THE WKB APPROXIMATION AND THE PLASMA RING OF JUPITER

(*Letter to the Editor*)

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Abstract. In this note we study the behaviour of hydromagnetic oscillations along the field lines of Jupiter's magnetosphere crossing the Io plasma ring. We compare the shape and period of these oscillations, as found by a direct numerical calculation, with those obtained with the WKB method, in order to show the unaccuracy of this approximation.

1. Introduction

Asymptotic solutions for the equations of Mathematical Physics usually work much better than one has a right to expect; although one may cunningly choose the coefficients to show the limitations of these methods, the fact remains that in most cases the difference between the true and the approximated solutions is much smaller than the a priori bounds given in the books. The purpose of this note is to study and actual and up-to-date phenomenon where the WKB approximation fails completely in its predictions. We think examples of this sort are a useful companion to the more numerous ones where everything works beautifully, to avoid the student being misguided into blind confidence. Low-frequency pulsations of the Earth's magnetosphere were observed as early as 1859 and identified later by Dungey (1954) as standing modes of the linearized magnetohydrodynamic equations along a field line anchored in the ionosphere. Later work confirmed and clarified this model (cf. Chen and Hasegawa, 1974; Lanzerotti and Southwood, 1979; Warner and Orr, 1979; Kivelson *et al.*, 1984). In Warner and Orr (1979) or Kivelson *et al.* (1984), for instance, a simplified model and the WKB approximation were used to obtain satisfactory values for the periods of the pulsations. Thanks to the Voyager spacecraft measurements, this interpretation has been extended to other magnetospheres, notably to Jupiter's (cf. Glassmeier *et al.*, 1989). A special feature of Jupiter is the plasma ring which surrounds the planet around the orbit of Io, where material density is roughly 700 times greater than elsewhere. Hydromagnetic pulsations associated to field lines crossing this torus are strongly influenced by it, making, as we shall see, useless the WKB method to obtain their periods.

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2. The Physical Background

Instead of discussing at length how the linearized magnetohydrodynamic equations apply to magnetospheric oscillations, we start from the equations in Singer *et al.* (1981) and refer the reader to the already mentioned papers for full details. For small perturbations $\mathbf{b}$ of the magnetic field orthogonal to the equilibrium $\mathbf{B}_0$, the component ξ_α of the material displacement in the direction of the vector field α satisfies the second-order equation

$$\frac{\partial^2}{\partial s^2}\left(\frac{\xi_\alpha}{h_\alpha}\right) + \frac{\partial}{\partial s}(\log(h_\alpha^2 B_0))\frac{\partial}{\partial s}\left(\frac{\xi_\alpha}{h_\alpha}\right) + \frac{\mu_0 \rho \omega^2}{B_0^2}\left(\frac{\xi_\alpha}{h_\alpha}\right) = 0, \tag{1}$$

where ρ is the density, ω the time frequency, s the arc length along the field line and h_α a geometrical factor measuring the separation between adjacent field lines in the α-direction. The perturbed field b_α may be obtained from the formula

$$b_\alpha = h_\alpha B_0 \frac{\partial}{\partial s}\left(\frac{\xi_\alpha}{h_\alpha}\right),$$

and the plasma velocity, from

$$u_\alpha = i\omega\xi_\alpha .$$

We shall use a dipole field as a model for the field lines crossing the Io plasma torus. Therefore (cf. Jackson, 1975), we have

$$\mathbf{B}_0 = M\left(\frac{2\cos\theta}{r^3}, \frac{\sin\theta}{r^3}\right),$$

where M is Jupiter's magnetic moment and θ the colatitude. The field lines satisfy $r = L\sin^2\theta$, r being the radial distance from the dipole, L the distance at the Equator. In this case the factor h_α for the lines in the same magnetic shell, corresponding to the so-called toroidal oscillations, is $r\sin\theta$, whereas for lines in the same meridian plane (poloidal oscillations) its value is

$$\frac{1}{rB_0\sin\theta}.$$

The field lines are anchored in the ionosphere of Jupiter at an angle θ such that

$$\sin\theta = \sqrt{\frac{1}{L}}.$$

Since the conductivity of the ionosphere is practically infinite in comparison with the outer regions, the natural boundary condition is to set there the displacement as zero. The plasma ring of Io lies at a distance of 6 Jovian radii from the planet; the density along a field line with $L = 6$ has a Gaussian distribution which drops

