0) = \max_{1 \leqq k \leqq l} \exp\{-\inf_{Q \in \mathcal{Q}_k} K(Q, P_\alpha)\},$$

where

$$\mathcal{Q}_k = \{Q\colon E^Q \log(f_k(X)/g_i(X)) \geqq K_i + L_0, i=1, \ldots, l\}.$$

The definition of L_0 implies that $Q_k \in \mathcal{Q}_k$ for all k. Therefore

$$\rho_\alpha(K_1, \ldots, K_l, L_0) \geqq \max_{1 \leqq k \leqq l} \exp\{-K(Q_k, P_\alpha)\}. \tag{3.6}$$

Now if an adaptive test exists then (3.3) holds and for any $i=1, \ldots, l$ (3.6) implies that

$$\exp(-K_i) \geqq \max_{1 \leqq k \leqq l} \exp\{-K(Q_k, P_i)\}$$

or

$$K_i \leqq \min_k K(Q_k, P_i),$$

which is equivalent to (3.5).

Thus an adaptive test exists if and only if the discrimination $K(Q_i, P_i)$ between members of one family does not exceed the discrimination $K(Q_k, P_i)$, $k \neq i$, between members of two different families.

It is easy to see that if condition (3.5) is met then the test with critical region of the form

$$\left\{\max_\alpha \sum_i^n \log f_\alpha(x_j) \geqq \max_\alpha \left[nK_\alpha + \sum_i^n \log g_\alpha(x_j)\right] - n \max_\alpha K_\alpha\right\}$$

is adaptive. This is a modified maximum likelihood ratio test with weights of the values of the nuisance parameter α proportional to $\exp(nK_\alpha)$.

Notice that the traditional maximum likelihood ratio test with critical region of the form

$$\left\{\max_\alpha \sum_1^n \log f_\alpha(x_j) \geqq c_n \max_\alpha \sum_1^n \log g_\alpha(x_j)\right\}$$

does not have to be adaptive. Moreover it can fail to be adaptive even when adaptive tests exist.

4. Example

Let distributions P_k and Q_k be members of an exponential family over Euclidean space, i.e., the densities f_k and g_k have the form

$$f_k(x)=\exp\{\xi_k' x - X(\xi_k)\},$$
$$g_k(x)=\exp\{\eta_k' x - X(\eta_k)\}, \qquad k=1, \ldots, l.$$

An easy calculation shows that

$$K(Q_\alpha, P_i)=X(\eta_i)-X(\xi_\alpha)+(\xi_\alpha-\eta_i)' \nabla X(\xi_\alpha),$$

where ∇X denotes the vector of partial derivatives of the function X. In particular

$$K_\alpha=X(\eta_\alpha)-X(\xi_\alpha)+(\xi_\alpha-\eta_\alpha)' \nabla X(\xi_\alpha).$$

Thus an adaptive test exists if and only if

$$X(\eta_\alpha)-X(\xi_\alpha)+(\xi_\alpha-\eta_\alpha)' \nabla X(\xi_\alpha)=\min_k [X(\eta_\alpha)-X(\xi_k)+(\xi_k-\eta_\alpha)' \nabla X(\xi_\alpha)] \qquad (4.1)$$

for all $\alpha=1, \ldots, l$.

For instance, if f_k and g_k are multivariate normal densities with means θ_k and μ_k respectively and common covariance matrix Σ, then

$$X(\xi)=\xi' \Sigma \xi/2.$$

Condition (4.1) means that

$$\min_{k,i} [(\xi_k-\eta_i)' \Sigma(\xi_k-\eta_i)-(\xi_i-\eta_i)' \Sigma(\xi_k-\eta_i)]=0,$$

where $\xi_k=\Sigma^{-1}\theta_k$, $\eta_k=\Sigma^{-1}\mu_k$. Thus an adaptive test exists if and only if for any i

$$(\theta_i-\mu_i)' \Sigma^{-1}(\theta_i-\mu_i)=\min_k (\theta_k-\mu_i)' \Sigma^{-1}(\theta_k-\mu_i).$$

As another specification of (4.1) let us consider the case when f_k and g_k are univariate normal densities with parameters θ_k, σ_k and μ_k, τ_k respectively. Then $\xi_k=(\sigma_k^{-2}, \theta_k\sigma_k^{-2})$, $\eta_k=(\tau_k^{-2}, \mu_k\tau_k^{-2})$ and for $\xi=(v, z)$, $v>0$

$$x(\xi)=[z^2 v^{-1}-(\log v)]/2.$$

An easy calculation shows that (4.1) means that for all $\alpha=1, \ldots, l$

$$\begin{aligned}&[(\mu_\alpha^2\tau_\alpha^{-2}-\theta_\alpha^2\sigma_\alpha^{-2})/2-\log(\sigma_\alpha/\tau_\alpha)-\sigma_\alpha^2(\sigma_\alpha^{-2}-\tau_\alpha^{-2})/2+\theta_\alpha^2(\sigma_\alpha^{-2}+\tau_\alpha^{-2})-\theta_\alpha\mu_\alpha/\tau_\alpha^2]\\ &\quad=\min_k[(\mu_\alpha^2\tau_\alpha^{-2}-\theta_k^2\sigma_k^{-2})/2-\log(\sigma_k/\tau_\alpha)-\sigma_k^2(\sigma_k^2-\tau_\alpha^{-2})/2\\ &\qquad+\theta_k^2(\sigma_k^{-2}+\tau_\alpha^{-2})-\theta_k\mu_\alpha/\tau_\alpha^2].\end{aligned}$$

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