Touching Triangle Representations for 3-Connected Planar Graphs

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Abstract. A touching triangle graph (TTG) representation of a planar graph is a planar drawing Γ of the graph, where each vertex is represented as a triangle and each edge e is represented as a side contact of the triangles that correspond to the end vertices of e. We call Γ a proper TTG representation if Γ determines a tiling of a triangle, where each tile corresponds to a distinct vertex of the input graph. In this paper we prove that every 3-connected cubic planar graph admits a proper TTG representation. We also construct proper TTG representations for parabolic grid graphs and the graphs determined by rectangular grid drawings (e.g., square grid graphs). Finally, we describe a fixed-parameter tractable decision algorithm for testing whether a 3-connected planar graph admits a proper TTG representation.

1 Introduction

Planar graphs are of interest in theory and in practice as they correspond to naturally occurring structures, such as skeletons of convex polytopes and duals of maps, and contain subclasses of interest, such as trees and grids. While traditionally graphs are represented by node-link diagrams, alternative representations also have a long history. There is a large body of work about representing planar graphs as contact graphs, i.e., graphs whose vertices are represented by geometric objects with edges corresponding to two objects touching in some specified fashion. Early results, such as Koebe's 1936 theorem [9] that all planar graphs can be represented by touching disks, deal with *point contacts*. Similarly, de Fraysseix *et al.* [5] construct representation of planar graphs with vertices as triangles, where the edges correspond to point contacts between triangles.

In this paper, we consider *side contact* representations of graphs, where vertices are represented by simple polygons, with an edge occurring whenever two polygons have non-trivially overlapping sides. Side contact representations of planar graphs have been studied by representing vertices with octagons [8,10] and hexagons [6].

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Certain subclasses of planar graphs admit even simpler side contact representations. Buchsbaum *et al.* [2] give an overview on the state of the art concerning rectangle contact graphs, which are often referred to as rectangular layouts. Graphs allowing rectangular layouts have been fully characterized [2,12,13] with linear-time constructive algorithms.

The simplest side-contact representation of a graph, in terms of the complexity of polygons used, is the triangle contact representation. Gansner *et al.* [7] show certain necessary and sufficient conditions for such representations, however a complete characterization turns out to be surprisingly difficult and is not yet known. It is known that every outerplanar graph admits a TTG representation that may not be proper, and every graph that is a weak dual of some maximal planar graph admits a proper TTG representation [7].

In this paper we examine only the proper TTG representations, i.e., the TTG representation must determine a tiling of some triangle and every tile must correspond to a distinct vertex of the input graph; see Figs. 1(a–b). Phillips [11] enumerates all possible tilings of a triangle into five subtriangles, which helps us to list all non-isomorphic connected planar graphs with less than six vertices that do not admit proper TTG representations; see Fig. 1(g).

Our Contributions: We give an algorithm to construct proper TTG representations of 3-connected cubic planar graphs. We then show that parabolic grid graphs and the graphs determined by rectangular grid drawings (e.g., square grid graphs) have proper TTG representations. Finally, we describe a fixed-parameter tractable decision algorithm for testing whether a 3-connected planar graph with n vertices admits a proper TTG representation. We use the maximum degree Δ, the number of outer vertices and the number of inner vertices with degree greater than three as fixed parameters. Specifically, if $\Delta = 4$, then this can be done in $O^*(4^{k_1}6^{k_2})$ time[1], where k_1 is the number of degree-4 inner vertices and k_2 is the number of vertices on the outerface, which results in a polynomial-time algorithm when $k_1 + k_2 = O(\log n)$.

2 Preliminaries

A *weak dual* of a planar graph G is a subgraph induced by the vertices of the dual graph of G that correspond to the inner faces of G. The weak dual D of every maximal planar graph M is a subcubic planar graph, where only three vertices of D have degree two. Therefore, by definition any straight-line drawing of M is a proper TTG representation of D. Constructing a proper TTG representation for a 3-connected cubic planar graph G may initially seem easy since it differs from the dual of a maximal planar graph by only one vertex. But a careful look at Figs. 1(c–f) reveals that it is not obvious how to construct a proper TTG representation of a 3-connected cubic planar graph from its corresponding maximal planar graph.

A *straight-line drawing* Γ of a planar graph G is a planar drawing of G, where each vertex is drawn as a point and each edge is drawn as a straight line

[1] O^* ignores the polynomial terms [14, Section 2].