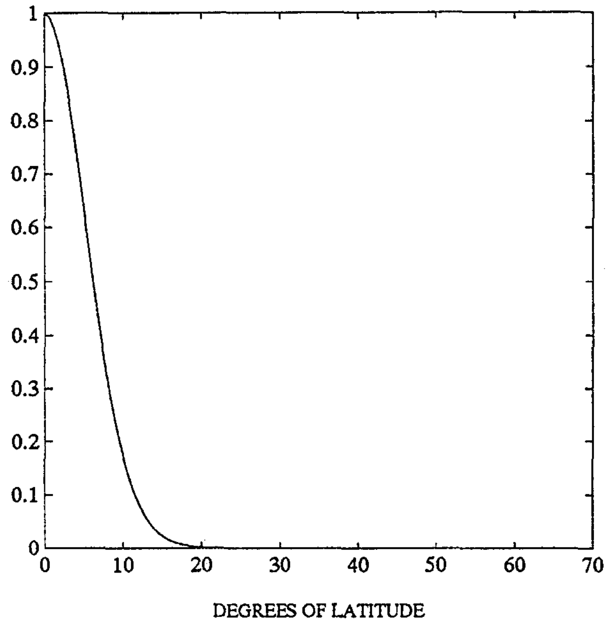

Fig. 1. Density along a field line with $L = 6$.

to (1/3700) of its value at the Equator at a latitude of 20°, and remains practically constant in the remaining field line. An adequate model for the density is therefore

$$\rho = \frac{1}{3700} + e^{20(\sin^6\theta - 1)}$$

(see Fig. 1).

3. Discussion of the Problem

Substituting $z = \cos\theta$ in Equation (1), and taking the above given values for h_α and B_0, we obtain that

$$u = \frac{\xi_\alpha}{h_\alpha}$$

satisfies the equations

$$\frac{d^2u}{dz^2} - \frac{6z}{1 + 3z^2}\frac{du}{dz} + \lambda^2\rho(1 - z^2)^6 u = 0 \qquad (2)$$

for the poloidal oscillations, and

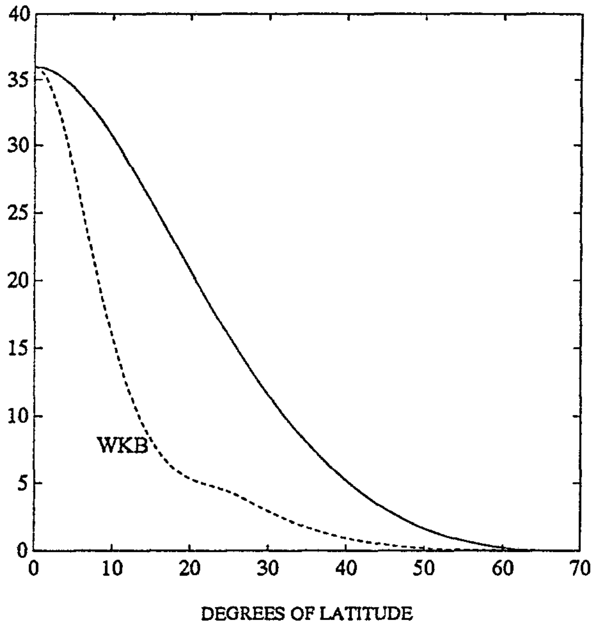

Fig. 2. First eigenmode. The dotted line is the WKB approximation.

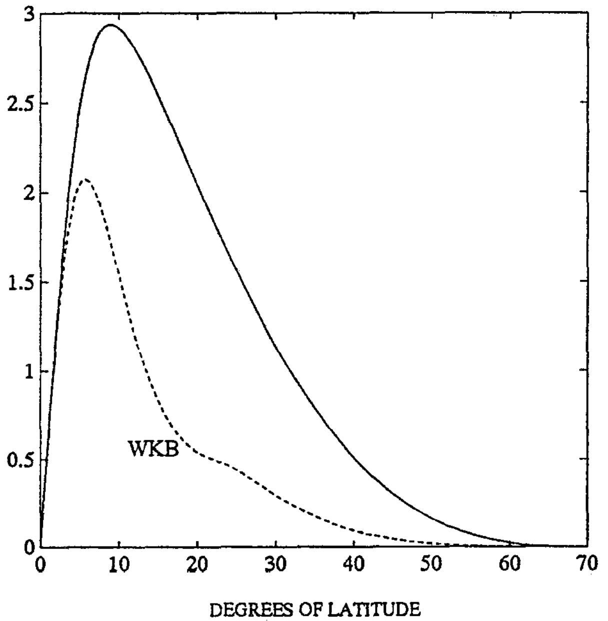

Fig. 3. Second eigenmode.

