How to Draw a Tait-Colorable Graph

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Abstract. Presented here are necessary and sufficient conditions for a cubic graph equipped with a Tait-coloring to have a drawing in the real projective plane where every edge is represented by a line segment, all of the lines supporting the edges sharing a common color are concurrent, and all of the supporting lines are distinct.

1 Introduction

The complete bipartite graph $K_{3,3}$ has 9 edges and the Pappus configuration has 9 lines. In fact, we may embed $K_{3,3}$ in the Pappus configuration, as in Figure 1.

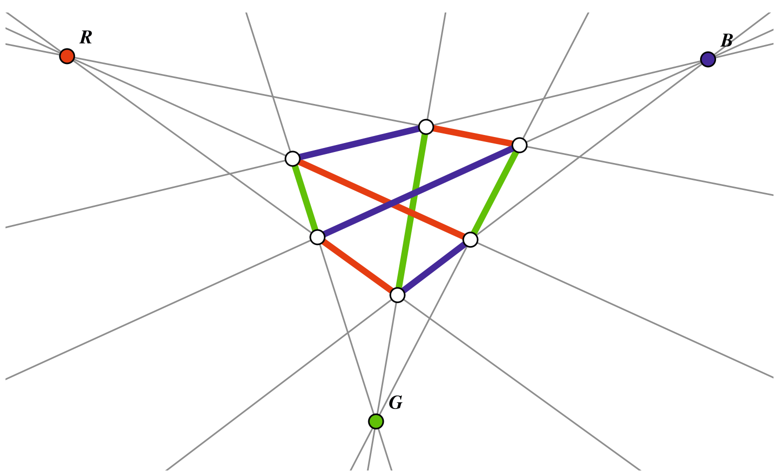

Fig. 1. How to draw $K_{3,3}$

Notice that $K_{3,3}$ has a Tait-coloring, meaning that the graph is cubic and the edges may be colored with three colors in such a way that edges of all three colors meet at every vertex [13]. Given a graph G with a Tait-coloring, we ask in general whether it is possible to represent G by this type of drawing. Specifically, we require that the drawing in the real projective plane have the properties that (a) every edge is represented by a line segment, (b) all of the lines supporting the edges sharing a common color are concurrent, and (c) every supporting line contains exactly 2 vertices of the graph. The main result here

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gives combinatorial necessary and sufficient conditions for when such a graph has this type of drawing. Since $K_{3,3}$ appears as a special case, this is a generalization of Pappus's classical hexagon theorem.

This is a particular type of straight-line drawing. Since every bipartite cubic graph is Tait-colorable, the main result yields the fact that every 3-connected bipartite cubic graph has a drawing with slope number less than or equal to 3. Moreover, the main result here singles out a particular class of graphs studied in [9], where it is proved that every graph with degree not exceeding 3 has slope number less than or equal to 5. A significant source for inspiration for this work was the recent study [5], [6] on xyz graphs and xyz polyhedra. According to [5], an xyz graph is a cubic graph which can be represented in $\mathbb{R}^3$ in such a way that every axis-parallel line contains either zero or two points of the graph. Every xyz graph is Tait-colorable, and it is well known that every cubic graph equipped with a Tait-coloring yields a corresponding compact surface, cf. [4]. Hence, an xyz surface is the compact surface associated to an xyz graph. Obviously the definitions of xyz graphs and projective drawings are closely related. Thus, for example, one sees that every xyz graph has a projective drawing, but a graph which has a projective drawing may fail to be an xyz graph. The complete bipartite graph $K_{3,3}$ furnishes an example of this. Thus, our class extends the class of xyz graphs.

The question of drawing graphs in this way arose during a reading of [11], where it is proved that realization spaces of 4-dimensional convex polytopes are universal. In the course of this proof, it is necessary to construct a 4-dimensional polytope with a non-prescribable octagonal face. In particular, imposing certain incidences among the edges and vertices forces slopes of the edges of the octagon to be a harmonic set. Having seen the construction of this octagon, it is natural to inquire how many incidences may be similarly prescribed.

By serendipity, the main result here is a close relative of a theorem which was discovered while studying a phenomenon which may be called "ghost symmetry". Roughly, one says that a subset of a Euclidean space has a ghost symmetry if some projection of that object has an unexpected symmetry group. The main result in [10] gives a necessary and sufficient condition for a cubic graph equipped with a Tait-coloring to correspond to a subset of the plane having ghost symmetries specified by the coloring. Even though the idea behind this "ghost symmetry prescribability theorem" is not nearly as intuitive as that of drawing a graph, it is stated below for the sake of completeness.

