Right Angle Crossing Graphs and 1-Planarity*

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Abstract. A Right Angle Crossing Graph (also called RAC graph for short) is a graph that has a straight-line drawing where any two crossing edges are orthogonal to each other. A 1-planar graph is a graph that has a drawing where every edge is crossed at most once. We study the relationship between RAC graphs and 1-planar graphs in the extremal case that the RAC graphs have as many edges as possible. It is known that a maximally dense RAC graph with $n > 3$ vertices has $4n - 10$ edges. We show that every maximally dense RAC graph is 1-planar. Also, we show that for every integer i such that $i \geq 0$, there exists a 1-planar graph with $n = 8 + 4i$ vertices and $4n - 10$ edges that is not a RAC graph.

1 Introduction

A *drawing* of a graph G maps each vertex u of G to a distinct point p_u in the plane, each edge (u, v) of G to a Jordan arc connecting p_u and p_v and not passing through any other vertex, and is such that any two edges have at most one point in common. A 1*-planar drawing* is a drawing of a graph where every edge can be crossed by at most one other edge. A 1*-planar graph* is a graph that has a 1-planar drawing. A *straight-line drawing* is a drawing of a graph such that every edge is a straight-line segment. A *Right Angle Crossing drawing* (or *RAC drawing*, for short) is a straight-line drawing where any two crossing edges form right angles at their intersection point. A *Right Angle Crossing graph* (or *RAC graph*, for short) is a graph that has a RAC drawing.

Pach and Tóth prove that 1-planar graphs with n vertices have at most $4n - 8$ edges, which is a tight upper bound [9]. Korzhik and Mohar prove that recognizing 1-planar graphs is NP-hard [8]. Suzuki studies the combinatorial properties of the so-called *optimal* 1*-planar* graphs, i.e. those n-vertex 1-planar graph having $4n - 8$ edges [10]. A limited list of additional papers on 1-planar graphs includes [4,7]. Didimo et al. show that a RAC graph with $n > 3$ vertices has at most $4n - 10$ edges and that this bound is tight [5]. Argyriou at al. prove that recognizing RAC graphs is NP-hard [2]. For recent references about RAC graphs and their variants see also [1,3,6,11].

This paper studies the relationship between RAC graphs and 1-planar graphs in the extremal case that the RAC graphs are as dense as possible. A RAC graph is *maximally dense* if it has $n > 3$ vertices and $4n - 10$ edges. While, at a first glance, one might think that, in order to maximize the number of edges in a RAC graph, a good strategy is that each edge should be crossed many times, we prove the following.

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Theorem 1. *Every maximally dense RAC graph is* 1*-planar. Also, for every integer* i *such that* $i \geq 0$*, there exists a* 1*-planar graph with* $n = 8 + 4i$ *vertices and* $4n - 10$ *edges that is not a RAC graph.*

We observe that the first part of Theorem 1 is trivially true if the maximally dense RAC graph has exactly 4 vertices. Namely, the maximally dense RAC graph with 4 vertices is K_4 which is planar and hence 1-planar. We prove that a maximally dense RAC graph with at least 5 vertices is also 1-planar by showing that all RAC drawings with $4n - 10$ edges are such that no edge is crossed twice. For reasons of space, some proofs are omitted or sketched in this abstract.

2 Red-Blue-Green Coloring of Maximally Dense RAC Graphs

Let G be a maximally dense RAC graph and let D be any RAC drawing of G. Let E be the set of the edges of D. In [5] the following 3-coloring of the edges of D (and hence of G) is described. Every edge of D is either a *red edge* or a *blue edge*, or a *green edge*. An edge is red if and only if it is not crossed by any other edge; a blue edge is only crossed by green edges, and a green edge is only crossed by blue edges. We call this 3-coloring of the edges of D a *red-blue-green coloring* of D and denote it as Π_{rbg}. Let $D_{rb} = (V, E_r \cup E_b)$ be the sub-drawing of D consisting of the red and blue edges and let G_{rb} be the corresponding subgraph of G. We call G_{rb} the *red-blue subgraph* of G induced by Π_{rbg} and we call D_{rb} the *red-blue sub-drawing* of D induced by Π_{rbg}. Note that, by construction, D_{rb} has no crossing edges and thus G_{rb} is a planar graph. We will always consider G_{rb} as a planar embedded graph, where the planar embedding is given by D_{rb}. Analogously, we define the *red-green* subgraph of G induced by Π_{rbg} denoted as G_{rg}, and the *red-green* sub-drawing of D induced by Π_{rbg} denoted as D_{rg}. Also G_{rg} has the planar embedding of D_{rg}, and thus G_{rg} and G_{rb} have the same external face.

