Embedding Plane 3-Trees in $\mathbb{R}^2$ and $\mathbb{R}^3$

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Abstract. A point-set embedding of a planar graph G with n vertices on a set P of n points in $\mathbb{R}^d$, $d \geq 1$, is a straight-line drawing of G, where the vertices of G are mapped to distinct points of P. The problem of computing a point-set embedding of G on P is NP-complete in $\mathbb{R}^2$, even when G is 2-outerplanar and the points are in general position. On the other hand, if the points of P are in general position in $\mathbb{R}^3$, then any bijective mapping of the vertices of G to the points of P determines a point-set embedding of G on P. In this paper, we give an $O(n^{4/3+\epsilon})$-expected time algorithm to decide whether a plane 3-tree with n vertices admits a point-set embedding on a given set of n points in general position in $\mathbb{R}^2$ and compute such an embedding if it exists, for any fixed $\epsilon>0$. We extend our algorithm to embed a subclass of 4-trees on a point set in $\mathbb{R}^3$ in the form of nested tetrahedra. We also prove that given a plane 3-tree G with n vertices, a set P of n points in $\mathbb{R}^3$ that are not necessarily in general position and a mapping of the three outer vertices of G to three different points of P, it is NP-complete to decide if G admits a point-set embedding on P respecting the given mapping.

1 Introduction

A *plane graph* is a planar graph with a fixed planar embedding. A *straight-line drawing* of a plane graph G in $\mathbb{R}^d$, $d \geq 1$, is a planar drawing of G, where the vertices of G are drawn as points in $\mathbb{R}^d$ and edges of G are drawn as noncrossing straight line segments. Although two straight line segments meet at their common endpoints if their corresponding edges are adjacent, we do not consider such a meeting point to be a crossing point. Given a plane graph G with n vertices and a set P of n points in $\mathbb{R}^d$, a *point-set embedding* of G on P is a straight-line drawing of G, where each vertex of G is mapped to a distinct point of P. See Figure 1 for an illustration of point-set embeddings in $\mathbb{R}^2$ and $\mathbb{R}^3$.

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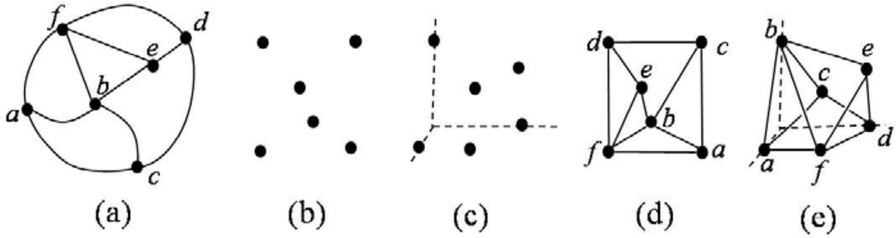

Fig. 1. (a) A plane graph G, (b) a set P of points in $\mathbb{R}^2$, (c) a set P' of points in $\mathbb{R}^3$, (d) a point-set embedding of G on P, and (e) a point-set embedding of G on P'

The problem of embedding planar graphs on fixed vertex locations has been studied for many years [1,4,8,9,12]. Every outerplanar graph with n vertices admits a point-set embedding on any set of n points in $\mathbb{R}^2$, where the points are in general position, i.e, no three points are collinear [4]. Bose et al. gave efficient algorithms to compute point-set embeddings of trees and outerplanar graphs in $O(n \log n)$-time [2] and $O(n \log^3 n)$-time [1], respectively. Recently, Nishat et al. [11] gave an $O(n^2 \log n)$-time algorithm that can decide if a plane 3-tree admits a point-set embedding on a given set of points in $\mathbb{R}^2$, even when the points are not in general position, and computes such an embedding if it exists. Although the point-set embeddability problem in $\mathbb{R}^2$ is polynomial-time solvable for outerplanar graphs and plane 3-trees, Cabello [3] proved that this problem is NP-complete for 2-outerplanar graphs, even when the given points are in general position. On the other hand, given a graph G with n vertices and a set P of n points in $\mathbb{R}^3$, where the points are in general position, i.e., no four points are coplanar, G always admits a point-set embedding on P.

In this paper, we give an $O(n^{4/3+\epsilon})$-expected time algorithm to compute a point-set embedding of a plane 3-tree with n vertices on a set of n points in $\mathbb{R}^2$ if such an embedding exists, for any fixed $\epsilon > 0$. We extend the algorithm to embed a subclass of 4-trees on a point set in $\mathbb{R}^3$ in the form of nested tetrahedra. We also prove that given a plane 3-tree G with n vertices, a set P of n points in $\mathbb{R}^3$ not necessarily in general position and a mapping of the three outer vertices of G to three points of P, it is NP-complete to decide whether G admits a point-set embedding on P for the given mapping of the outer vertices. This negative result is interesting since the problem is solvable in polynomial time in $\mathbb{R}^2$ [11]. Cabello [3] also asked: What is the complexity of the point-set embeddability problem for 3-connected plane graphs in $\mathbb{R}^2$? Since a plane 3-tree is 3-connected, our hardness result answers the analogous question for $\mathbb{R}^3$.

2 Preliminaries

In this section we give some definitions that will be used throughout the paper.

