FURTHER CONSIDERATIONS ON CONTRACTING SOLAR NEBULA

J. J. RAWAL
Nehru Planetarium, Nehru Centre, Worli, Bombay, India

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Abstract. Prentice (1978a) in his modern Laplacian theory of the origin of the solar system has established the scenario of the formation of the solar system on the basis of the usual laws of conservation of mass and angular momentum and the concept of supersonic turbulent convection that he has developed. In this, he finds the ratio of the orbital radii of successively disposed gaseous rings to be a constant $\simeq 1.69$. This serves to provide a physical understanding of the Titius–Bode law of planetary distances. In an attempt to understand the law in an alternative way, Rawal (1984) starts with the concept of Roche limit. He assumes that during the collapse of the solar nebula, the halts at various radii are brought about by the supersonic turbulent convection developed by Prentice and arrives at the relation: $R_p = R_\odot a^p$, where R_p are the radii of the solar nebula at various halts during the collapse, $R_\odot$ the radius of the present Sun and $a = 1.442$. 'a' is referred here as the Roche constant. In this context, it is shown here that Kepler's third law of planetary system assumes the form: $T_p = T_0(a^{3/2})^p$, where T_p are the orbital periods at the radii R_p, $T_0 \simeq 0.1216\,\mathrm{d} \simeq 3\,\mathrm{h}$, and a the Roche constant. We are inclined to interpret 'T_0' to be the rotation period of the Sun at the time of its formation when it attained the present radius. It is also shown that the oribital periods T_p corresponding to the radii R_p submit themselves to the Laplace's resonance relation.

1. Introduction

Since the time Copernicus discovered that the planets revolve around the Sun, astronomers have been trying to understand the origin of the solar system. Numerous theories for the origin of the solar system have so far been advanced (ter Haar and Cameron, 1963; ter Haar, 1967; Williams and Cremine, 1968; Woolfson, 1969; Alfvén and Arrhenius, 1970a,b; Nieto, 1972; Reeves, 1978; and Prentice, 1978a, b). Among all these theories of the formation of the solar system, Laplace's nebular hypothesis is favoured (see Reeves, 1978; Rawal, 1984). However, it faces few problems (for full details, see above mentioned references). The difficulties faced by Laplacian hypothesis are considered by Prentice (1978a, b). He presents an outline of the Laplacian theory, which he calls 'modern Laplacian theory' for the origin of the solar system. He considers the influence of a supersonic turbulent stress on the cloud and shows how this stress leads to the formation and detachment of a discrete system of gaseous rings, the ratio of the orbital radii R_p of successively disposed gaseous rings being a constant forming a geometric progression similar to the Titius–Bode law of planetary distances (ter Haar, 1950; Dermott, 1968; Nieto, 1972; and Rawal, 1978). To be more precise, on the basis of supersonic turbulent convection and the usual laws of conservation of mass and angular momentum, Prentice, in his modern Laplacian theory, gets the ratio of the orbital radii R_p of successively disposed gaseous rings to be a constant given by