The next lemmas will particularly focus on the size and the coloring of some specific faces of the red-blue graph G_{rb}. We will consider its external face, denoted as f_{ext}, and its *fence faces*, defined as those internal faces that share at least one edge with f_{ext}. In the proofs that follow, we denote with m_r the number of red edges, with m_b the number of blue edges, and with m_g the number of green edges. Without loss of generality, we will assume from now on that our red-blue-green coloring is such that $m_b \geq m_g$. Also, we denote with f_{rb} the number of faces of G_{rb} and with n the number of its vertices.

Lemma 1. *[5] Every internal face of* G_{rb} *has at least two red edges. Also, all edges of* f_{ext} *are red.*

All remaining lemmas of this section assume that the maximally dense RAC graph G has at least 5 vertices.

Lemma 2. *Face* f_{ext} *is a* 3*-cycle.*

Sketch of Proof: By Lemma 1, every internal face of G_{rb} has at least two red edges and all edges of f_{ext} are red. Hence, denoting with $|f_{ext}|$ the number of edges of f_{ext}, we have $m_r \geq (f_{rb} - 1) + \frac{|f_{ext}|}{2}$. Since G_{rb} is a planar graph, Euler's formula implies that

$m_r + m_b \leq n + f_{rb} - 2$. It follows $m_b \leq n - 1 - \frac{|f_{ext}|}{2}$. Since also the red-green subgraph of G is planar and it has the same external face of G_{rb}, by Euler's formula we also have that $m_r + m_g \leq 3n - 3 - |f_{ext}|$. It follows that $m_r + m_b + m_g \leq 4n - 4 - \frac{3|f_{ext}|}{2}$. Observe that $|f_{ext}| \geq 5$ would imply $m_r + m_b + m_g < 4n - 10$, which is impossible because G is a maximally dense RAC graph. We now show that the external face of G_{rb} cannot be a 4-cycle either. By contradiction, assume that $|f_{ext}| = 4$. Consider first the case that some fence face of G_{rb} has more than 3 edges: Since $|f_{ext}| = 4$ and a fence face has size at least 4, we have $m_r + m_b \leq 3n - 8$. By the inequalities above, we also have $m_r \geq f_{rb} + 1$ and $m_b \leq n - 3$. Since G is maximally dense, we have $m_r + m_b + m_g = 4n - 10$. It follows that $m_r + m_g \geq 3n - 7 > m_r + m_b$, which is however impossible because we are assuming $m_b \geq m_g$. Lastly, consider the case that $|f_{ext}| = 4$ and all fence faces are 3-cycles. Note that there must be four fence faces: If there were only three fence faces there would be a vertex of degree at most three in G, which is impossible in a maximally dense graph with at least 5 vertices. Since in every RAC drawing of G each fence face is drawn as a triangle, for at least one of these four triangles the angle opposite to the edge that belongs to f_{ext} must be larger than or equal to $\frac{\pi}{2}$. This observation, together with Lemma 1, implies that at least one of the fence faces consists of all red edges in any red-blue-green coloring. We therefore have the following: $m_r \geq (f_{rb} - 2) + \frac{|f_{ext}|}{2} + \frac{3}{2} = f_{rb} + \frac{3}{2}$. Since m_r is an integer, we have $m_r \geq f_{rb} + 2$. By $m_r + m_b \leq n + f_{rb} - 2$ we obtain $m_b \leq n - 4$, and by $m_r + m_b + m_g = 4n - 10$ we obtain $m_r + m_g \geq 3n - 6$. However, G_{rg} is a planar graph and it has the same external face as G_{rb}, that has size 4; so, G_{rg} cannot be a maximal planar graph, a contradiction. It follows that f_{ext} must be a 3-cycle. □

Lemma 3. *Graph G_{rb} is biconnected.*

Lemma 4. *Graph G_{rb} has three fence faces. Also, each fence face of G_{rb} is a 3-cycle.*