A plane graph divides the plane into connected regions called *faces*. The unbounded region is the *outer face* and all other faces are *inner faces*. The vertices on the outer face are *outer vertices* and all other vertices are *inner vertices*. A *triangular face* contains only three vertices on its boundary. If all the faces of a plane graph G are triangular, then G is a *triangulated plane graph*.

For a cycle C in G, $G(C)$ denotes the subgraph of G induced by the vertices inside and on the boundary of C. If a cycle contains only three vertices a, b, c on its boundary then we denote the cycle by C_{abc}. A graph G with $n \geq 3$ vertices is a *plane* 3*-tree* if it satisfies the following properties. (a) G is a triangulated plane graph. (b) If $n > 3$, then G has a vertex of degree three whose removal gives a plane 3-tree with $n - 1$ vertices.

Any plane 3-tree G has exactly one inner vertex p, which is the common neighbor of the outer vertices of G. We call p the *representative vertex* of G. Plane 3-trees are also known as Apollonian networks and stacked polytopes [6].

Let P be a set of points. We denote by $|P|$ the number of points in P. Let a, b and c be three points that do not necessarily belong to P. By $P(abc)$ we denote the points of P, which are on the boundary and inside of triangle abc.

3 Point-Set Embeddings of Plane 3-Trees in $\mathbb{R}^2$

In this section we give an $O(n^{4/3+\epsilon})$-expected time algorithm to embed a plane 3-tree with n vertices on a set of n points in general position in $\mathbb{R}^2$, where $\epsilon > 0$ is fixed.

Nishat et al. [11] gave an $O(n^2)$-time algorithm for computing a point-set embedding of a plane 3-tree with n vertices on a set of n points in general position. Recently, Moosa et al. [10] tried to give a faster algorithm for computing point-set embeddings of plane 3-trees using a range search data structure of Chazelle et al. [5]. Their algorithm takes $O(n^{4/3+\epsilon} \log n + n^{4/3+\epsilon} \log(l/s))$ time, where $\epsilon > 0$, l is the largest distance between any two points in the point-set and s is the distance between the closest pair of points. Consequently, finding an algorithm for computing point-set embeddings of plane 3-trees with improved running time, where the time complexity is only a function of n, was open.

Like Moosa et al. we also use the range search data structure of Chazelle et al. [5]. Using randomization, however, the expected running time of our algorithm is bounded by a function of n alone for any set of n points in general position in $\mathbb{R}^2$ and independent of the corresponding parameters l and s. Before describing our algorithm we need the following lemma.

Lemma 1. *Let abc be a triangle with a set P of $n > 0$ points in its proper interior, where the points are in general position and preprocessed to answer any triangular range counting query in $f(n)$ time. Let $k \leq n$ be a positive integer. Then in $O(f(n) \log n)$ expected time we can find a point q on bc such that $|P(abq)| = k$.*

Proof. We first set $x = b$ and $y = c$. We then execute the following steps.
Step 1. Randomly choose a point[1] $w \in P(axy)$. Let z denote the intersection point of xy and the line passing through a and w.
Step 2. If $|P(abz)| < k$, then set $x = z$ and go to Step 1. If $|P(abz)| > k$, then set $y = z$ and go to Step 1. Otherwise, set $q = z$.

It is straightforward to observe that Steps 1–2 correctly find the required point q. We now analyze the running time. Consider some iteration i of Steps 1–2. Let P_i be the points of $P(axy)$ at the beginning of the i-th iteration. Let X_j be the indicator random variable such that $X_j = 1$ if point $p_j \in P_i$ remains inside triangle axy after the i-th iteration, and $X_j = 0$ otherwise. Since any point p_j is removed from further consideration with probability 1/2, therefore $E[X_j] = 1/2$. Consequently, the expected number of points that remains in axy after the i-th iteration is $E[X] = \sum_{\forall p_j \in P_i} E[X_j] = |P_i|/2$. Since at each iteration the number of points to consider is reduced by a factor of 1/2, the expected number of iterations is $O(\log n)$. At each iteration, Steps 1–2 take $O(f(n))$-time. Therefore, the total expected running time is $O(f(n) \log n)$. □

Theorem 1. *Let G be a plane 3-tree with n vertices and let P be a set of n points in general position in $\mathbb{R}^2$. We can decide in $O(n^{4/3+\epsilon})$ expected time, for any fixed $\epsilon > 0$, whether G admits a point-set embedding on P and compute such an embedding if it exists.*

Proof. Let a, b and c be the three outer vertices of G and let p be the representative vertex of G. We use the following steps of Nishat et al. [11] to test and compute point-set embedding of G on P.

Step 1. Let C be the convex hull of P. If the number of points on the boundary of C is not exactly three, then G does not admit a point-set embedding on P.
Step 2. For the possible six different mappings of vertices a, b, c to the three points x, y, z on C, execute Step 3.
Step 3. Let n_1, n_2 and n_3 be the number of vertices of $G(C_{abp}), G(C_{bcp})$ and $G(C_{cap})$, respectively. Without loss of generality assume that the current mapping of a, b and c is to x, y and z, respectively. Find the unique mapping of the representative vertex p of G to a point $w \in P$ such that the triangles xyw, yzw and zxw properly contain exactly n_1, n_2 and n_3 points, respectively. If no such mapping of p exists, then G does not admit a point-set embedding on P for the