The drawings which appear here were produced using Cinderella [12].

2 Theory of Tait-Colored Graphs

Suppose n is a positive integer. Define a *Tait-colored graph* as a pair $G = (S, T)$, where $S = \{1, 2, 3, ..., 2n-1, 2n\}$, and $T : \{1, 2, 3\} \to H$ is a function into the set $H \subset \mathrm{Perm}(S) \cong S_{2n}$ of all fixed-point-free permutations of S of order 2. It is convenient to designate the image $T(\{1, 2, 3\})$ by $\{b, g, r\}$ and write $T = \{b, g, r\}$.

In graph-theoretic terminology, H coincides with the set of all perfect matchings of the complete graph with vertex set S. Define the *Θ-graph* as the unique Tait-colored graph with 2 vertices.

Notice that we use the term "Tait-colored graph" instead of the more cumbersome phrase "cubic graph equipped with a Tait-coloring". Indeed, every graph which we consider is already equipped with a Tait-coloring, and we do not address the issue of Tait-colorability. The reason we regard a Tait-colored graph as a triple of permutations is due to the connection to ghost symmetries described below.

Define a *component* of a graph $G = (S, T)$ as an orbit in the group generated by T. Call a graph *connected* if it has exactly one component. Define a *monochromatic pair* of a graph $G = (S, T)$ as a pair of transpositions appearing in the cycle decomposition of one of the elements of T. Graph-theoretically, a monochromatic pair is a pair of edges having the same color. Thus, one may designate a monochromatic pair by $\pi = \{(x, \sigma x), (y, \sigma y)\}$, where $\sigma \in T$ and x and y are vertices. A pair of edges of a connected graph G is a *2-edge cut set* if deleting the edges disconnects the graph. If G is Tait-colored and π is a 2-edge cut set, then a 2-edge cut set is necessarily a monochromatic pair.

The language required in the case a graph is not bipartite is streamlined by using the bipartite double cover of a graph, as defined, for example, in [2]. We specialize this to Tait-colored graphs. Thus, suppose $G = (S, T)$ is a Tait-colored graph. Introduce disjoint sets S_1 and S_2 such that there exist bijections $\beta_i : S \to S_i$. The bipartite double cover of G has vertex set $S_1 \cup S_2$ and edges $\{(\beta_1(z), \beta_2(\sigma z)) : z \in S, \sigma \in T\}$. Alternatively, one constructs the bipartite double cover by introducing a monochrome pair $\{(\beta_1(z), \beta_2(\sigma z)), (\beta_2(z), \beta_1(\sigma z))\}$ for every edge $(z, \sigma z)$ of G. Notice that the Tait-coloring of G furnishes a natural Tait-coloring of the bipartite double cover.

In general, a connected graph is bipartite if and only if its bipartite double cover is disconnected; if a graph is bipartite, then the bipartite double cover is a disjoint union of two copies of the original graph. Call a Tait-colored graph $G = (S, T)$ is *strongly non-bipartite* if the deleted graph $G \backslash \pi$ is non-bipartite for every monochromatic pair π. Notice that a graph is strongly non-bipartite if the bipartite double cover of $G \backslash \pi$ is connected for every monochromatic pair π. Finally, note that the bipartite double cover of a connected non-bipartite graph G has a 2-edge cut set if and only if $G \backslash \pi$ has both a bipartite component and a non-bipartite component for some monochromatic pair π.

3 Projective Drawings

The purpose of this section is to state and prove the main results of this article.

Suppose $G = (S, T)$ is a Tait-colored graph and assume $T = \{b, g, r\}$. Define a *parallel drawing* of G as a pair (ι, ϕ) where $\iota : S \to \mathbb{R}^2$ is a function and $\phi = \{\phi_b, \phi_g, \phi_r\}$ is a triple of projections to distinct 1-dimensional subspaces of $\mathbb{R}^2$ such that such that $\phi_\sigma(\iota(z)) = \phi_\sigma(\iota(\sigma z))$ for all $z \in S$ and all $\sigma \in T$. Call a monochromatic pair $\{(x, \sigma x), (y, \sigma y)\}$ *degenerate* for (ι, ϕ) if the lines $\overline{\iota(x), \iota(\sigma x)}$

and $\overline{\iota(y), \iota(\sigma y)}$ coincide. Given a Tait-colored graph $G = (S, T)$, define the *trivial drawing* of G by specifying $\iota(z) = 0$ for all $z \in S$. Clearly every monochromatic pair of a graph is degenerate for its trivial drawing. Call a parallel drawing *faithful* if it has no degenerate pairs. In this terminology, we are interested in general in when one may find a faithful parallel drawing of G.