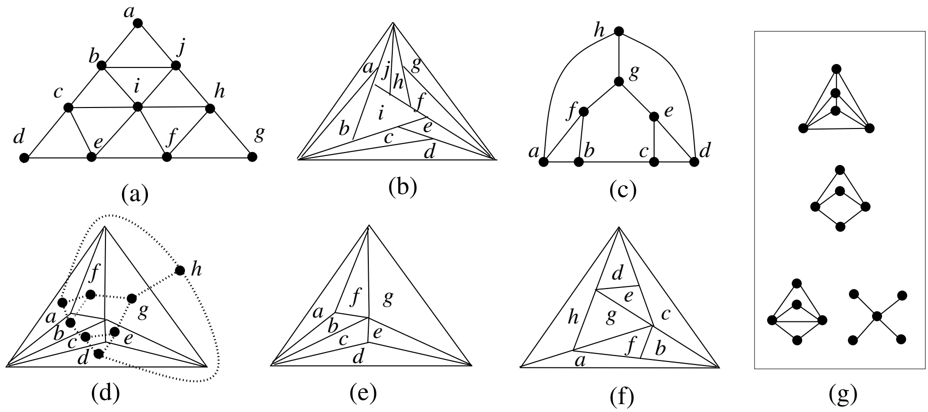

Fig. 1. (a) A planar graph G. (b) A proper TTG representation of G. (c) A 3-connected cubic planar graph G'. (d) The dual graph M of G', where G is shown in dotted lines. (e) A straight-line drawing of M is a proper TTG representation of its own weak dual. (f) A proper TTG representation of G'. (g) All planar graphs with less than six vertices that do not admit proper TTG representations.

segment. A path $v_1, v_2, \ldots, v_k$ is *stretched* in Γ if all the vertices on the path are collinear in Γ. Two paths are *non-crossing* if they do not have an internal vertex in common. A *path covering* of G is an edge covering of G by non-crossing edge-disjoint paths.

Theorem 1 (de Fraysseix and de Mendez [3]). *A path covering $\mathcal{P}$ of a plane graph $\mathcal{G}$ is stretchable if and only if each subset $\mathcal{S}$ of $\mathcal{P}$ with at least two paths has at least three free vertices, where a free vertex in the graph H induced by $\mathcal{S}$ is a vertex on the outerface of H that is not internal to any path of $\mathcal{S}$.*

By a *k-cycle* in G we denote a cycle of k vertices in G. By len(f) we denote the length (i.e., the number of vertices on the boundary) of a face f of G.

Throughout the paper we only examine the proper touching triangle representations. We also assume that the combinatorial embedding of the input graph is fixed, i.e., a *plane graph*.

3 Proper TTG Representations of Cubic Graphs

In this section we describe an algorithm for constructing a proper TTG representation of a 3-connected cubic planar graph. In particular, every 3-connected cubic planar graph G can be constructed starting with a K_4 and then applying one of the three "growth" operations [1]; see Figs. 2(a–c). We use this inductive construction of G to construct its TTG representation. While constructing G, we maintain a plane graph G' that corresponds to the TTG representation of G. We also define a path covering $P(G')$ of G' such that any planar embedding of G' with every path in $P(G')$ stretched, is a TTG representation of G.

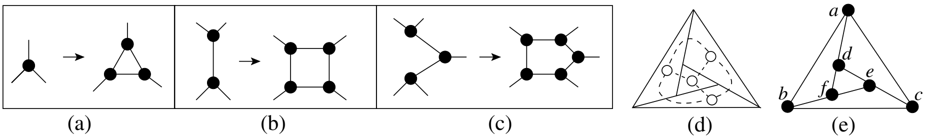

Fig. 2. (a–c) Growth operations 1–3; (d) G, its proper TTG representation; (e) G'

We start with $G = K_4$, and the graph G' that corresponds to the TTG representation of G; see Figs. 2(d–e). Throughout the algorithm G' will have exactly three inner faces incident to its three outer edges, each of which is a 4-cycle. We call these faces the *quads* of G'. For every quad we will define a *stick*, which is a path of three vertices on the corresponding quad. No two sticks in G' will have an edge in common. All the inner faces of G' other than the quads will be 3-cycles, which we call the *ordinary faces*.

In Fig. 2(e), the 4-cycles $[a, b, f, d]$, $[b, c, e, f]$ and $[c, a, d, e]$ are the quads of G', where $\langle a, d, f\rangle$, $\langle b, f, e\rangle$ and $\langle c, e, d\rangle$ are their sticks, respectively. The *path covering* $P(G')$ consists of the sticks and all the edges of G' that are not covered by the sticks, i.e., $P(G') = \{\langle a, d, f\rangle, \langle b, f, e\rangle, \langle c, e, d\rangle, \langle a, b\rangle, \langle b, c\rangle, \langle c, a\rangle\}$.