[1] A simplex range searching data structure based on partition trees and cutting trees (such as that of Chazelle et al. [5]) can be augmented to return a range selection query in $f(n)$ time without any asymptotic increase in space or preprocessing time. That is, each of the t distinct range selection queries on triangle pqr, where $t = |P(pqr)|$, returns a distinct element of $P(pqr)$. The ordering of elements is determined by the trees' internal structures; the specific order is unimportant, so long as there is a bijection between selection queries and elements returned for a given query triangle. By choosing a value uniformly at random in $\{1, 2, \ldots, t\}$ and retrieving the corresponding element using a range selection query, we can select a point $w \in P(pqr)$ at random.

current mapping of a, b, c to x, y, z; hence go to Step 2 for the next mapping. Otherwise, recursively compute point-set embeddings of $G(C_{abp}), G(C_{bcp})$ and $G(C_{cap})$ on $P(xyw), P(yzw)$ and $P(zxw)$, respectively.

The time complexity is dominated by the cost of Step 3 and the bottleneck is the recursive computation of the mappings of the representative vertices. It is straightforward to observe that the recurrence relation for the time taken in Step 3 is $T(n) = T(n_1) + T(n_2) + T(n_3) + \mathcal{T}$, where $\mathcal{T}$ denotes the time required to find the mapping of the representative vertex.

We speed up the mapping of the representative vertex as follows: We use a data structure to preprocess the points of P in $O(g(n))$ time to answer any triangular range reporting query in $O(f(n)+k)$ time and triangular range counting query in $O(f(n))$ time, where k is the number of points reported. Let the outer vertices a, b, c be mapped to points x, y, z, respectively, and let n_1, n_2 and n_3 be the number of vertices of $G(C_{abp}), G(C_{bcp})$ and $G(C_{cap})$, respectively. We need to find a mapping of p to a point $w \in P$ such that triangles xyw, yzw and zxw properly contain exactly n_1, n_2 and n_3 points, respectively. Without loss of generality assume that $n_2 \leq \min\{n_1, n_3\}$.

By Lemma 1, we find two points u and v on yz such that $P(xyu)=n_1 + 3$ and $P(xzv)=n_3 + 3$ in $O(f(n) \log n)$ time. It is straightforward to show that if $dist(z, v) > dist(z, u)$, then p does not have the required mapping. Otherwise, if p has the required mapping to a point $w \in P$, then $w \in P(xuv)$. Since $|P(xuv)|=O(n_2)$, we can enumerate all the points of $P(xuv)$ in $O(f(n) + n_2)$ time. For each point $q \in P(xuv)$, we check if $|P(xyq)|=n_1 + 3, |P(yzq)|=n_2 + 3$ and $|P(zxq)|=n_3+3$ in $O(f(n))$ time. Hence, $\mathcal{T} = O(f(n) \log n)+O(f(n)+n_2)+O(n_2 \cdot f(n))$ and $T(n) = T(n_1) + T(n_2) + T(n_3) + O(\min\{n_1, n_2, n_3\} f(n) \log n)$. This recurrence solves to $T(n) = O(nf(n) \log^2 n)$.

For n points in $\mathbb{R}^d$, the data structure of Chazelle et al. [5] takes $g(n) = O(m^{1+\epsilon})$ preprocessing time and $f(n) = O(n^{1+\epsilon}/m^{1/d})$ time for range counting queries, where $n < m < n^d$ and $\epsilon > 0$. Here $d = 2$ and for the best bound, we choose $m = n^{4/3}$. We thus get $T(n) = (n^{4/3+\epsilon} \log^2 n)$ and $g(n) = O(n^{4/3+4\epsilon/3})$. Therefore, we need $O(n^{4/3+\epsilon'} \log^2 n)$ time in total, where $\epsilon' = 4\epsilon/3 > 0$.

Observe that for any $\epsilon' > 0$, $n^{4/3+\epsilon'} \log^2 n = O(n^{4/3+\epsilon''})$ for any $\epsilon'' > \epsilon'$. □

4 Tetrahedral Embeddings of Tetrahedral 4-Trees

In this section we introduce tetrahedral 4-trees and extend Theorem 1 to $\mathbb{R}^3$.

Let a, b, c and d be four points in general position in $\mathbb{R}^3$. By $T(abcd)$ we denote the tetrahedron defined by points a, b, c and d. A *vertex insertion* operation on $T(abcd)$ places a vertex p interior to $T(abcd)$ and adds edges from p to a, b, c, d, such that $T(abcp)$, $T(abdp)$, $T(bcdp)$ and $T(cadp)$ define four new tetrahedra. By a *tetrahedral embedding* we denote a straight-line embedding formed by starting with a tetrahedron and then applying vertex insertion operations recursively on zero or more newly generated tetrahedra. A graph G with $n \geq 4$ vertices is a *tetrahedral 4-tree* if it admits a tetrahedral embedding. A *tetrahedral point-set embedding* of G on a set P of n points is a tetrahedral embedding of G, where the vertices of G are mapped to distinct points of P.

Let G be a tetrahedral 4-tree with n vertices. Then by definition, G satisfies the following properties.

(a) G is a 4-tree.

(b) Let Γ be a tetrahedral embedding of G. Then the convex hull of the points of Γ is a tetrahedron $T(s_1 s_2 s_3 s_4)$, where s_1, s_2, s_3, s_4 are the four points on the convex hull. By the *surface vertices* of G we denote the vertices u_1, u_2, u_3, u_4 of G that correspond respectively to the points s_1, s_2, s_3, s_4.