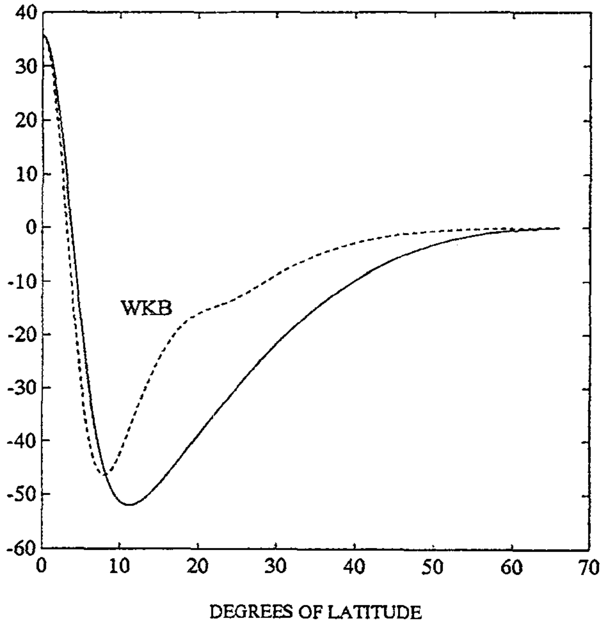

Fig. 4. Third eigenmode.

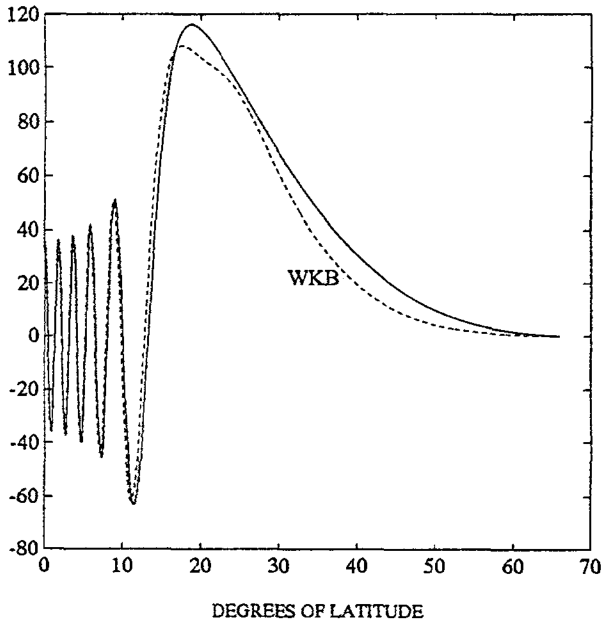

Fig. 5. Twenty-first eigenmode.

$$\frac{d^2u}{dz^2} + \lambda^2 \rho (1 - z^2)^6 u = 0 \tag{3}$$

for the toroidal ones, where

$$\lambda = \frac{\omega L^4}{M} \sqrt{\mu_0}\,.$$

The WKB approximation (see for example, Olver, 1974 or Bender and Orszag, 1984) to the displacement ξ is the same for both (2) and (3): i.e.,

$$\xi \sim \frac{c_1}{\rho^{1/4}} \cos\left(\lambda \int_0^z (1 - z^2)^3 \sqrt{\rho}\, \mathrm{d}z)\right) + \frac{c_2}{\rho^{1/4}} \sin\left(\lambda \int_0^z (1 - z^2)^3 \sqrt{\rho}\, \mathrm{d}z\right); \tag{4}$$

and so the eigenvalues for the solution vanishing in

$$\cos \theta = \pm \sqrt{1 - 1/L}$$

are

$$\lambda_n = \frac{n\pi}{2 \int_0^{\bar{z}} (1 - z^2)^3 \sqrt{\rho}\, \mathrm{d}z}\,, \quad n = \pm 1, \pm 2, \ldots$$

The first eigenvalues are given by

$$\lambda_1 = 9.849917135883,$$
$$\lambda_2 = 19.699834271767,$$
$$\lambda_3 = 29.549751407650.$$

On the other hand, Equations (2) and (3) have been solved numerically by an adaptative Runge-Kutta 7–8 method, combined with the Müller algorithm to find the solutions vanishing at the boundaries. The first eigenvalues are, for the poloidal case

$$\lambda_1^p = 2.426700796652,$$
$$\lambda_2^p = 13.982085660047,$$
$$\lambda_3^p = 24.330132731001;$$

and for the toroidal one

$$\lambda_1^t = 3.336848012754,$$
$$\lambda_2^t = 14.274714553018,$$
$$\lambda_3^t = 24.524602360100;$$

whereas the graphics of the real eigenmodes, as compared with the WKB ones, are given in Figures 2 to 4. Both the eigenvalues and the eigenmodes agree better as λ grows, as it should be; for instance $\lambda_{21} = 206.84825985355$, $\lambda_{21}^p =$

204.12738958705 (see Fig. 5); but in most natural phenomena only the first harmonics are recognizable, and the same happens in this case (cf. Glassmeier *et al.*, 1989). The most spectacular disagreement lies in the fundamental mode: the WKB approximation gives an artifically low-period (high frequency) oscillation, whereas the real one has a period almost four times higher. Also the WKB eigen-oscillations are more closely confined to the plasma torus, and an apparent inflexion occurs near its boundary, which, however, is totally spurious. The approximation is worse at the areas of high density gradient, as one could guess beforehand: there the coefficients differ more markedly from constants. Obviously an experimenter trusting the WKB model would be driven to false expectations.

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