Assume inductively that we have a 3-connected cubic planar graph $\mathcal{G}$, its corresponding graph $\mathcal{G}'$ and path covering $P(\mathcal{G}')$, where one of the three growth operations of Figs. 2(a–c) on $\mathcal{G}$ produces the input graph G. In Lemmas 1–3 we show how to construct the graph G' and its path covering $P(G')$.

Lemma 1. *Assume that G is produced from $\mathcal{G}$ by an application of Operation 1. Then the graph G' and its path covering $P(G')$ can be constructed by a constant number of insertion/deletion on $\mathcal{G}'$ and $P(\mathcal{G}')$, respectively.*

Proof. First consider the case when vertex v of $\mathcal{G}$, on which we apply Operation 1, corresponds to an ordinary face T of $\mathcal{G}'$. We then add a vertex x inside T and connect the vertex with the three vertices on the boundary of T. Let the resulting graph be G'. It is easy to verify that the vertices on the cycle that replaces v correspond to the three new ordinary faces in G'; see Figs. 3(a–b). The path cover $P(G')$ consists of all the paths of $P(\mathcal{G}')$ along with the three paths that correspond to the three new edges incident to x.

Next consider the case when vertex v of $\mathcal{G}$, on which we apply Operation 1, corresponds to a quad $T = [a, b, c, d]$ of $\mathcal{G}'$. Without loss of generality assume that the stick of T is $\langle a, d, c\rangle$ and the outer edge of T is (b, c). We then add a vertex x inside T and add the edges (a, x), (b, x) and (d, x); see Figs. 3(c–d). Let the resulting graph be G'. The 4-cycle $[b, x, d, c]$ is a quad in G' and $\langle b, x, d\rangle$ is its stick. Since $\mathcal{G}'$ contains exactly three quads, G' also contains exactly three quads (i.e., $[b, x, d, c]$ replaces $[a, b, c, d]$ and all other quads remain the same). The path cover $P(G')$ consists of all the paths of $P(\mathcal{G}') \setminus \langle a, d, c\rangle$ along with the paths $\langle a, d\rangle, \langle d, c\rangle, \langle a, x\rangle, \langle b, x, d\rangle$. □

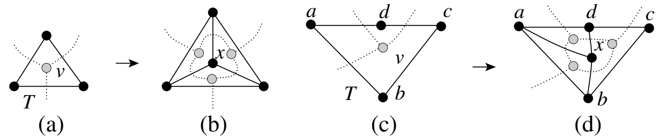

Fig. 3. (a–d) Illustration for Operation 1. $\mathcal{G}$ and G are shown in dotted lines as weak duals of $\mathcal{G}'$ and G', respectively.

Since the path covering $P(G')$ consists of the sticks and all the edges of G' that are not covered by the sticks, in Lemmas 2 and 3, we will only define the sticks in G', instead of defining $P(G')$ explicitly.

Lemma 2. *Assume that G is produced from $\mathcal{G}$ by an application of Operation 2. Then the graph G' and its path covering $P(G')$ can be constructed by a constant number of insertion/deletion on $\mathcal{G}'$ and $P(\mathcal{G}')$, respectively.*

Proof. Assume that the vertices v and u of $\mathcal{G}$ on which we apply Operation 2 correspond to two faces T_1 and T_2 of $\mathcal{G}'$. Then T_1 and T_2 must share an edge, which we denote by e'. We distinguish three cases.

Case 1 (T_1 and T_2 are ordinary faces): Here we subdivide e' with a vertex x and connect x with the vertices on T_1 and T_2 that are not already adjacent to x. The resulting graph is G'; see Figs. 4(a–b). The new faces are ordinary, and hence the quads and sticks of G' coincide with the quads and sticks of $\mathcal{G}'$.

Case 2 (Exactly one of T_1 and T_2 is a quad): Without loss of generality assume that the outer boundary of the union of T_1 and T_2 is a, b, c, d, e, T_1 is the quad and $\langle a, c, d\rangle$ is its stick; see Fig. 4(c). We now subdivide e' with a vertex x. If (d, e) is the outer edge, then we add the edges $(x, b), (x, e)$. Otherwise (a, e) is the outer edge and we add the edges $(x, b), (x, d)$. The resulting graph is G'; see Figs. 4(c)–(f). The quad $[a, c, d, e]$ of $\mathcal{G}'$ does not determine quad for G'. The new quad of G' is $[x, c, d, e]$ (resp., $[a, x, d, e]$), where $\langle x, c, d\rangle$ (resp., $\langle a, x, d\rangle$) is its stick, as shown in Fig. 4(d) (resp., Fig. 4(f)). The four new faces in G' correspond to the four vertices of the cycle that replace u and v of $\mathcal{G}$.