(c) If $n{>}4$, then there exists a point p in Γ which is adjacent to the points s_1, s_2, s_3, s_4. By the *core vertex* of G we denote the vertex v that corresponds to p.

(d) Removal of v, u_1, u_2, u_3, u_4 splits G into four (possibly empty) components C_1, C_2, C_3 and C_4, respectively. Then the vertices of C_i along with $\{v, u_1, u_2, u_3, u_4\} \setminus \{u_i\}$ induce a tetrahedral 4-tree, which is placed inside $T(pabc)$ in Γ, where $\{a, b, c\} \subseteq \{\{s_1, s_2, s_3, s_4\} \setminus \{s_i\}\}$.

If G admits a tetrahedral point-set embedding on a given set of points in $\mathbb{R}^3$, then we can prove that the mapping of the core vertex is unique. Using the range search data structure of Chazelle et al. [5] we can preprocess the points in $O(n^{(1+\epsilon)9/4})$ time, where any triangular range counting query takes $O(n^{1/4+\epsilon})$ time, $\epsilon > 0$. Therefore, we can find the mapping of the core vertex in $O(n \cdot n^{1/4+\epsilon}) = O(n^{5/4+\epsilon})$ time. Since we need to find $O(n)$ such mappings in a recursive fashion, the total time required is $O(n^{9/4+\epsilon})$. We thus have the following theorem.

Theorem 2. *Let G be a tetrahedral 4-tree with n vertices and let P be a set of n points in general position in $\mathbb{R}^3$. We can decide in $O(n^{9/4+\epsilon})$ time, for any fixed $\epsilon > 0$, whether G admits a tetrahedral point-set embedding on P and compute such an embedding if it exists.*

5 Point-Set Embeddings of Plane 3-Trees in $\mathbb{R}^3$

Given a plane 3-tree G with n vertices, a set P of n points (not necessarily in general position) in $\mathbb{R}^2$ and a mapping for the outer vertices of G to three points in P, Nishat et al. [11] gave an $O(n^2 \log n)$-time algorithm for testing whether G admits a point-set embedding on P for the given mapping of the outer vertices. In this section we prove that the corresponding decision problem is NP-complete when the points are in $\mathbb{R}^3$. A formal definition of the problem is as follows:

Problem: Three Dimensional Point-Set Embedding (3DPSE)

Instance: A plane 3-tree G with n vertices, a set P of n points (not necessarily in general position) in $\mathbb{R}^3$ and a mapping of the three outer vertices of G to three different points in P.

Question: Does G admit a point-set embedding on P that respects the given mapping of the outer vertices?

We prove NP-hardness of 3DPSE by reduction from a strongly NP-complete problem 3-Partition [7], which is defined as follows.

Instance: A set of $3m$ nonzero positive integers $S=\{a_1, a_2, \ldots, a_{3m}\}$ and an integer $B > 0$, where $a_1+a_2+\ldots+a_{3m} = mB$ and $B/4 < a_i < B/2, 1 \le i \le 3m$.

Question: Can S be partitioned into m subsets $S_1, S_2, \ldots, S_m$ such that $|S_1| = |S_2| = \ldots = |S_m| = 3$ and the sum of the integers in each subset is equal to B?

Here is an outline of our proof for NP-hardness. For a given instance $\mathcal{I} = \{S, m, B\}$ of 3-Partition, we construct a point set $\mathcal{P}$, a plane 3-tree $\mathcal{G}$ and a mapping of the three outer vertices of $\mathcal{G}$ to the three points of $\mathcal{P}$. We prove that $\mathcal{G}$ admits a point-set embedding on $\mathcal{P}$ respecting the mapping of the outer vertices if and only if $\mathcal{I}$ has an affirmative answer.

We first assume that $\mathcal{I}$ has an affirmative answer, and then show a construction of a point-set embedding of $\mathcal{G}$ on $\mathcal{P}$ respecting the mapping of the outer vertices. The other direction of the claim is: if $\mathcal{G}$ admits the required embedding on $\mathcal{P}$, then $\mathcal{I}$ has an affirmative answer. We prove the contrapositive. We assume that $\mathcal{I}$ has a negative answer, and then prove that $\mathcal{G}$ does not admit a point-set embedding on $\mathcal{P}$ respecting the mapping of the outer vertices. To prove this, we show that the mapping of the outer vertices of $\mathcal{G}$ restricts some vertices of G to map onto some special points of $\mathcal{P}$. This mapping leaves m groups of B points unmapped, where the remaining vertices of $\mathcal{G}$ are to be mapped. These remaining vertices of $\mathcal{G}$ correspond to the integers in S. If $\mathcal{G}$ admits the required embedding on $\mathcal{P}$, then those remaining vertices admit a mapping to the unmapped groups of points. Each group corresponds to a subset of the solution of $\mathcal{I}$. Since we assumed that $\mathcal{I}$ has a negative answer, this gives a contradiction.