The space of parallel drawings of a fixed graph $G = (S, T)$ carries an action by the group $GL(2, \mathbb{R})$, induced by the usual action on $\mathbb{R}^2$. Thus, suppose (ι, ϕ) is a drawing of G and $g \in GL(2, \mathbb{R})$. Define a pair (ι^g, ϕ^g) by $\iota^g(x) = g\iota(x)$ and $(\phi^g)_\sigma = (g^{-1})^T \phi_\sigma$ for all $x \in S$ and $\sigma \in T$. Then (ι^g, ϕ^g) is a drawing of T. Notice (ι^g, ϕ^g) is faithful whenever (ι, ϕ) is faithful. The action of $GL(2, \mathbb{R})$ factors through to an action of $PGL(2, \mathbb{R})$ which is triply transitive on $\mathbb{R}P^1$. Thus, one may prescribe $\{\phi_b, \phi_g, \phi_r\}$ to be the projections to any three distinct 1-dimensional subspaces of $\mathbb{R}^2$. For this reason, we assume throughout that these projections are fixed and suppress mention of them unless necessary.

The main results are handled in two cases, depending on whether or not a given graph is bipartite:

Theorem 1. *A connected, bipartite, Tait-colored graph which is not the Θ-graph admits a faithful parallel drawing if and only if it does not have a 2-edge cut set.*

There is a similar characterization when the graph is non-bipartite:

Theorem 2. *A connected, non-bipartite, Tait-colored graph admits a faithful parallel drawing if and only if it is strongly non-bipartite and its bipartite double cover does not have a 2-edge cut set.*

In the proofs of each of these, one must establish two directions. The "combinatorial" direction is to show that if a graph has a certain pathology, then the graph cannot be drawn faithfully. The converse "constructibility" direction is to show that if a graph lacks the pathology, then one may draw it faithfully.

Call a monochromatic pair of a Tait-colored graph G *forced* if it is degenerate in every parallel drawing of G. Forced pairs allow an alternate statement of the main results:

Corollary 3. *Suppose G is a connected, Tait-colored graph with at least 4 vertices and π is a monochromatic pair of G. (a) If G is bipartite, then π is forced if and only if π is a cut set. (b) If G is non-bipartite, then π is forced if and only if $G\backslash\pi$ has a bipartite component.*

In particular, a monochromatic pair π of a connected non-bipartite graph G can be forced in one of two ways. Either $G\backslash\pi$ is connected and bipartite or π is a cut set and exactly one of the two components of $G\backslash\pi$ is bipartite.

Given a Tait-colored graph $G = (S, T)$, define a *projective drawing* of G as a pair (ι, λ) where $\iota : S \to \mathbb{R}P^2$ is a function and $\lambda : T \to \mathbb{R}P^2$ is an injection such that λ_σ lies on $\overline{\iota(z), \iota(\sigma z)}$ for all $z \in S$ and $\sigma \in T$. Call a projective drawing *faithful* if the lines $\overline{\iota(z), \iota(\sigma z)}$ correspond bijectively with the edges $(z, \sigma z)$. A projective drawing becomes a parallel drawing when the three points λ are collinear. Since faithfulness and non-collinearity are open conditions, one may perturb a parallel drawing so that the three "vanishing points" λ are non-collinear while preserving faithfulness. In fact, if a graph admits a faithful parallel drawing, then one may find a projective drawing for any choice of three distinct points λ. The converse of this, however, is not true. For example, the complete quadrangle is a projective drawing of the complete graph K_4 equipped with its unique Tait-coloring, and it is often taken as an axiom in projective geometry that the three diagonal points not lie on a common line.