Case 3 (Both T_1 and T_2 are quads): Without loss of generality assume that the outer boundary of the union of T_1 and T_2 is a, b, c, d, e, f, and $\langle a, d, e\rangle$, $\langle b, c, d\rangle$ are the sticks of T_1, T_2, respectively. By induction, every quad in $\mathcal{G}'$ contains an outer edge. Since b and e are distinct vertices, both (a, b) and (e, f) cannot be the outer edges of $\mathcal{G}'$. Consequently, (a, b) and (a, f) are the outer edges of T_1 and T_2, respectively; see Fig. 4(g).

We now subdivide e' with a vertex x and add the edges $(x, c), (x, e)$; see Fig. 4(h). The quads $[a, b, c, d]$ and $[a, d, e, f]$ of $\mathcal{G}'$ are not the quads for G'. The quads of G' are $[a, b, c, x]$ and $[a, x, e, f]$, where $\langle b, c, x\rangle$ and $\langle a, x, e\rangle$ are their corresponding sticks. □

The following lemma examines Operation 3, whose proof is omitted due to space constraints.

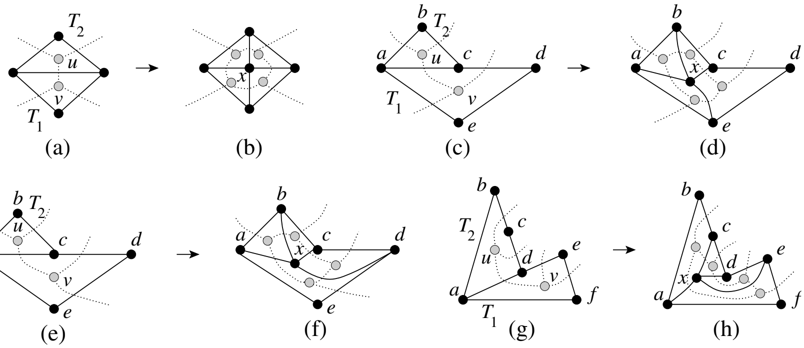

Fig. 4. (a–h) Illustration for Operation 2. $\mathcal{G}$ and G are shown in dotted lines as weak duals of $\mathcal{G}'$ and G', respectively.

Lemma 3. *Assume that G is produced from $\mathcal{G}$ by an application of Operation 3. Then the graph G' and its path covering $P(G')$ can be constructed by a constant number of insertion/deletion on $\mathcal{G}'$ and $P(\mathcal{G}')$, respectively.*

Theorem 2. *Every 3-connected cubic planar graph admits a proper TTG representation.*

Proof. Let G be the input graph. We use Lemmas 1–3 to construct the corresponding graph G' and path covering $P(G')$. Since G' contains G as its weak dual, if G' admits a straight-line drawing Γ, where all the faces are drawn as triangles, then Γ must be a proper TTG representation of G.

By construction G' has exactly three inner faces that are of length four (i.e., the quads). All the other faces are of length three. Consequently, if G' admits a straight-line drawing Γ, then all the inner faces except the three quads must be drawn as triangles. If the three sticks of G' are stretched in Γ, then every face of Γ must be a triangle, and hence Γ must be a proper TTG representation of G. In other words, any planar embedding of G', where every path in $P(G')$ is stretched, must be a proper TTG representation of G.

It now suffices to prove that G' admits a planar embedding, where each path in $P(G')$ is stretchable. It is straightforward to verify that each subset of $P(G')$ with at least two paths has at least three free vertices. Hence by Theorem 1, G' admits a planar drawing, where every path in $P(G')$ is stretched; such a drawing can be computed by solving a barycentric system [3][2]. □

4 Proper TTG Representations of Grid Graphs

In this section we give an algorithm to construct proper TTG representations for square grid graphs and parabolic grid graphs. Note that Gansner *et al.* [7] gave an

[2] The authors believe that instead of relying on de Fraysseix and de Mendez's result [3], one can adapt well known straight-line planar graph drawing techniques (e.g. shift method [4]) to construct such a drawing of G' on an integer grid with small area.

algorithm to construct TTG representations for square grids and its subgraphs, where the outerface takes the shape of an astroid, (also called cubocycloid), and hence the TTG representations were not proper. On the other hand, our algorithm constructs proper TTG representations.