We now describe the formal reduction. Let m and B be two nonzero positive integers. We first define a set $\mathcal{P}_{m,B}$ of $2mB + 10m - 4$ points as follows:

(a) Two points p and r at $(0, 5, 4m)$ and $(mB+2(m-1), 0, 5m)$, respectively.
(b) The set P_z of $4m$ collinear points on line $x = y = 0$, where $P_z = \{(0, 0, i) | 0 \le i \le 4m - 1\}$. By q we denote the point at $(0, 0, 0)$.
(c) The set P_y of $mB + 2(m - 1)$ points on line $y - 1 = z = 0$, where $P_y = \{(i, 1, 0) | 1 \le i \le mB + 2(m - 1)\}$.
(d) Points $P_u = \{u_1, u_2, \ldots, u_{m-1}\}$, where point $u_i, 1 \le i \le m-1$, is the intersection point of the plane z=1 with the line joining p and the midpoint of the line segment between $(i(B+2)-1, 1, 0)$ and $(i(B+2), 1, 0)$. See Figure 2(a).
(e) Points $P_v = \{v_1, v_2, \ldots, v_{m-1}\}$, where point $v_i, 1 \le i \le m-1$, is the intersection point of the plane z=$4m$+1 with the line joining r and point $u_i \in P_u$.
(f) Points $P_w = \{w_1, w_2, \ldots, w_{mB+2(m-1)}\}$, where point $w_i, 1 \le i \le mB + 2(m - 1)$, is the intersection point of the plane $z = 4m$ with the line joining r with point $p_i \in P_y$. See Figure 2(b).

Observe that $|P_z|$=$4m$, $|P_y|$=mB+$2(m-1)$, $|P_u|$=$m-1$, $|P_v|$=$m-1$ and $|P_w| = mB$+$2(m-1)$. Thus the number of points in $\mathcal{P}_{m,B}$ along with p, r is $2mB$+$10m-4$. We now have the following lemma.

Lemma 2. *Let l_1 be a line segment joining points a and b, where $a \in P_y$ and $b \in P_z$. Let l_2 be another line segment joining points a' and b', where $a' \in P_y, b' \in P_z$ and $\{a', b'\} \ne \{a, b\}$. Then l_1 and l_2 do not cross.*

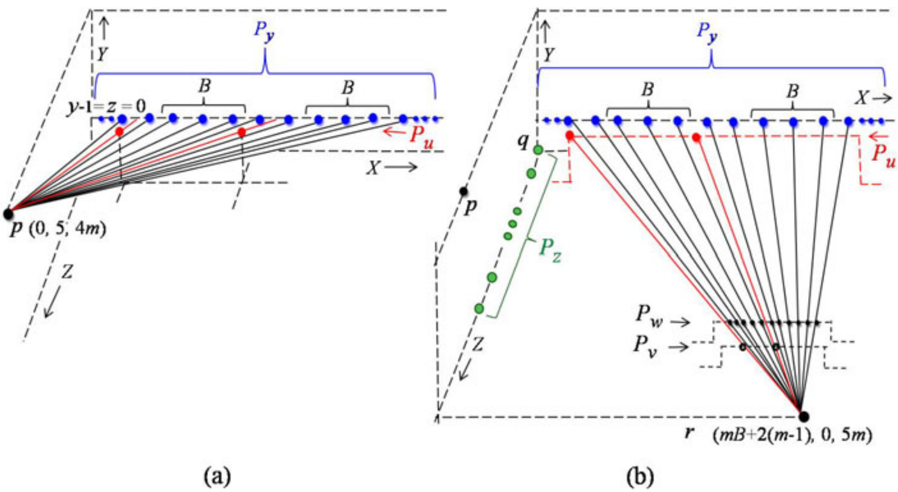

Fig. 2. $\mathcal{P}_{m,B}$, where sets P_u, P_z and P_y are shown in red, green and blue, respectively

Let $x_0, x_1, \cdots, x_n$ be a path of $n+1$ vertices. We add two vertices l, r to the path by adding the edges (l, x_i), (r, x_i), where $0 \le i \le n$. We call the resulting graph a *butterfly* and denote it by W_{n+1}. We call l, r the *wings of* W_{n+1} and path $x_0, x_1, \ldots, x_n$ the *spine of* W_{n+1}. We call x_0 and x_n the two *ends* of the spine. Figure 3(a) depicts a butterfly W_4. Let m and B be two nonzero positive integers and let $S = \{a_1, a_2, \ldots, a_{3m}\}$ be a set of $3m$ nonzero positive integers. We now construct a plane graph $\mathcal{G}_{m,B,S}$ with $2mB + 10m - 4$ vertices as follows:

1. Construct a butterfly W_{4m}. Let a and c be its wings. Add an edge between a and c. Any plane embedding Γ of W_{4m} keeping a and c on the outer face will have one end of the spine on the outer face, which we denote by b. Without loss of generality assume $b = x_0$. See Figure 3(b).

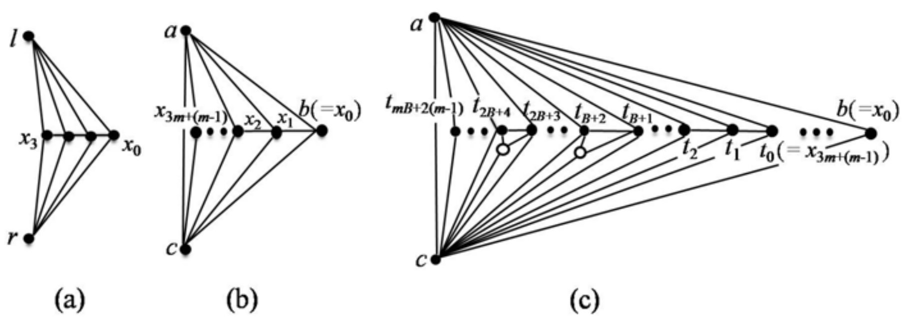

Fig. 3. (a) W_4, (b) W_{4m}, together with an edge (a, c) between the wings a and c, and (c) illustration for G', where the buds are shown by empty circles