3.1 The Combinatorial Direction

Proposition 4. *Suppose $G = (S, T)$ is a connected Tait-colored graph and π is a monochromatic pair of G. (a) If G is bipartite and π is a cut set, then π is forced. (b) If G is non-bipartite and $G \backslash \pi$ has a bipartite component, then π is forced.*

Proof. (a) Let $\{S_1, S_2\}$ be the bipartition of S and let V be the vertices of one of the components of $G \backslash \pi$ Without loss of generality, assume that $\pi = \{(x, bx), (y, by)\}$, where $x \in S_1$ and $y \in S_2$. Let ι be any drawing of G. Notice $g(S_1 \cap V) = S_2 \cap V$ and $g(S_2 \cap V) = S_1 \cap V$, so we may write

$$\sum_{z \in S_1 \cap V} \phi_g(\iota(z)) = \sum_{z \in S_1 \cap V} \phi_g(\iota(gz)) = \sum_{z \in S_2 \cap V} \phi_g(\iota(z)) \ .$$

By a similar token, we also have

$$\sum_{z \in S_1 \cap V} \phi_r(\iota(z)) = \sum_{z \in S_1 \cap V} \phi_r(\iota(rz)) = \sum_{z \in S_2 \cap V} \phi_r(\iota(z)) \ .$$

Since ϕ_g and ϕ_r are linearly independent, this yields

$$\sum_{z \in S_1 \cap V} \iota(z) = \sum_{z \in S_2 \cap V} \iota(z) \ .$$

In particular, this implies

$$\sum_{z \in S_1 \cap V} \phi_b(\iota(z)) = \sum_{z \in S_2 \cap V} \phi_b(\iota(z)) \ .$$

Subtracting off all terms corresponding to blue edges of G, this yields $\phi_b(\iota(x)) = \phi_b(\iota(y))$. However, ι is a drawing of G, so $\phi_b(\iota(x)) = \phi_b(\iota(bx))$ and $\phi_b(\iota(y)) = \phi_b(\iota(by))$. Therefore π is a forced pair.

Part (b) follows in an analogous manner, although there are two parts to show. If $G \backslash \pi$ has two components, then one may use the argument above on the bipartite component. If $G \backslash \pi$ is connected, then it is bipartite and there is a uniquely determined bipartition $\{S_1, S_2\}$ of the vertices of $G \backslash \pi$. Again one may show as above that the centroid of $\iota(S_1)$ coincides with the centroid of $\iota(S_2)$. One then projects according to ϕ_b, subtracts terms corresponding to blue edges, and concludes that the vertices of π must be collinear. $\square$

This establishes the combinatorial direction for both theorems.

3.2 The Constructibility Direction

Given a Tait-colored graph $G = (S, T)$, define a *cycle* of G as a sequence $\gamma = (z_1, z_2, ..., z_{2k})$ of vertices such that $z_{i+1} = \sigma_i z_i$ for some $\sigma_i \in T$ for each i, regarding subscripts modulo $2k$. Call a cycle of length $2k$ *simple* if it has $2k$ distinct vertices.

Suppose (ι, ϕ) is a parallel drawing of $G = (S, T)$ and $\gamma = (z_1, z_2, ..., z_{2k})$ is a simple cycle of G. Assuming $T = \{b, g, r\}$, choose non-zero vectors $v_\sigma \in \ker(\phi_\sigma)$ such that $v_b + v_g + v_r = 0$. Define $\tau_i \in T\backslash\{\sigma_{i-1}, \sigma_i\}$, the unique third color at z_i which is not equal to σ_{i-1} or σ_i. Next, assign $s : \{1, 2, 3, ..., 2k\} \to \{-1, 1\}$ as follows. First let $s_1 = 1$. Then, assuming s_i has been defined, let

$$s_{i+1} = \begin{cases} s_i & \text{if } \tau_{i+1} = \tau_i \ , \\ -s_i & \text{if } \tau_{i+1} \neq \tau_i \ . \end{cases}$$

For ease of notation, write $v_i = v_\tau$ if $\tau = \tau_i$ for each i. Next, for $t \in \mathbb{R}$, let $\iota_{\gamma,t}(z) = \iota(z)$ for all $z \notin \{z_1, z_2, ..., z_{2k}\}$ and

$$\iota_{\gamma,t}(z_i) = \iota(z_i) + t s_i v_i$$

for all $i \in \{1, 2, 3, ..., 2k\}$. It is routine to verify that $\iota_{\gamma,t}$ is a parallel drawing of G for any choice of cycle γ satisfying the assumptions above. Call $\iota_{\gamma,t}$ a *perturbation* of ι along γ. Notice that a perturbation is defined only for cycles of even length.