A *square grid graph* $G_{m,n}$, where $m, n \geq 1$, is the graph determined by an integer grid I of dimension $m \times n$. By a vertex $u_{x,y}$ of $G_{m,n}$ we denote the vertex that corresponds to the point (x, y) of I. See Fig. 5(a), where $u_{2,1}$ corresponds to the point c. By $x(v)$ (respectively, $y(v)$) we denote the x-coordinate (respectively, y-coordinate) of the point v. Let $v_1, v_2, \ldots, v_k$ be a polygonal chain such that $x(v_1) < x(v_2) < \ldots < x(v_k)$, $y(v_2) > y(v_3) > \ldots > y(v_k) > y(v_1)$ and $v_2, v_3, \ldots, v_k, v_2$ forms a strictly convex polygon; see Fig. 5(b). We call such a polygonal chain a *ripple of* k *vertices* and denote it by R_k.

Theorem 3. *Every* $G_{m,n}$*,* $m, n \geq 1$*, admits a proper TTG representation.*

Proof. We first construct $G_{m,1}$ as follows. Construct a ripple $R_{m+2} = (v_1, v_2, \ldots, v_{m+2})$. Then add a point b below R_{m+2} and draw straight line segments from b to each vertex in R_{m+2}. We make sure that such that $x(b) = x(v_{m+2}) + \epsilon, \epsilon > 0$, and the drawing is planar. Now add a point t above R_{m+2} with $x(t) = x(v_2)$ and draw straight line segments from t to each vertex in R_{m+2}. We place t with sufficiently large y-coordinate so that the drawing remains planar and the vertices t, v_{m+2}, b become collinear. The resulting drawing is a proper TTG representation of $G_{m,1}$; see Fig. 5(c). Assume inductively that $G_{m,i}$, $i < n$, admits a proper TTG representation such that the following conditions hold.

(a) The topmost vertex t in the drawing is adjacent to a ripple R_{m+2} and the triangles incident to t correspond to the vertices of the ith row of $G_{m,i}$.
(b) The triangle below the edge $(v_j, v_{j+1}), 1 \leq j \leq m+1$, corresponds to the vertex $u_{j-1,i-1}$ of $G_{m,i}$.
(c) The bottommost vertex b of the drawing has the largest x coordinate in the drawing and it is adjacent to the leftmost and the rightmost vertices of R_{m+2}.
(d) One can decrease the y coordinate of b and redraw its adjacent edges to obtain another proper TTG representation of $G_{m,i}$.

The above conditions hold for the base case. We now construct the proper TTG representation of $G_{m,n}$ from the proper TTG representation Γ of $G_{m,n-1}$.

Let R_{m+2}=$(v_1, v_2, \ldots, v_{m+2})$ be the ripple that is adjacent to the topmost vertex t. Delete t from Γ to obtain another drawing Γ'. Now draw another ripple R'_{m+2}=$(v'_1(= v_1), v'_2, \ldots, v'_{m+2}(= v_{m+2}))$ such that $x(v'_2) = x(v_2)$, $y(v'_2) > y(v_2)$ and $v'_j, 2 < j < m+2$, is the midpoint of the line segment $v'_{j-1}v_j$; see Fig. 5(d). The triangles incident to R'_{m+2} correspond to a new row of m vertices, i.e, the $(n-1)$th row $G_{m,n}$. We now add a point t' above R'_{m+2} with $x(t')$=$x(v'_2)$ and draw straight line segments from t' to each vertex in R'_{m+2}. Conditions (c) and (d) help us to install t' with sufficiently large y-coordinate such that the drawing remains planar and the vertices t', v_{m+2} and the bottommost point b become collinear; see Fig. 5(e). Observe that the resulting drawing is the proper TTG representation of $G_{m,n}$ for which the conditions (a)–(d) hold. □

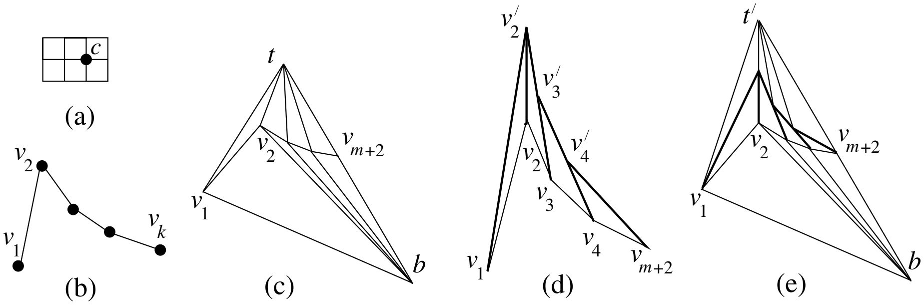

Fig. 5. (a) $G_{3,2}$. (b) R_5. (c) Proper TTG representation of $G_{m,1}$. (d) Construction of the triangles for the $(n-1)$th row of $G_{m,n}$. R'_{m+2} is shown in bold. (e) Installing t'.

