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Probability Theory and Related Fields

Remark on Exit Times from Cones in $\mathbb{R}^n$ of Brownian Motion

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Summary. Our purpose is to show how the asymptotics in Corollary 1.3 of [2] can be obtained under much weaker hypotheses. It turns out the problem essentially reduces to showing that if $R(s)$ is a Bessel process, $u>0$ and $\alpha>0$, then $P\left(\int_0^t R(s)^{-2}ds \leqq u\right)=O(t^{-\alpha})$ as $t\to\infty$. We provide a simple proof of this fact.

Let Θ_t be Brownian motion on S^{d-1}, the unit sphere in $\mathbb{R}^d$. Thus Θ_t is governed by half the Laplace-Beltrami operator $L_{S^{d-1}}$ on S^{d-1}. $L_{S^{d-1}}$ is a nice elliptic operator that is self-adjoint with respect to normalized area measure $d\sigma$ on S^{d-1}. Thus for any open $D\subseteq\bar{D}\subseteq S^{d-1}\backslash(0,\ldots,0,1)$ with regular boundary, we get for $y\in D$ and $\eta_D=\inf\{t>0\colon \Theta_t\notin D\}$,

$$P_y(\eta_D>t)\sim e^{-\lambda_D t/2}\,m_D(y)\int_D m_D\,d\sigma \qquad \text{as } t\to\infty. \tag{1.1}$$

Here (λ_D, m_D) is the first eigenvalue-eigenfunction pair of $L_{S^{d-1}}$ on D with Dirichlet boundary conditions. See Port and Stone [4], pp. 121–127.

The Brownian motion B_t on $\mathbb{R}^d$ can be represented by the skew product $(R(t), \Theta(T(t)))$, where $R(t)$ is a Bessel process with parameter d and generator $\frac{1}{2}\left(\frac{d^2}{dr^2}+\frac{d-1}{r}\frac{d}{dr}\right)$, independent of $\Theta(t)$ and $T(t)=\int_0^t R(s)^{-2}ds$ (see Itô and McKean [3], §7.15). Thus for any open cone $C\subseteq\mathbb{R}^d$ with vertex 0, for $G=C\cap S^{d-1}$, and for $\tau_C=\inf\{t>0\colon B_t\notin C\}$ we have

$$\begin{aligned} P_x(\tau_C>t) &= P_x(\Theta(T(s))\in G \quad \text{for all } s\in[0,t]) \\ &= P_x(\eta_G>T(t)) \\ &= \int_0^\infty P_x(\eta_G>u)\,d_u P_x(T(t)\leqq u). \end{aligned} \tag{1.2}$$

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Let C_θ be a right circular cone (with vertex 0) of angle $\theta \in (0, \pi)$. If

$$\gamma(\lambda) := \{-(d-2) + [(d-2)^2 + 4\lambda]^{1/2}\}/2 \tag{1.3}$$

then by Theorem 1.2 and Corollary 1.3 in [2] and the application in Burkholder [1], pp. 192–193, the mapping

$$\theta \in (0, \pi) \to a_\theta := \gamma(\lambda_\theta) \tag{1.4}$$

is strictly decreasing and continuous with range $(0, \infty)$. Moreover

$$P_x(\tau_\theta > t) \sim B_\theta [|x|^{-2} t]^{-a_\theta/2} m_\theta(x/|x|) \quad \text{as } t \to \infty \tag{1.5}$$

where we have replaced C_θ subscripts by θ and

$$B_\theta = 2^{-a_\theta/2} \frac{\Gamma((a_\theta + d)/2)}{\Gamma(a_\theta + d/2)} \int_{S^{d-1} \cap C_\theta} m_\theta \, d\sigma. \tag{1.6}$$

We use this to study $P_x(T(t) \leqq u)$.