Lemma 5. *G_{rb} and G_{rg} are both maximal planar graphs.*

Sketch of Proof: By Lemmas 2 and 4, f_{ext} is a 3-cycle consisting of red edges and the three fence faces are all 3-cycles. By simple geometric arguments it follows that in any red-blue-green coloring of a RAC drawing of G, at least two of the triangles representing these fence faces consist of red edges. We therefore have: $m_r \geq (f_{rb} - 3) + \frac{|f_{ext}|}{2} + \frac{3}{2} + \frac{3}{2}$, which implies $m_r \geq f_{rb} + 2$. By $m_r + m_b \leq n + f_{rb} - 2$, we obtain $m_b \leq n - 4$. By $m_r + m_b + m_g = 4n - 10$ we have $m_r + m_g \geq 3n - 6$. Since G_{rg} is a planar graph, it has exactly $3n - 6$ edges and so does G_{rb} because $m_b \geq m_g$. It follows that G_{rb} and G_{rg} are are both maximal planar graphs. □

3 Proof of Theorem 1

The following lemma is the key for proving the first part of Theorem 1.

Lemma 6. *Every RAC drawing of a maximally dense RAC graph is also a 1-planar drawing.*

Proof. The proof is immediate if the maximally dense RAC graph has 4 vertices. Let G be a maximally dense RAC graph with at least 5 vertices, let D be a RAC drawing of G and consider any red-blue-green coloring of the edges of D. Let e be a blue edge of D. By Lemma 5, every blue edge $e = (u, v)$ of G_{rb} is shared by two internal triangular faces, that we denote as f and f'. Let u, v, w be the vertices of f and u, v, w' be the vertices of f'. Since by Lemma 1 every face of G_{rb} has two red edges, we have that edges (u, w) and (w, v) are not crossed by any other edge; similarly, edges (u, w') and (w', v) of f' are both red. Since every blue edge is crossed by some green edges, we have that there can be only one green edge crossing e, namely edge (w, w'). It follows that the RAC drawing D is also a 1-planar drawing. □

To show the second part of Theorem 1, we describe an infinite family of 1-planar graphs that have the same edge density as the maximally dense RAC graphs but are not maximally dense RAC graphs. Consider first the graph G_0 of of Figure 1 (a). Clearly it is 1-planar; also, it has $n = 8$ vertices and $4n - 10 = 22$ edges.

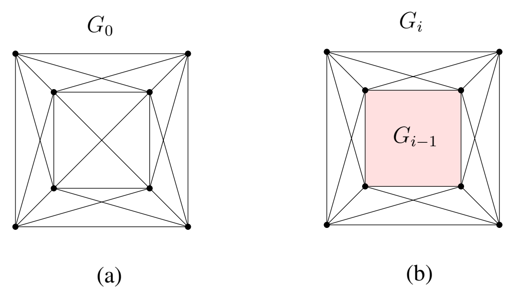

Fig. 1. (a) Graph G_0; (b) Constructing graph G_i from G_{i-1}

Lemma 7. *Graph G_0 is not a RAC graph.*

Proof. Observe that G_0 has the following properties: (1) Every vertex of G_0 has degree at least five and at most six; (2) For every 3-cycle of G_0 with vertices u, v, w, there exists a fourth vertex z such that the subgraph induced by u, v, w, z is the complete graph K_4; (3) There is a 4-cycle through the remaining four vertices of G_0, i.e. the vertices that do not form this K_4.

Suppose, for a contradiction, that G_0 had a RAC drawing D_0. By Lemma 2, the external face of D_0 is a triangle; let u, v, w be the vertices of this external face. Let z be the vertex such that the sub-drawing of D induced by vertices u, v, w, z is a planar representation of K_4. Let f_0, f_1, and f_2 be the three internal faces of this sub-drawing. Let v_0, v_1, v_2, v_3 be the remaining four vertices of G_0. They can be either all inside the same face, or they can be in two faces, or they can be in three faces. The three cases are illustrated in Figure 2.

Assume that v_0, v_1, v_2, v_3 are all in a same face, say f_0. Refer to Figure 2 (a). By Lemma 4, D_0 has three fence faces and these faces are triangles. As discussed in the proof of Lemma 5, in any red-blue-green coloring of D the edges of at least two of these three triangles are red. Since f_1 and f_2 are both fence faces, either (w, z) is a red edge or (u, z) is a red edge. Assume, w.l.o.g. that (w, z) is red. Since vertex v has degree at least five and (w, z) is red, there must be at least two edges that connect v to one of the vertices inside f_0; both such edges must cross (u, z) (see the dotted edges in Figure 2 (a)). However, by Lemma 6, D_0 is also a 1-planar drawing and (u, z) cannot be crossed twice; a contradiction.