One may quickly establish:

Proposition 5. *Suppose $G = (S, T)$ is a Tait-colored graph, ι is a parallel drawing of G, and γ is a simple cycle of G. (a) If z is a vertex of γ and $t \neq 0$, then $\iota_{\gamma,t}(z) \neq \iota(z)$. (b) If z is not a vertex of γ, then $\iota_{\gamma,t}(z) = \iota(z)$ for all t. (c) If $(z, \sigma z)$ is an edge of γ and $t \neq 0$, then $\overline{\iota_{\gamma,t}(z), \iota_{\gamma,t}(\sigma z)} \neq \overline{\iota(z), \iota(\sigma z)}$. (d) If z is a vertex of γ and $(z, \sigma z)$ is not an edge of γ, then $\overline{\iota_{\gamma,t}(z), \iota_{\gamma,t}(\sigma z)} = \overline{\iota(z), \iota(\sigma z)}$ for all t.*

This proposition indicates how the constructibility direction of the proof works. Our aim in each case is to show that one may perturb the trivial drawing along simple cycles and inductively remove every degenerate pair under the given hypotheses. In each case, one chooses the perturbation carefully so as not to introduce any further degeneracies. Note that if $\pi = \{(x, \sigma x), (y, \sigma y)\}$ is a degenerate pair for a drawing ι and there exists a pair (γ, t) such that π is not a degenerate pair for $\iota_{\gamma,t}$, then the general perturbation of ι along γ has strictly fewer degenerate pairs than ι. This follows because the condition for a pair to be degenerate for $\iota_{\gamma,t}$ is a linear equation in t. Since there are only finitely many monochromatic pairs, the number of values of t for which $\iota_{\gamma,t}$ has more or equal numbers of degenerate pairs than ι is finite.

The Bipartite Case. Suppose first that $G = (S, T)$ is a connected, bipartite, Tait-colored graph. Furthermore, assume that G does not have a 2-edge cut set. Let ι be any drawing of G and suppose $\pi = \{(x, \sigma x), (y, \sigma y)\}$ is degenerate for ι. Since π is not a 2-edge cut set, there is a simple cycle γ in G which contains the edge $(x, \sigma x)$, but not the edge $(y, \sigma y)$. Hence, if G has a degenerate pair, one may always choose a non-zero perturbation $\iota_{\gamma,t}$ which has fewer degenerate pairs than ι. This completes the proof of the main result in the case when G is bipartite.

The Non-Bipartite Case. This follows analogously, although it is more cumbersome. Suppose $G = (S, T)$ is a connected, non-bipartite, Tait-colored graph, ι is a drawing of G, and $\pi = \{(x, \sigma x), (y, \sigma y)\}$ is a degenerate pair for ι. Assume moreover that the deleted graph $G\backslash\pi$ is non-bipartite and does not have a bipartite component. Showing the existence of a desirable cycle depends on whether or not π is a cut set.

Suppose first that $G\backslash\pi$ is connected. Since $G\backslash\pi$ is not bipartite, there is a simple cycle γ in G having the property that $(x, \sigma x)$ is an edge of γ and $(y, \sigma y)$ is not an edge of γ. (If such a cycle didn't exist, then $G\backslash\pi$ would be bipartite or disconnected.) Hence, one may perturb along γ to eliminate the degeneracy along π without introducing more degeneracies.

Suppose instead that $G\backslash\pi$ is disconnected, consisting of components G_1 and G_2. Assume x is a vertex of G_1 and σx is a vertex of G_2. Since neither G_1 nor G_2 is bipartite, there are integers j, k and cycles $\gamma_1 = (x = z_1, z_2, ..., z_{2j-1}, z_{2j} = x)$ of length $2j - 1$ in G_1 and $\gamma_2 = (\sigma x = z_{2j+1}, z_{2j+2}, ..., z_{2j+2k} = \sigma x)$ of length $2k - 1$ in G_2. Without loss of generality, assume that γ_1 and γ_2 are simple. Define a cycle γ of length $2j + 2k$ by

$$\gamma = (z_1, z_2, ..., z_{2j-1}, z_{2j}, z_{2j+1}, ..., z_{2j+2k}) \ .$$

Hence γ is a cycle in G which uses the edge $(x, \sigma x)$ but not $(y, \sigma y)$. We must provide a definition of the perturbation along γ because it has repeated vertices $z_1 = z_{2j} = x$ and $z_{2j+1} = z_{2j+2k} = \sigma x$. In this case, we define $\iota_{\gamma,t}$ as above on the non-repeated vertices, but also specify

$$\iota_{\gamma,t}(z_1) = \iota(z_1) + t(s_1 v_1 + s_{2j} v_{2j})$$

and

$$\iota_{\gamma,t}(z_{2j+1}) = \iota(z_{2j+1}) + t(s_{2j+1} v_{2j+1} + s_{2j+2k} v_{2j+2k}) \ .$$