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TABLE I

R_p, the radius of the contracting Solar nebula in units of AU	Annular ring (R_{p-1}, R_p)	Width $R_p - R_{p-1}$ of the annular ring (R_{p-1}, R_p) in units of AU	Mean radius of the annular ring (R_{p-1}, R_p) in units of AU	Known Object in the annular ring (R_{p-1}, R_p)	Observed mean distance of the known object in the annular ring (R_{p-1}, R_p) in units of AU
$R_\odot = 0.004652$	$(R_\odot, R_1)$	0.0024	0.006	–	–
$R_1 = 0.007$	(R_1, R_2)	0.003	0.008	–	–
$R_2 = 0.01$	(R_2, R_3)	0.004	0.012	–	–
$R_3 = 0.014$	(R_3, R_4)	0.006	0.017	–	–
$R_4 = 0.02$	(R_4, R_5)	0.01	0.025	–	–
$R_5 = 0.03$	(R_5, R_6)	0.012	0.036	–	–
$R_6 = 0.042$	(R_6, R_7)	0.018	0.051	–	–
$R_7 = 0.06$	(R_7, R_8)	0.027	0.073	–	–
$R_8 = 0.087$	(R_8, R_9)	0.043	0.11	–	–
$R_9 = 0.13$	(R_9, R_{10})	0.05	0.155	–	–
$R_{10} = 0.18$	(R_{10}, R_{11})	0.08	0.22	–	–
$R_{11} = 0.26$	(R_{11}, R_{12})	0.12	0.32	–	–
$R_{12} = 0.38$	(R_{12}, R_{13})	0.16	0.46	Mercury	0.4
$R_{13} = 0.54$	(R_{13}, R_{14})	0.26	0.67	Venus	0.7
$R_{14} = 0.8$	(R_{14}, R_{15})	0.33	0.965	Earth	1
$R_{15} = 1.13$	(R_{15}, R_{16})	0.47	1.365	Mars	1.5
$R_{16} = 1.6$	(R_{16}, R_{17})	0.75	1.975	Asteroids	–
$R_{17} = 2.35$	(R_{17}, R_{18})	1.05	2.875	Asteroids	2.8
$R_{18} = 3.4$	(R_{18}, R_{19})	1.5	4.15	Asteroids	–
$R_{19} = 4.9$	(R_{19}, R_{20})	2.1	5.95	Jupiter	5.2
$R_{20} = 7$	(R_{20}, R_{21})	3.15	8.575	Saturn	9.5
$R_{21} = 10.15$	(R_{21}, R_{22})	4.5	12.4	Chiron	13.7
$R_{22} = 14.65$	(R_{22}, R_{23})	6.47	17.88	Uranus	19.2
$R_{23} = 21.12$	(R_{23}, R_{24})	9.34	25.79	Neptune	30.1
$R_{24} = 30.46$	(R_{24}, R_{25})	13.44	37.18	Pluto	39.4
$R_{25} = 43.9$					

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$$R_p/R_{p+1} = \left[1 + \frac{m}{Mf}\right]^2 = \text{const.} \simeq 1.69, \tag{1.1}$$

where m is the mass of a disposed ring, M the remaining mass of protosolar nebula and, f, the moment-of-inertia coefficient.

In an attempt to understand the Titius–Bode law in an alternative way, Rawal (1984) starts with the concept of Roche limit. He assumes that during the collapse of the solar nebula the halts at various radii are brought about by the supersonic turbulent convection developed by Prentice and arrives at the relation

$$R_p = R_\odot a^p, \tag{1.2}$$

where R_p are the radii of the solar nebula at various halts during the collapse, $R_\odot$ the radius of the present Sun and $a = 1.442$ is referred here as the Roche constant. Rawal (1984) tabulates all R_p (see Table I). He points out that the ring-structure-feature is a common and natural feature of the heavenly bodies, in particular, of the major members of the solar system (also see Rawal, 1981, 1982). He then reconciles his work with that of Prentice and finds that

$$R_p/R_{p+1} = \left[1 + \frac{m}{Mf}\right]^2 = 1.442 = a. \tag{1.3}$$

His discussion supports modern Laplacian theory of Prentice, and in turn, modern Laplacian theory provides an understanding between supersonic turbulent convection and Roche limit, in that, the rotational instability at the equator of the protosolar nebula arises at various stages of its contraction precisely by the step of Roche constant which is the same as Bode's constant of modern Laplacian theory leading to the formation and detachment of a discrete system of gaseous rings, the whole process being controlled by the phenomenon of supersonic turbulent convection. The usefulness of the work being that once the radius of the primary is known, the relation can be set up very simply and uniquely. The discussion could be considered as an alternative way of deriving the Titius–Bode law.

In the present paper, it is proposed to derive and discuss the Kepler's third law of planetary system in relation to this concept. It is also proposed to discuss the resonant structure in the solar system in this context.

2. Kepler's Third Law

A systematic search for regularity in the major satellite systems by Dermott (1968) has revealed that the orbital periods of the regular satellites are closely approximated by the relation

$$T_n = T_0 A^n, \tag{2.1}$$

where T_n is the orbital period of the nth satellite. It must be allowed, though, that in any one system, there are a small number of vacancies. For all systems A is the square root of a small integer and it is suggested that T_0 is related to the rotational period of the primary. The relation can be applied to the planetary system but there are some anomalies. It is also suggested that this regularity, which is related to the preference for near commensurability among pairs of mean motions in the solar system, is a condition of formation rather than the result of evolution and thus could be of considerable cosmogonic importance.