A *rectangular grid drawing* $\mathbb{G}_{m,n}$ is a planar drawing of some graph, where each vertex is drawn as a point on the $m \times n$ grid, each edge is drawn as either a horizontal or a vertical straight line segment and each face takes the shape of a rectangle. We now generalize the proof of Theorem 3 to prove the following.

Theorem 4. *Let G be a planar graph with only four vertices of degree two. If G admits a rectangular grid drawing, then G admits a proper TTG representation.*

Proof. Let $\mathbb{G}_{m,n}$, $m, n>1$, be a rectangular grid drawing and let $\mathbb{G}_{m,j}$, $j \le n$, be the subgraph of $\mathbb{G}_{m,n}$ induced by the vertices of the jth row and all the rows below it. A vertex u is *unsaturated* in $\mathbb{G}_{m,j}$ if u has a neighbor in $\mathbb{G}_{m,n}$ that does not belong to $\mathbb{G}_{m,j}$. Otherwise, u is *saturated* in $\mathbb{G}_{m,j}$.

We first construct a ripple R_k, where k is the number of vertices in the lowest row of $\mathbb{G}_{m,n}$. Observe that R_k is a TTG representation (not necessarily proper) of $\mathbb{G}_{m,0}$. We then incrementally construct the TTG representation $\Gamma_{m,i}$ (not necessarily proper) for $\mathbb{G}_{m,i}$, $i<n$, and finally add the triangles for the nth row such that the resulting drawing becomes a proper TTG representation of $\mathbb{G}_{m,n}$. While constructing $\Gamma_{m,i}$, $i<n$, we maintain the following invariants.

(a) Let $u_1, u_2, \ldots, u_t$ be the unsaturated vertices of $\mathbb{G}_{m,i}$. Then the outer boundary of $\Gamma_{m,i}$ while walking clockwise from the leftmost to the rightmost vertex of $\Gamma_{m,i}$ is a ripple $R_{t+1}=(v_1, v_2, \ldots, v_{t+1})$. The triangle below the edge $(v_j, v_{j+1}), 1 \le j \le t$, corresponds to the vertex u_j.
(b) The bottommost vertex b of the drawing has the largest x coordinate in the drawing and it is adjacent to the leftmost and the rightmost vertices of R_{t+1}.
(c) One can decrease the y coordinate of b and redraw its adjacent edges to obtain another TTG representation (not necessarily proper) of $G_{m,i}$.

Observe that the invariants are similar to invariants we used in the proof of Theorem 3. Consequently, we can install the nth row in a similar way. □

A *parabolic grid of n lines* is the graph determined by the arrangement of line segments $l_0, l_1, \ldots, l_n$, where l_i, $1 \le i \le n-1$, has endpoints at $(0, i)$ and $(n-i, 0)$, and the endpoints of l_0 and l_n are $(0,0)$, $(n-1,0)$ and $(0,n-1)$, $(0,0)$, respectively. We now have the following theorem whose proof is omitted.

Theorem 5. *Every parabolic grid graph admits a proper TTG representation.*

5 Fixed-Parameter Tractability

Let G be a 3-connected plane graph with maximum degree four. We give an $O^*(4^{k_1}6^{k_2})$-time algorithm to decide whether G admits a proper TTG representation, where k_1 and k_2 are the number of inner vertices of degree four and the number of outer vertices in G, respectively.

Here is an outline of our algorithm. Given a 3-connected max-degree-4 plane graph G, we first construct a set of graphs $\mathcal{D}$ such that every graph $H \in \mathcal{D}$ contains G as its weak dual. We then prove that G admits a proper TTG representation if and only if some graph $H \in \mathcal{D}$ admits a straight-line drawing, where each face of H is a triangle; see Lemma 4. For each H we construct a set of feasible path coverings such that H admits a straight-line drawing with each face of H as a triangle if and only if one of these path coverings is stretchable; see Lemma 5. We show that the stretchability of each path covering can be tested in polynomial time; see Lemma 6. We show that $|\mathcal{D}| = O^*(2^{k_2})$ and the number of path coverings is $O^*(4^{k_1}3^{k_2})$. Therefore, the algorithm takes $O^*(4^{k_1}6^{k_2})$ time.