As above, it is routine to verify that this defines a 1-parameter family of parallel drawings of G. However, one must verify that this perturbation removes the degeneracy along π without introducing any others. Since the vertices of γ_1 and γ_2 are distinct, we have $\tau_1 \neq \tau_{2j}$ and $\tau_{2j+1} \neq \tau_{2j+2k}$. This in turn implies that $s_1 \neq s_{2j}$ and $s_{2j+1} \neq s_{2j+2k}$. Hence, up to a sign,

$$\iota_{\gamma,t}(z_1) = \iota(z_1) \pm t(v_1 - v_{2j})$$

and

$$\iota_{\gamma,t}(z_{2j+1}) = \iota(z_{2j+1}) \pm t(v_{2j+1} - v_{2j+2k}) \ .$$

Now, recall that the subspaces $\{v_b^{\perp}, v_g^{\perp}, v_r^{\perp}\}$ are distinct. This implies that the vectors $v_1 - v_{2j}$ and $v_{2j+1} - v_{2j+2k}$ are both non-zero. Hence, a non-zero perturbation of ι along γ may remove the degeneracy along π without introducing any additional degeneracies.

This completes the proof of the theorem in the case G is non-bipartite.

4 Subsequent Results

The purpose of this section is to highlight some general observations concerning projective drawings of Tait-colored graphs. These results are not required in the proofs of the main results above, but they are nevertheless interesting in their own right. Given that the main theorem is a generalization of Pappus's theorem, it is thought that these results may play some role in studying geometric configurations.

4.1 Quasi-Faithful Drawings

A faithful projective drawing of a Tait-colored graph is the nicest possible because all supporting lines are distinct. However, even though a certain graph may not admit a faithful drawing, it may admit one that is nearly faithful. Thus, call a drawing ι of a Tait-colored graph $G = (S, T)$ *quasi-faithful* if ι restricts to a bijection on S. For a quasi-faithful drawing, we do not require that the supporting lines $\overline{\iota(z), \iota(\sigma z)}$ be distinct. Call a Tait-colored graph *quasi-faithful* if it admits at least one quasi-faithful drawing but it does not admit a faithful drawing. For example, the graph in Figure 2 is quasi-faithful because the edges $(5, 7)$ and $(2, 10)$ lie on the same supporting line in every drawing.

Whether or not a graph is quasi-faithful depends only on the combinatorics of its forced pairs. For example, if a graph has only one forced pair, then one may always perturb to find a quasi-faithful drawing. In fact, the main results above show that a graph fails to have a quasi-faithful drawing exactly when it has at least two forced pairs π_1 and π_2 with different colors and at least two common vertices.

The graph of a triangular prism, for example, does not even have a quasi-faithful drawing. For, let $G = (S, T)$, where $S = \{1, 2, 3, 4, 5, 6\}$ and $T = \{b, g, r\}$ with $b = (1, 2)(3, 4)(5, 6)$, $g = (1, 6)(2, 3)(4, 5)$, and $r = (1, 3)(2, 5)(4, 6)$. Then G is non-bipartite and both of the graphs $G\backslash\{(1, 2), (5, 6)\}$ and $G\backslash\{(2, 3), (4, 5)\}$ are bipartite. According to the theorem in the case G is non-bipartite, the pairs $\{(1, 2), (5, 6)\}$ and $\{(2, 3), (4, 5)\}$ are forced. Since the points $\{1, 2, 5, 6\}$ are always collinear and the points $\{2, 3, 4, 5\}$ are always collinear, the points $\{2, 5\}$ common to both always coincide in every drawing of G. By symmetry, the points $\{1, 4\}$ are always coincident as are $\{3, 6\}$.

4.2 Forced Triples

For some projective drawings, one sees extraneous points where three supporting lines always concur. For example, in every projective drawing of the graph appearing in Figure 2, the lines $\overline{1, 2}$, $\overline{4, 9}$ and $\overline{6, 7}$ are concurrent. The purpose of this section is to characterize when this happens. Suppose $G = (S, T)$ is a Tait-colored graph and assume $T = \{b, g, r\}$. Call a triple $\tau = \{(x, bx), (y, gy), (z, rz)\}$ of edges with 6 distinct vertices *forced* if the lines $\overline{\iota(x), \iota(bx)}$, $\overline{\iota(y), \iota(gy)}$, and $\overline{\iota(z), \iota(rz)}$ are concurrent in every drawing ι of G.