In order that Equation (1.2) derived by Rawal (1984) reconciles with Kepler's third law we have

$$T_p = T_0(a^{3/2})^p, \tag{2.2}$$

where T_p are the orbital periods at the radii R_p, $T_0 \simeq 0.1216\,d \simeq 3$ hours and 'a' the Roche constant. We are inclined to interpret 'T_0' to be the rotation period of the Sun at the time of its formation at the present radius for the reasons given below.

Hoyle (1978) considers the slow spin of the Sun as the most remarkable character of the formation process of the solar system and tries to get an estimate of the final period of rotation of the solar nebula when it has attained the diameter of the present Sun by giving the following order of magnitude condition:

$$\frac{\text{Initial period of rotation}}{\text{Final period of rotation}} \simeq \left(\frac{\text{Initial diameter}}{\text{Final diameter}}\right)^2. \tag{2.3}$$

He estimates the initial diameter of the solar nebula to be of the order 10^{17} cm and the initial period of rotation to be about 3×10^7 yr and arrives at the final period of rotation to be 0.0212 days. He then points out that this rotation period corresponds to the rotation speed of $2.4 \times 10^3\ \mathrm{km\,s^{-1}}$ which is 1200 times the actual rotation speed of the present Sun at the equator and remarks that the Sun could not in fact spin as fast as that. If the actual Sun were made to spin faster and faster, it would become unstable at a rotational speed of about $400\ \mathrm{km\,s^{-1}}$. His calculations indicated a speed about six times faster than this, which means that the solar nebula could not shrink to its present diameter, rotational forces would become dominant at a much larger diameter, in fact, at 6^2 times the present diameter of the Sun.

Larson (1969) in his study of collapse of the solar nebula estimates the initial period of rotation of the cloud to be $\sim 10^9$ yr and its initial diameter to be $\sim 3.2 \times 10^{17}$ cm and arrives at the final period of rotation to be ~ 0.07 days. Other authors (for example, Bok and Reilly, 1947; Prentice and ter Haar, 1971; Mestel, 1977; Prentice, 1978b) in their study of star formation and stellar clouds arrive at more or less the same value for the dimension of a star-forming cloud (solar nebula) but differ in the value of its initial rotation period by an order of magnitude one or two. If we take the initial period of rotation of the solar nebula to be $\sim 3 \times 10^8$ yr and its initial diameter to be $\sim 10^{17}$ cm and recalculate the final period of rotation, then it turns out to be $\sim 0.1216\,d$ as against Hoyle's value of $0.0212\,d$ and Larson's value of $0.07\,d$. The value of the final rotation period calculated by us corresponds to the rotational speed of about $416\ \mathrm{km\,s^{-1}}$ which is just the critical rotational speed. Alternatively, we may employ Kepler's third law to calculate the orbital period of a particle moving around the centre of the nebula at a distance equal to the radius of the Sun and find it to be $\sim 0.1175\,d$. We, therefore, make an assumption here that the Sun at the time of its formation when it attained the present radius was rotating almost at the critical break-up speed of $400\,\mathrm{km\,s^{-1}}$ and interpret $T_0 \simeq 0.1216\,d \simeq 3\,\mathrm{h}$ to be its rotation period at the time.

TABLE II

This table shows various T_p obtained from relation (2.2) of the text. Corresponding R_p are shown in Table I.

T_p	Its period in days or years	T_p	Its period in days or years
T_0	0.1175 d		
T_1	0.2032 d	T_{14}	261.1 d
T_2	0.3524 d	T_{15}	444.1 d
T_3	0.6102 d	T_{16}	769.1 d
T_4	1.057 d	T_{17}	3.735 yr
T_5	1.8300 d	T_{18}	6.320 yr
T_6	3.1700 d	T_{19}	10.9400 yr
T_7	5.4870 d	T_{20}	18.9500 yr
T_8	9.5060 d	T_{21}	32.8200 yr
T_9	16.4600 d	T_{22}	56.8400 yr
T_{10}	28.5100 d	T_{23}	98.4500 yr
T_{11}	49.3800 d	T_{24}	170.5000 yr
T_{12}	85.5100 d	T_{25}	295.2000 yr
T_{13}	148.1000 d		

In Table II are shown various T_p at the corresponding R_p shown in Table I.