2. Let the spine vertices of W_{4m} starting from b in Γ be $b(=x_0), x_1, \ldots, x_{3m+(m-1)}$. We now add a second butterfly $W_{mB+2m-1}$ in the triangular face $acx_{3m+(m-1)}$, where a, c are the wings of $W_{mB+2m-1}$ and $t_0(=x_{3m+(m-1)}), t_1, \ldots, t_{mB+2(m-1)}$ is the spine. Insert a vertex in each triangular face $ct_{i(B+2)-1}t_{i(B+2)}$, $1 \leq i \leq m-1$, and add three edges to connect the inserted point with $c, t_{i(B+2)-1}$ and $t_{i(B+2)}$. Let G' be the subgraph of $\mathcal{G}_{m,B,S}$ bounded by the triangular face $acx_{3m+(m-1)}$. We call each of these inserted vertices a *bud*. See Figure 3(c).
3. For each triangular face $ax_{3m+i}x_{3m+i-1}$, $1 \leq i \leq m-1$, in Γ, insert three vertices l_i, m_i, n_i inside that face and add edges $(l_i, m_i), (m_i, n_i), (n_i, l_i), (a, l_i)$, $(a, m_i), (a, n_i), (x_{3m+i}, l_i), (x_{3m+i}, n_i), (x_{3m+i-1}, n_i)$ avoiding crossing. See Figure 4(a). We call each of these inserted triples of vertices a *trigon*.
4. For each triangular face ax_ix_{i-1}, $1 \leq i \leq 3m$, in Γ, create a butterfly W_{a_i} inside that face with wings a and x_i. Then add an edge between x_{i-1} and one end of the spine of W_{a_i} avoiding crossing. See Figure 4(b). We denote all $W_{a_i}, 1 \leq i \leq 3m$, by *butterflies of* $\mathcal{G}_{m,B,S}$. The graph defined by the resulting embedding is $\mathcal{G}_{m,B,S}$. See Figure 4(c).

Note that $\mathcal{G}_{m,B,S}$ is an embedded plane graph (not necessarily a straight-line embedding). We used Γ only to define the plane embedding of $\mathcal{G}_{m,B,S}$.

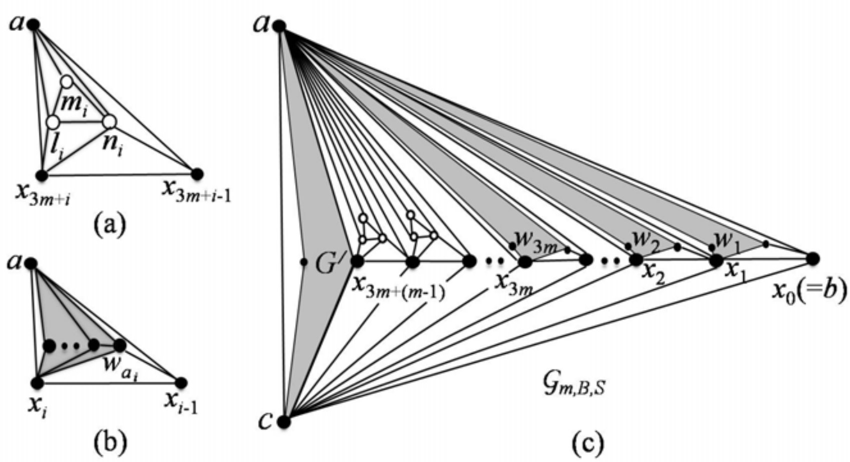

Fig. 4. (a) Insertion of a trigon, (b) illustration for W_{a_i}, and (c) $\mathcal{G}_{m,B,S}$, where vertices of the trigons are shown by empty circles

Observe that $x_0, x_1, \ldots, x_{3m+(m-1)}$ is a sequence of $|P_z|$ vertices of $\mathcal{G}_{m,B,S}$ and $t_1, t_2, \ldots, t_{mB+2(m-1)}$ is a sequence of $|P_w|$ vertices of $\mathcal{G}_{m,B,S}$. The number of buds in $\mathcal{G}_{m,B,S}$ is $|P_v|$ and the number of vertices in the trigons and spines of butterflies in $\mathcal{G}_{m,B,S}$ is $|P_y| + |P_u|$. Therefore, the number of vertices in $\mathcal{G}_{m,B,S}$ along with a, c is equal to the number of points in $\mathcal{P}_{m,B}$, i.e., $2mB + 10m - 4$. We now have the following lemma.

Lemma 3. *$\mathcal{G}_{m,B,S}$ is a plane 3-tree.*

We now use $\mathcal{P}_{m,B}$ and $\mathcal{G}_{m,B,S}$ to prove the following theorem.

Theorem 3. *3DPSE is NP-complete.*

Proof. Given a mapping of the vertices of a plane 3-tree G to the points of P, it is straightforward to check if the drawing determined by this mapping is a straight-line drawing of G in polynomial time. Therefore, the problem is in NP.