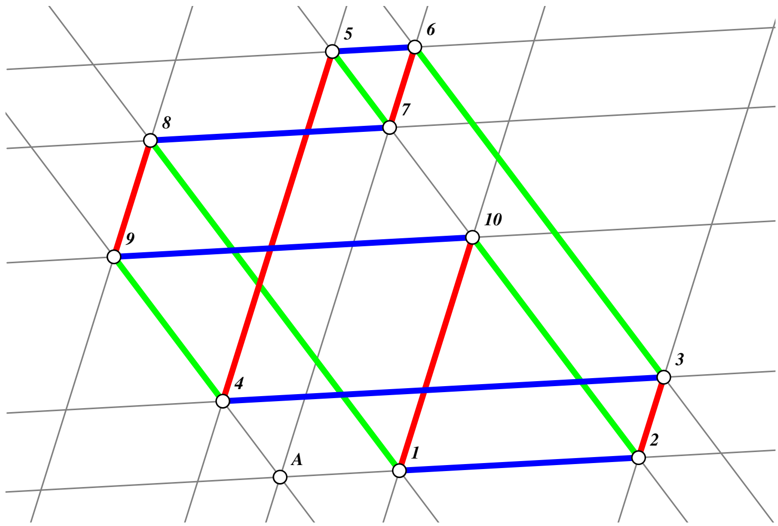

Fig. 2. A quasi-faithful graph with a forced triple

Proposition 6. *Suppose $G = (S, T)$ is a connected Tait-colored graph and τ is a triple of edges with 6 distinct vertices. (a) If G is bipartite, then τ is forced if and only if τ is a cut set. (b) If G is non-bipartite, then τ is forced if and only if $G \backslash \tau$ has a bipartite component.*

Notice in particular that a triple τ in a connected non-bipartite graph G can be forced in one of two ways. Either $G \backslash \tau$ is connected and bipartite or τ is a cut set and exactly one of the two components of $G \backslash \tau$ is bipartite.

The proof of this follows in a manner similar to the main results above. In the combinatorial direction, one uses the style of argument which was used for proposition 4. In the constructibility direction, one shows that if a drawing has a coincident triple of lines which does not arise in one of these ways, then one may perturb the drawing along a carefully chosen cycle to remove the degeneracy.

The presence of a forced triple provides a way to decompose a drawing of a graph into "subdrawings". Suppose for instance that $G = (S, T)$ is connected and bipartite and $\tau = \{(x, bx), (y, gy), (z, rz)\}$ is a forced triple. Then $G \backslash \tau$ has two components. Let S_1 and S_2 be the sets of vertices for each of these components and assume $x, y, z \in S_1$ and $bx, gy, rz \in S_2$. Then one obtains two smaller graphs G_1 and G_2 by introducing vertices, say w_1 and w_2, and edges $\{(x, w_1), (y, w_1), (z, w_1)\}$ in G_1 and $\{(bx, w_2), (gy, w_2), (rz, w_2)\}$ in G_2. The graphs G_1 and G_2 inherit Tait-colorings from G, but they are not subgraphs of G. Nevertheless, due to the proposition above, every projective drawing of G automatically contains projective drawings of G_1 and G_2. There is a similar decomposition when G is non-bipartite.

4.3 Realization Spaces

The purpose here is to discuss the space of all drawings of a given graph. The main point is that such spaces are always topologically trivial.

Given a Tait-colored graph $G = (S, T)$, let $\mathcal{D}(G)$ denote the set of all parallel drawings of G with a fixed triple of projections. Reading the definition, one sees that $\mathcal{D}(G)$ is a vector space. Using perturbations, one may show:

Proposition 7. *Suppose n is a positive integer, $|S| = 2n$, and $G = (S, T)$ is a Tait-colored graph. (a) If G is bipartite, then $\mathcal{D}(G)$ has dimension $n + 1$. (b) If G is non-bipartite, then $\mathcal{D}(G)$ has dimension n.*

Given G, let $\mathcal{F}(G)$ denote the space of all faithful parallel drawings of G. Again, due to the linearity in the definition, one sees that $\mathcal{F}(G)$ is the complement of an arrangement of hyperplanes in $\mathcal{D}(G)$. This yields:

Proposition 8. *Suppose n is a positive integer, $|S| = 2n$, and $G = (S, T)$ is a Tait-colored graph which admits a faithful parallel drawing. (a) If G is bipartite, then $\mathcal{F}(G)$ is a disjoint union of open cells of dimension $n + 1$. (b) If G is non-bipartite, then $\mathcal{F}(G)$ is a disjoint union of open cells of dimension n.*

Currently the number of connected components of $\mathcal{F}(G)$ as a function of G is unknown.