Babinet (1861) pointed out that the Sun is spinning far too slowly with a period of some 25 days instead of few hours expected for the Laplacian theory. Present work shows that he was right. At present, there is a growing group of theorists who believe that the interior of the Sun may still be rotating quite rapidly (Dicke, 1970, 1983).

Thus, A – a constant of Dermott having the property that it is the square root of a small integer is replaced by 3/2 power of the Roche constant, and the opinion of Dermott that T_0 is related to the rotational period of primary is replaced by the period of rotation of the primary at the time of its formation.

3. Resonant Structure in the Solar System

Resonant theory in the planetary system states that if n_1, n_2, n_3 ($n_i = 2\pi/T_i$, T_i being the corresponding orbital period $n_1 > n_2 > n_3$), denote the mean motions of three secondaries going around a primary (orbits assumed circular and coplanar), then a necessary condition for the frequent occurrence of mirror configuration (Dermott, 1973; Rawal, 1981; 1982) is of the form

$$\alpha n_1 - (\alpha + \beta)n_2 + \beta n_3 \simeq 0, \tag{3.1}$$

TABLE III

Resonance in the triads of successive R_p shown in Table I. Corresponding T_p are found from the relation (2.2) of the text and shown in Table II.

Triad	RMMR $\beta/(\beta + \alpha)$	Triad	RMMR $\beta/(\beta + \alpha)$
$R_{\odot}$ R_1 R_2	$\simeq 2/3$	R_{13} R_{14} R_{15}	$\simeq 2/3$
R_1 R_2 R_3	$\simeq 2/3$	R_{14} R_{15} R_{16}	$\simeq 2/3$
R_2 R_3 R_4	$\simeq 2/3$	R_{15} R_{16} R_{17}	$\simeq 2/3$
R_3 R_4 R_5	$\simeq 2/3$	R_{16} R_{17} R_{18}	$\simeq 2/3$
R_4 R_5 R_6	$\simeq 2/3$	R_{17} R_{18} R_{19}	$\simeq 2/3$
R_5 R_6 R_7	$\simeq 2/3$	R_{18} R_{19} R_{20}	$\simeq 2/3$
R_6 R_7 R_8	$\simeq 2/3$	R_{19} R_{20} R_{21}	$\simeq 2/3$
R_7 R_8 R_9	$\simeq 2/3$	R_{20} R_{21} R_{22}	$\simeq 2/3$
R_8 R_9 R_{10}	$\simeq 2/3$	R_{21} R_{22} R_{23}	$\simeq 2/3$
R_9 R_{10} R_{11}	$\simeq 2/3$	R_{22} R_{23} R_{24}	$\simeq 2/3$
R_{10} R_{11} R_{12}	$\simeq 2/3$	R_{23} R_{24} R_{25}	$\simeq 2/3$
R_{11} R_{12} R_{13}	$\simeq 2/3$		
R_{12} R_{13} R_{14}	$\simeq 2/3$		

where α and β are small positive integers. It follows from Equations (3.1) that in a reference frame revolving with the mean motion of any one of the three secondaries, the relative mean motions n_i' of the other two are commensurate and that in a frame I (that of the innermost secondary), we have $n_2' = n_1 - n_2$ and $n_3' = n_1 - n_3$ and the ratio of these relative mean motions is given as follows

$$n_2'/n_3' = (n_2 - n_1)/(n_3 - n_1) = \beta/(\beta + \alpha). \tag{3.2}$$

In terms of revolution periods, Equation (3.2) becomes

$$n_2'/n_3' = (T_2 - T_1)T_3/(T_3 - T_1)T_2 = \beta/(\beta + \alpha). \tag{3.3}$$

For a stable three-body resonance, the relative mean motion ratio (RMMR), Equation (3.3), has the value 2/3. This case is called Laplace's resonance relation and three successive orbits following this relation represent stable motion.

Here we apply the resonance theory among R_p. Results are shown in Table III. From Table III, we see that R_p submit themselves to the Laplace's resonance relation without any exception. Rawal (1981) has discussed the resonance theory among all the members of the solar system and concluded that all the planets of the solar system follow the Laplace's resonance relation. Here it is found that all R_p of the Equation (1.2) are in agreement with his aforesaid conclusion regarding planets.

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