Let $w_1, w_2, \ldots, w_t$ be the outer vertices of G in clockwise order. Construct a graph G' by inserting G into a cycle $c_1, c_2, \ldots, c_t$ of t vertices and adding the edges (c_i, w_i), $1 \leq i \leq t$. Let G^* be the weak dual of G'; see Fig. 6(a). Consider now the set of graphs D that are obtained by contracting at most $t-3$ outer edges of G^*. Since G is 3-connected, D contains all the 3-connected plane graphs that contain G as their weak dual. For every graph $D' \in D$, we construct a set $D'_i, i \in \{0, 1, 2, 3\}$, of $\binom{k_2}{i}$ graphs that are obtained from D' by subdividing i outer edges of D' (with one division vertex per edge); see Figs. 6(b–c). Let $\mathcal{D} = \bigcup_{\forall D' \in D}(D'_0 \cup D'_1 \cup D'_2 \cup D'_3)$. Observe that every graph that satisfies the following conditions belongs to $\mathcal{D}$.

(a) At most three outer vertices of H are of degree two.
(b) For every outer vertex v of degree two in H, if we contract an edge that is incident to v, then the resulting plane graph H' must be a 3-connected planar graph that contains G as its weak dual.

We now have the following lemma, whose proof is omitted due to space constraints.

Lemma 4. *G admits a proper TTG representation if and only if some graph $H \in \mathcal{D}$ admits a straight-line drawing, where each face of H is a triangle.*

Let Γ be a straight line drawing of a plane graph H and let f be a face in Γ. By a *corner at v in f* we denote the angle at v interior to f, which is formed by the edges incident to v on f. A corner at v is *bold* if v is an internal vertex in Γ. A corner at v is *stretched* in Γ, if the corresponding angle is equal to 180°. A corner at v is *concave* in Γ, if the corresponding angle is greater than 180°. We call an inner face f a *semi-outer face* of H, if f contains an outer vertex on its boundary. Otherwise, f is a *full-inner* face of H. See Fig. 6.

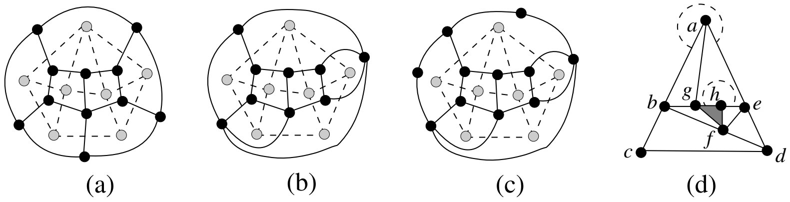

Fig. 6. (a) G and G^*, where G is shown in dashed lines. (b) A member D' of D. (c) A member of D'_2. (d) A straight-line drawing Γ, where a concave and a stretched corner is shown at vertex a and h, respectively. Every corner in Γ that is incident to an inner vertex (i.e., f, g or h) is a bold corner. All the inner faces in Γ are semi-outer except the shaded face, which is a full inner face.

Observe that for every $H \in \mathcal{D}$, if f is a semi-outer face in H, then $\text{len}(f) \in \{3, 4, 5, 6\}$; Fig. 6(c) shows an example where each of these values appears at least once. For every other inner face f, $\text{len}(f) \in \{3, 4\}$. Moreover, if Γ is a straight-line drawing of H, where all the faces are drawn as triangles, then every face f in Γ contains exactly $\text{len}(f) - 3$ stretched corners. The following lemma computes an upper bound on the number of ways the corners of H can be stretched to have such a straight-line drawing.

Lemma 5. *The number of ways in which the corners of H can be stretched to obtain a straight-line drawing Γ such that every face f in Γ contains len$(f) - 3$ stretched corners is $O^*(4^{k_1}3^{k_2})$, where k_1 and k_2 are the number of inner vertices of degree four and the number of outer vertices in Γ, respectively.*

Every candidate of Lemma 5, marks some of the corners of H as "stretched". The following lemma shows how to test the feasibility of such a marking.