We now create an instance of 3DPSE from an instance $B, S=\{a_1, a_2, \ldots, a_{3m}\}$, of 3-Partition. We construct a point-set $\mathcal{P}_{m,B}$ and a plane 3-tree $\mathcal{G}_{m,B,S}$. For convenience we denote $\mathcal{P}_{m,B}$ and $\mathcal{G}_{m,B,S}$ by $\mathcal{P}$ and $\mathcal{G}$, respectively. Since 3-Partition is strongly NP-complete, i.e., it remains NP-complete even when B is bounded by a polynomial in m. Therefore, $\mathcal{G}$ has a polynomial number of vertices and $\mathcal{P}$ has a polynomial number of points. Furthermore, the coordinates of p are bounded by polynomials. Consequently, we can construct $\mathcal{P}$ and $\mathcal{G}$ in polynomial time. Recall the points p, q, r of $\mathcal{P}$ and vertices a, b, c of $\mathcal{G}$. We now ask whether $\mathcal{G}$ admits a point-set embedding on $\mathcal{P}$, where the vertices a, b and c are mapped respectively to the points p, q and r. In the following we prove that such a point-set embedding is possible if and only if the given instance of 3-Partition has an affirmative answer.

Case 1: The given instance of 3-Partition has an affirmative answer.

We construct a point-set embedding of $\mathcal{G}$ on $\mathcal{P}$, where the vertices a, b, c are mapped respectively to the points p, q, r, as follows:

1. Map the buds of G' to the points of P_v consecutively. Map the internal vertices of G' other than the buds of G' to the points of P_w consecutively. Since the points of P_v are visible from p and the points of P_v and P_w are visible from r, no two internal edges of G' cross. See Figure 5(a).
2. Map the vertices $b(= x_0), x_1, x_2, \ldots, x_{3m+(m-1)}$ to the points of P_z starting from $(0, 0, 0)$. The points of P_z are visible from points p and r since these visibilities are not occluded by the edges of G'. Therefore, we can draw the edges joining a and c to $b(= x_0), x_1, x_2, \ldots, x_{3m+(m-1)}$ without creating any crossing.
3. Map each trigon l_i, m_i, n_i of $\mathcal{G}$ respectively to the points $(i(B+2)-1, 1, 0), u_i$, $(i(B+2), 1, 0)$, where $u_i \in P_u$ and $1 \le i \le m-1$. Observe that the points of P_y and P_u are still visible from p. See Figure 5(b). Moreover, by Lemma 2, the edges joining points from P_z and P_y do not create any crossing. Therefore, we can draw the edges joining vertices $x_{3m}, x_{3m+1}, \ldots, x_{3m+(m-1)}$ and vertex a to the trigons without creating any crossing.
4. Observe that there are m groups of consecutive B points on P_y. Denote these groups by $B_1, B_2, \ldots, B_m$. Let $S_1, S_2, \ldots, S_m$ be the solution of the given instance of 3-Partition. Since each S_i, $1 \le i \le m$, contains three integers a_j, a_k and a_l that sum to B, we can map the spines of the corresponding three butterflies W_{a_j}, W_{a_k} and W_{a_l} to B_i. Observe that the points of B_i are visible to p. See Figure 5(b). Moreover, by Lemma 2, the edges joining

points from P_z and P_y do not create any crossing. Therefore, we can draw the edges joining vertices $x_0, x_1, \ldots, x_{3m}$ and a to the spine vertices of the butterflies without creating any edge crossing.

Case 2: The given instance of 3-PARTITION has a negative answer and hence the set S cannot be partitioned into m subsets, where each subset contains exactly three integers and the sum of the integers in each subset is equal to B.

In the following we prove that in this case $\mathcal{G}$ does not admit a point-set embedding on $\mathcal{P}$, where vertices a, b, c are mapped respectively to points p, q, r.