Assume that v_0, v_1, v_2 are in f_0 and v_3 is in f_2. Refer to Figure 2 (b). Since there is a cycle with vertices v_0, v_1, v_2, v_3, there are at least two edges incident to v_3 that cross the boundary of f_2. If both these edges cross edge (u, z), then the same argument as in the previous case applies. If one of these edges crosses (v, z), it must also cross (w, z) to reach any one of v_0, v_1, v_2 (see for example the dotted edge (v_2, v_3) in Figure 2 (b)). But this would violate Lemma 6, a contradiction.

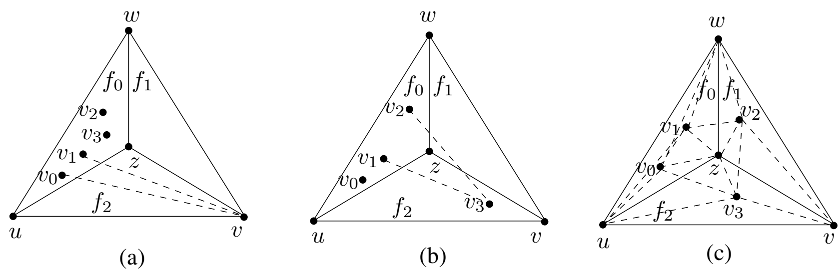

Fig. 2. The three cases in the proof of Lemma 7. (a) v_0, v_1, v_2, v_3 are all in f_0, (w, z) is a red edge, and two dotted edges cross (u, z); (b) v_3 is in f_2 and edge (v_2, v_3) violates the 1-planarity condition; (c) v_3 is in f_2, v_2 in f_1 and z has degree seven.

Finally, assume that v_0, v_1 are in f_0, v_2 is in f_1 and v_3 is in f_2, as depicted in Figure 2 (c). Since there is a 4-cycle with vertices v_0, v_1, v_2, v_3, there is an edge of this cycle crossing (u, z), one crossing (v, z), and one crossing (w, z). Again by Lemma 6, neither (u, z), nor (v, z), nor (w, z) can be crossed by any other edge. In order to guarantee that every vertex of G_0 has degree at least five, we must have that v_0 and v_1 are adjacent to all vertices of f_0, v_2 is adjacent to all vertices of f_1, and v_3 is adjacent to all vertices of f_3 (see the dotted edge v_2, v_3) in Figure 2 (c)). This implies that z has degree seven, which is however impossible because every vertex of G_0 has degree at most six. The statement of the lemma follows. □

Lemma 8. *For every integer i such that $i \geq 0$, there exists a 1-planar graph with $n = 8 + 4i$ vertices and $4n - 10$ edges that is not a RAC graph.*

Proof. Let $\mathcal{G}$ be a family of graphs defined as follows. G_0 is a graph of $\mathcal{G}$. Graph G_i of $\mathcal{G}$ is obtained from G_{i-1} by adding four vertices to the external face of G_{i-1} and 16 edges as described in Figure 1 (b). Observe that every graph in $\mathcal{G}$ is 1-planar and it has $n = 8 + 4i$ vertices and $4n - 10$ edges. Suppose that G_i had a RAC drawing D_i. Since any sub-drawing of a RAC drawing is RAC drawing too, the sub-drawing of D_i representing graph G_0 should also be a RAC drawing of G_0, contradicting Lemma 7. It follows that no graph of $\mathcal{G}$ is a RAC graph, which proves the lemma. □

Lemma 6 and Lemma 8 prove Theorem 1. Note that Theorem 1 does not hold if we drop the requirement of maximal density. Consider, for example, the graph G formed from K_5 by adding 11 paths of length 3 between every pair of vertices. It can be proved that G is a RAC graph, but it is not 1-planar.

4 Open Problems

1. Establish whether recognizing maximally dense RAC graphs is computationally as difficult as recognizing RAC graphs in the general case [2].
2. Characterize 1-planar graphs that admit a RAC drawing.

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