Lemma 6. *Let H be a graph that belongs to $\mathcal{D}$. Assume that for every face f in H, exactly len$(f) - 3$ corners of f are marked "stretched". Then one can decide in polynomial time whether H admits a straight-line drawing Γ, where all the corners marked "stretched" are stretched.*

Proof. If two different corners at the same vertex are marked stretched, then H cannot have a straight-line drawing such that both of those corners are stretched simultaneously. We thus assume that every vertex can have at most one corner that is marked stretched. We now construct a set P of paths, as follows.

- The three corners that are not marked on the outer face of H must be concave corners. Let the corresponding vertices be u, v and w in clockwise order on the outer face of H. Let S_{uv} be the path on the boundary of the outer face between the vertices u and v. Define S_{vw} and S_{wu} in a similar way. We add the paths S_{uv}, S_{vw} and S_{wu} to P.
- For every corner ϕ that is marked "stretched", we do the following. Let the vertex and edges that correspond to ϕ be v and $(v, x), (v, y)$, respectively. We add the path x, v, y to P.

- For every edge (x, y) of H, if (x, y) does not belong to any path of P, then we add the path x, y to P.
- For any two paths $u_1, u_2, \ldots, u_{k-1}, u_k$ and $v_1, v_2, \ldots, v_{t-1}, v_t$ in P, if $u_{k-1} = v_1$ and $u_k = v_2$, then we delete those paths from P and add the path $u_1, u_2, \ldots, u_{k-1}(= v_1), u_k(= v_2), \ldots, v_{t-1}, v_t$ to P. We assume that $u_1, u_2, \ldots, v_{t-1}, v_t$ do not create a cycle. Otherwise, each of the vertices on the cycle will contain a stretched corner and H will not have a straight-line drawing.

Observe that every edge in G is contained in a path of P. Furthermore, if H admits the required drawing Γ, then every path in P must be stretched in Γ. In the rest of the proof we show that every pair of paths in P is non-crossing and edge-disjoint, i.e., P is a path covering of H, and hence we can use Theorem 1 to test whether H admits the required drawing in polynomial time. □

Theorem 6. *Let G be a 3-connected plane graph with maximum degree four. Then one can decide in $O^*(4^{k_1}6^{k_2})$-time whether G admits a proper TTG representation, where k_1 and k_2 are the number of inner vertices of degree four and the number of outer vertices in G, respectively.*

One can adapt the decision algorithm of this section for more general classes of plane graphs as follows. Let G be 3-connected plane graph of max-degree-Δ. Then one can construct a set of graphs $\mathcal{D}$ such that every graph $H \in \mathcal{D}$ contains G as its weak dual, and G admits a proper TTG representation if and only if some graph $H \in \mathcal{D}$ admits a straight-line drawing, where each face of H is a triangle. Observe that the cardinality of such a set is independent of Δ and $|\mathcal{D}| = O^*(2^{k_2})$. Since the proof of Lemma 6 does not depend on Δ, we can use the same lemma to construct necessary path coverings and to test the stretchability of those path coverings. Observe that the number of path coverings of H that we need to check is bounded by the number of ways we can mark the corners of H such that for every face f in H, exactly $\text{len}(f) - 3$ corners of f are marked "stretched". Since $\text{len}(f) \leq \Delta + 2$, the number of path coverings is $O(\Delta^{3(k_1+k_2)})$, where k_1 is the number of inner vertices with degree greater than three. Consequently, the running time of the modified algorithm is $O^*(2^{k_2}\Delta^{3(k_1+k_2)})$, which is polynomial if $\Delta{=}O(1)$ and $k_1{+}k_2{=}O(\log n)$.

Theorem 7. *Let G be a 3-connected n-vertex plane graph with maximum degree Δ. Then one can decide in $O^*(2^{k_2}\Delta^{3(k_1+k_2)})$ time whether G admits a proper TTG representation, where k_1 and k_2 are the number of inner vertices of degree greater than three and the number of outer vertices in G, respectively.*

6 Conclusion and Open Problems

We presented algorithms for constructing proper TTG representations for 3-connected cubic planar graphs, and some grid graphs. Our results are strong in the sense that there exist 2-connected and 3-connected graphs with maximum degree four that do not admit proper TTG representations; see Fig. 1(g). We also described a fixed-parameter tractable decision algorithm for deciding proper

TTG representations. In all these cases, one can obtain the proper TTG representation (if it exists) by solving a barycentric system using the result of de Fraysseix and de Mendez [3]. Finding such representations on an integer grid with small area may be an interesting avenue to explore. The main open problem is of course whether recognizing graphs having proper TTG representation is NP-hard, for general planar graphs, or whether there exists a polynomial-time algorithm.

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