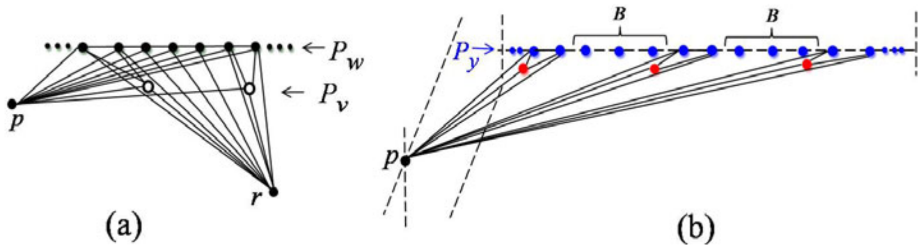

Fig. 5. Illustration for the proof of Theorem 3

1. Suppose for a contradiction that $\mathcal{G}$ admits a point-set embedding Γ on $\mathcal{P}$, where the vertices a, b and c are mapped respectively to the points p, q and r. We then claim that the trigons and spine vertices of the butterflies of $\mathcal{G}$ are mapped to the points of P_y and P_u in Γ. To justify the claim observe that c is not adjacent to the trigons and butterflies and the degree of c is $7m+mB-2$. On the other hand, all the points of $\mathcal{P}$ other than the points of P_y and P_u are visible to r and there are $7m + mB - 2$ such points. Since c is mapped to r, the trigons and the spine vertices of the butterflies $W_{a_i}, 1 \leq i \leq 3m$, of $\mathcal{G}$ must be mapped to the points of P_y and P_u in Γ.
2. Recall that vertex a is mapped to point p, and the vertices of all trigons and butterflies are adjacent to a. There are $m-1$ trigons in $\mathcal{G}$ and $m-1$ points in P_u. Denote these trigons by $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{m-1}$. Since the points of P_y are collinear, one vertex of each trigon $\mathcal{T}_j, 1 \leq j \leq m-1$, will be mapped to a point of P_u. This mapping associates each trigon with a distinct point of P_u.
3. Let $\mathcal{T}_j$ be a trigon with three vertices l_j, m_j, n_j and without loss of generality assume that m_j is mapped to $u_i \in P_u$. We then claim that l_j and n_j must be mapped to the points $\{(i(B+2)-1, 1, 0), (i(B+2), 1, 0)\}$ in Γ. Otherwise, assume that l_j or n_j is mapped to a point x, where $x \in P_y$ and $x \notin \{(i(B+2)-1, 1, 0), (i(B+2), 1, 0)\}$. Then the edge xu_i must cross either the edge determined by $p, (i(B+2)-1, 1, 0)$, or the edge determined by $p, (i(B+2), 1, 0)$. Therefore, the trigons must divide the points of $P_y \setminus \bigcup_{i=1}^{m-1}\{(i(B+2)-1, 1, 0), (i(B+2), 1, 0)\}$ into m groups each containing consecutive B points. See Figure 5(b). Let these groups be $B_1, B_2, \ldots, B_m$. Consequently, the spine vertices of the butterflies must be mapped to these m groups in Γ.

4. Observe that the number of spine vertices in each butterfly is greater than $B/4$ and less than $B/2$. Therefore, four or more butterflies contain more than B spine vertices cumulatively, and hence cannot be mapped to a single B_i. Similarly, less than three butterflies contain less than B spine vertices cumulatively, and hence cannot cover the points of a single B_i. Therefore, each B_i must contain the spine vertices of exactly three butterflies in Γ and the corresponding three integers must sum to B. Consequently, if we form subsets $S_i, 1 \leq i \leq m$, where each S_i consists of three integers that correspond to B_i, then we can find m subsets $S_1, S_2, \ldots, S_m$, where the sum of the integers in each subset is equal to B.

Observe that subsets $S_1, S_2, \ldots, S_m$ correspond to a solution to the given instance of 3-PARTITION, which contradicts the assumption that the given instance has a negative answer. □

6 Conclusion

In this paper we have given an $O(n^{4/3+\epsilon})$-expected time algorithm for computing point-set embeddings of plane 3-trees in $\mathbb{R}^2$. Since a planar 3-tree G has only a linear number of plane embeddings, we can check point-set embeddability for all the embeddings of G and determine whether G has a plane embedding on the given set of points in polynomial time. On the other hand, we have proved that this embeddability problem is NP-complete in $\mathbb{R}^3$, when a mapping for the outer vertices of the input graph is given and the given points are not necessarily in general position. The best known lower bound on time for computing point-set embeddings of plane 3-trees on the points in $\mathbb{R}^2$ is $\Omega(n \log n)$ [11]. Therefore, it would be interesting to find a faster algorithm as well as to improve the lower bound on the time required to find point-set embeddings of plane 3-trees in $\mathbb{R}^2$.

References

1. Bose, P.: On embedding an outer-planar graph in a point set. Computational Geometry: Theory and Applications 23(3), 303–312 (2002)
2. Bose, P., McAllister, M., Snoeyink, J.: Optimal algorithms to embed trees in a point set. Journal of Graph Algorithms and Applications 1(2), 1–15 (1997)
3. Cabello, S.: Planar embeddability of the vertices of a graph using a fixed point set is NP-hard. Journal of Graph Algorithms and Applications 10(2), 353–363 (2006)
4. Castañeda, N., Urrutia, J.: Straight line embeddings of planar graphs on point sets. In: Proc. of CCCG, pp. 312–318 (1996)
5. Chazelle, B., Sharir, M., Welzl, E.: Quasi-optimal upper bounds for simplex range searching and new zone theorems. Algorithmica 8(5&6), 407–429 (1992)
6. Demaine, E.D., Schulz, A.: Embedding stacked polytopes on a polynomial-size grid. In: Proc. of ACM-SIAM SODA, pp. 77–80 (2011)
7. Garey, M.R., Johnson, D.S.: Computers and intractability. Freeman, San Francisco (1979)

8. Giacomo, E.D., Didimo, W., Liotta, G., Meijer, H., Wismath, S.K.: Constrained point-set embeddability of planar graphs. International Journal of Computational Geometry and Applications 20(5), 577–600 (2010)
9. Kaufmann, M., Wiese, R.: Embedding vertices at points: Few bends suffice for planar graphs. Journal of Graph Algorithms and Applications 6(1), 115–129 (2002)
10. Moosa, T.M., Rahman, M.S.: Improved algorithms for the point-set embeddability problem for plane 3-trees. CoRR abs/1012.0230 (2010), `http://arxiv.org/abs/1012.0230`
11. Nishat, R.I., Mondal, D., Rahman, M. S.: Point-set embeddings of plane 3-trees. In: Brandes, U., Cornelsen, S. (eds.) GD 2010. LNCS, vol. 6502, pp. 317–328. Springer, Heidelberg (2011)
12. Pach, J., Wenger, R.: Embedding planar graphs at fixed vertex locations. Graphs and Combinatorics 17(4), 717–728 (2001)