5 Conclusion

5.1 Higher Degree

One may extend the notions above and ask about parallel drawings of d-edge-colored regular graphs. However, even the case of $d = 4$ appears very difficult to handle:

Conjecture 9. The decidability problem of whether or not a 4-edge-colored quartic graph has a faithful parallel drawing is polynomially equivalent to the existential theory of the reals.

See [3] for a definition. The intuition behind this conjecture is based on several facts. Consider that the space of faithful drawings of a Tait-colored graph is always a complement of a hyperplane arrangement, but this property does not appear to be predictable for quartic graphs. For example, there is no simple relationship between the dimension of the realization space and the number of vertices for quartic graphs. By a similar token, the realization space of a Tait-colored graph always contains realizations with integer coordinates, but, by contrast, the realization space of a quartic graph often uses an extension of the rationals. Another source of intuition comes from the theory of graph-encoded manifolds and crystallizations [7]. In this theory, one represents a pseudomanifold of dimension d with a $(d + 1)$-edge-colored graph of degree $d + 1$, and then uses these representations to study the pseudomanifold. Thus, each 4-edge-colored quartic graph represents a certain 3-dimensional pseudomanifold. Even in dimension 3, several decision problems are known to be **NP**-complete [1,8]. Thus, it is thought that decision problems in 3-dimensional manifolds must translate via polynomial-time algorithms to decision problems for 4-edge-colored quartic graphs. Finally, it was shown in [11] that any semialgebraic variety can be approximated by the realization space of a 4-dimensional convex polytope. This is in contrast to the case in dimension 3, where the realization space of every convex polytope is an open cell. Thus, things appear to "go bad" when one increases the dimension. Due to the connection to crystallizations of pseudomanifolds, increasing the degree from 3 to 4 is akin to increasing the dimension.

5.2 Ghost Symmetry in the Plane

This section explains the connection from the main result from [10] to parallel drawings. Suppose $G = (S, T)$ is a Tait-colored graph. Define a *GS realization* of G as a pair (ι, ϕ) where $\iota : S \to \mathbb{R}^2$ is a function and $\phi = \{\phi_b, \phi_g, \phi_r\}$ is a triple of projections to distinct 1-dimensional subspaces of $\mathbb{R}^2$ such that such that $\phi_\sigma(\iota(z)) = -\phi_\sigma(\iota(\sigma z))$ for all $z \in S$ and all $\sigma \in T$. The 1-dimensional subspaces in this definition are the *lines of ghost symmetry* because, while the 2-dimensional point configuration $\iota(S)$ may have a trivial symmetry group, there are three distinct 1-dimensional shadows $\phi_\sigma(\iota(S))$ which each have bilateral symmetry. Call a GS realization *faithful* if each of the projections ϕ_σ restricts to a bijection on $\iota(S)$. Here is the main theorem from [10]:

Theorem 10. *A Tait-colored graph admits a faithful GS realization if and only if it does not have a 2-edge cut set.*

Obviously the statement of this theorem is similar to the main results above on parallel drawings. The main difference is that the theorem does not resort to the notion of bipartiteness. One may prove this analogously. In the combinatorial direction, one shows that a 2-edge cut set makes faithfulness impossible, and in the constructibility direction one shows that one may perturb along cycles (of arbitrary length) to obtain a faithful drawing. Likewise, there are analogous statements for realization spaces and forced triples. Having seen this theorem, it should be clear why we regard a Tait-colored graph as a triple of fixed-point-free involutions: After projecting one of these configurations $\iota(S)$ down to one of the three prescribed subspaces, one obtains a linear configuration whose two-fold symmetry yields the corresponding permutation of those points.

Here is the connection to parallel drawings. Suppose $G = (S, T)$ is a connected, bipartite, Tait-colored graph which does not have a 2-edge cut set. Then G has a faithful GS realization, say ι. Let $\{S_1, S_2\}$ be the bipartition of the vertices. Define a map $\iota' : S \to \mathbb{R}^2$ by

$$\iota'(z) = \begin{cases} \iota(z) & \text{if } z \in S_1, \\ -\iota(z) & \text{if } z \in S_2. \end{cases}$$

Then ι' is a faithful parallel drawing of G.

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