Lemma. *For any $\alpha > 0$ and $u > 0$,*

$$P_x(T(t) \leqq u) = O(t^{-\alpha}) \quad \textit{as } t \to \infty.$$

Proof. Choose $\theta \in (0, \pi)$ such that $\gamma(\lambda_\theta) = 2\alpha$ and let $G_\theta = C_\theta \cap S^{d-1}$. By rotational invariance we may assume $x \in C_\theta$. In what follows, A will be constant independent of t which may change from line to line. As $t \to \infty$,

$$\begin{aligned}
P_x(T(t) \leqq s) &\leqq e^{\lambda_\theta s/2} E_x e^{-\lambda_\theta T(t)/2} \\
&= A \int_0^\infty e^{-\lambda_\theta u/2} \, d_u P_x(T(t) \leqq u) \\
&\leqq A \int_0^\infty P_x(\eta_{G_\theta} > u) \, d_u P_x(T(t) \leqq u) \quad \text{(by (1.1))} \\
&= A P_x(\tau_\theta > t) \quad \text{(by (1.2))} \\
&\leqq A t^{-a_\theta/2} \quad \text{(by (1.5))} \\
&= A t^{-\alpha}
\end{aligned}$$

as desired. □

With this it is easy to obtain the following result.

Theorem. *Let $G \subseteq \bar{G} \subseteq S^{d-1} \backslash (0, \ldots, 0, 1)$ have a regular boundary. Then for the cone $C = \bigcup_{r>0} rG$ and $x \in C$,*

$$P_x(\tau_C > t) \sim B(x) t^{-a_G/2}$$

where $a_G = \gamma(\lambda_G)$ *and*

$$B(x) = \frac{\Gamma((a_G+d)/2)}{\Gamma(a_G+d/2)} [|x|^2/2]^{a_G/2} m_G(x/|x|) \int_G m_G \, d\sigma.$$

Moreover if we replace B_t *by the solution* X_t *to* $dX_t = |X_t|^\beta dB_t$, $\beta \neq 1$, *then*

$$P_x(\tau_c > t) \sim \begin{cases} B_\beta(x) t^{-a_G/2(1-\beta)} & \text{if } \beta < 1 \\ B_\beta(x) t^{(2-d-a_G)/2(\beta-1)} & \text{if } \beta > 1 \end{cases} \quad \text{as } t \to \infty$$

where a_G *is as before and*

$$\begin{aligned} B_\beta(x) &= \frac{\Gamma((a_G+d-2\beta)/2(1-\beta))}{\Gamma((2a_G+d-2\beta)/2(1-\beta))} \\ &\quad \cdot \{|x|^{2(1-\beta)}/2(1-\beta)^2\}^{a_G/2(1-\beta)} m_G(x/|x|) \int_G m_G \, d\sigma \qquad \text{if } \beta < 1, \\ &= \frac{\Gamma((a_G+2\beta-2)/2(\beta-1))}{\Gamma((2a_G+2\beta+d-4)/2(\beta-1))} \\ &\quad \cdot \{2(\beta-1)^2 |x|^{2(\beta-1)}\}^{(2-d-a_G)/2(\beta-1)} m_G(x/|x|) \int_G m_G \, d\sigma \qquad \text{if } \beta > 1. \end{aligned}$$

Proof. By the Lemma, if f and g are bounded on $[0, \infty)$, where $f \sim g \sim e^{-\lambda u}$ as $u \to \infty$ for some $\lambda > 0$ and $Ef(T(t))$ decays polynomially in t, then $Ef(T(t)) \sim Eg(T(t))$ as $t \to \infty$. Then to prove the Theorem for the Brownian motion case, just chose $\theta \in (0, \pi)$ such that $a_\theta = \gamma(\lambda_\theta) = \gamma(\lambda_G) = a_G$ and use (1.1), (1.2) and (1.5).

In the case of the diffusion X_t, see now that X_t may be represented as the skew product $(\tilde{R}(t), \Theta(\tilde{T}(t)))$ where $\tilde{R}(t)$ is the diffusion on $(0, \infty)$ with generator $\frac{r^{2\beta}}{2}\left(\frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr}\right)$ independent of $\Theta(t)$ and $\tilde{T}(t) = \int_0^t R(s)^{2\beta-2} ds$. As in the Brownian case, the theorem reduces to the lemma with $\tilde{T}(t)$ replacing $T(t)$. To prove the Lemma for $\tilde{T}(t)$, as before it suffices to get the asymptotics of $P_x(\tau_\theta > t)$ (right circular cones) for X_t. With polar coordinates (r, θ) and a change of variables $s = r^{-2(1-\beta)}t$, separation of variables easily yields the asymptotic behavior of $P_x(\tau_\theta > t)$ as $t \to \infty$. □

References

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