

# Drawing Hamiltonian Cycles with No Large Angles

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**Abstract.** Let  $n \geq 4$  be even. It is shown that every set  $S$  of  $n$  points in the plane can be connected by a (possibly self-intersecting) spanning tour (Hamiltonian cycle) consisting of  $n$  straight line edges such that the angle between any two consecutive edges is at most  $2\pi/3$ . For  $n = 4$  and 6, this statement is tight. It is also shown that every even-element point set  $S$  can be partitioned into at most two subsets,  $S_1$  and  $S_2$ , each admitting a spanning tour with no angle larger than  $\pi/2$ . Fekete and Woeginger conjectured that for sufficiently large even  $n$ , every  $n$ -element set admits such a spanning tour. We confirm this conjecture for point sets in convex position. A much stronger result holds for large point sets *randomly* and uniformly selected from an open region bounded by finitely many rectifiable curves: for any  $\varepsilon > 0$ , these sets almost surely admit a spanning tour with no angle larger than  $\varepsilon$ .

## 1 Introduction

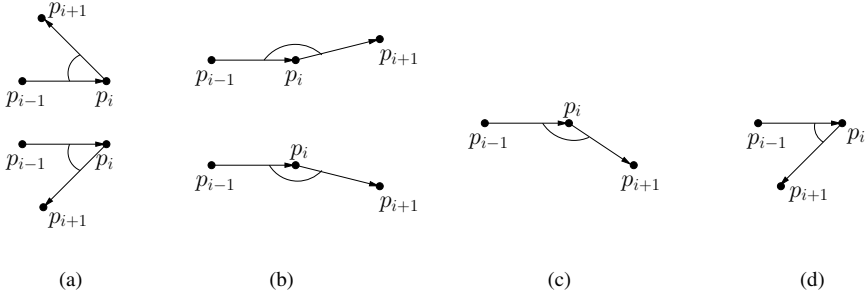
Consider a set of  $n \geq 2$  points. A *spanning tour* is a directed Hamiltonian cycle, drawn with straight line edges; if  $n = 2$  the tour consists of the two edges, with opposite orientations, connecting the two points. When three points,  $p_1$ ,  $p_2$ , and  $p_3$ , are traversed in this order, their *rotation angle*  $\angle p_1 p_2 p_3$  is the angle in  $[0, \pi]$  determined by segments  $p_1 p_2$  and  $p_2 p_3$ . If  $p_3$  is on the left (resp. right) side of the oriented line  $\overrightarrow{p_1 p_2}$  then we say that the tour, or path makes a *left* (resp. *right*) *turn* at  $p_2$ . If a tour (or path) makes only right turns, we call it *pseudo-convex*.

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\* Supported in part by NSF CAREER grant CCF-0444188, and by the Discrete and Convex Geometry project, in the framework of the European Community's "Structuring the European Research Area" program. Part of the research by this author was done at the Alfred Rényi Institute of Mathematics in Budapest, and at the Ecole Polytechnique Fédérale de Lausanne.

\*\* Research partially supported by NSF grant CCF-08-30272, grants from OTKA, SNF, and PSC-CUNY.

\*\*\* Supported by OTKA.



**Fig. 1.** (a) acute (b) obtuse (c) obtuse and pseudo-convex (d) acute and pseudo-convex

If all of its rotation angles are at most  $\pi/2$ , we call it an *acute* tour (or path). If all rotation angles are at least  $\pi/2$ , the tour (or path) is *obtuse*; see Figure 1.

Given a set  $A$  of angles, the *angle-restricted tour* (ART) problem is to decide whether a set  $S$  of  $n$  points in the plane allows a (possibly self-intersecting) spanning tour such that all the  $n$  angles between consecutive segments belong to the set  $A$ ; see [10].

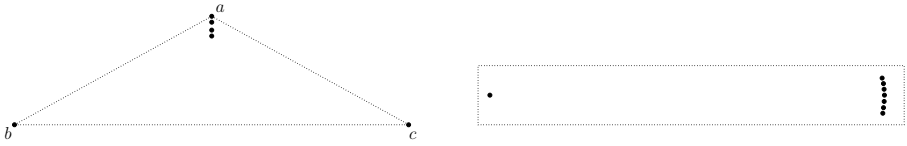
Fekete and Woeginger [10] proved that every finite set of at least *five* points admits a pseudo-convex tour and a non-intersecting pseudo-convex spanning path. They also noticed that every  $n$ -element point set  $S$  admits an acute spanning path. To see this, start at any point  $p_1 \in S$ . Assuming that the initial portion  $p_1 \dots p_i$  of such a path has already been defined and  $i < n$ , let  $p_{i+1}$  be an element of  $S \setminus \{p_1, \dots, p_i\}$  farthest away from  $p_i$ . It is easy to check that the resulting *path*  $p_1 \dots p_n$  is acute. It is also clear that such a path cannot be always completed to an acute *tour*. Indeed, if all points are on a line and  $n$  is *odd*, then along any (spanning) tour, one of the rotation angles must be equal to  $\pi$ .

The question arises: Does every *even*-element point set admit a tour with small rotation angles? More precisely, given an  $n$ -element point set  $S$  in the plane, where  $n$  is even, let  $\alpha = \alpha(S) \geq 0$  denote the smallest angle such that  $S$  admits a (spanning) tour with the property that all of its rotation angles belong to  $[0, \alpha]$ . Finally, let  $\alpha(n)$  be the maximum of  $\alpha(S)$  over all  $n$ -element point sets in the plane. Trivially,  $\alpha(2) = 0$ . The 4-element point set formed by the 3 vertices and the center of an equilateral triangle shows that  $\alpha(4) \geq 2\pi/3$ . The 6-point configuration depicted in Fig. 2 (left) shows that  $\alpha(6) \geq 2\pi/3$ .

In this note we show that  $\alpha(n) \leq 2\pi/3$ , for all even  $n \geq 4$ .

**Theorem 1.** *Let  $n \geq 4$  be even. Every set of  $n$  points in the plane admits a spanning tour such that all of its rotation angles are at most  $2\pi/3$ . This bound is tight for  $n = 4, 6$ . Such a tour can be computed in  $O(n^{4/3} \log^{1+\varepsilon} n)$  time, for every  $\varepsilon > 0$ .*

It remains open whether the bound  $2\pi/3$  can be replaced by  $\pi/2$ , for every even  $n \geq 8$ , as was conjectured in [10]. In other words, every  $n$ -element set may admit an acute tour, whenever  $n \geq 8$  is even. The point set depicted in Fig. 2 (right)



**Fig. 2.** Left:  $\triangle abc$  is an isosceles triangle with  $\angle bac = 2\pi/3$ . Point  $a$  and the 3 points below it are placed on the altitude of the triangle, and very closely inter-spaced. Every tour on these 6 points has a rotation angle of at least  $2\pi/3 - \varepsilon$ . Right:  $n - 1$  equidistant points very closely inter-spaced on a small arc of a circle, and one point at the center. Every tour on these  $n$  points has a rotation angle of at least  $\pi/2 - \varepsilon$ .

demonstrates that this statement, if true, cannot be improved. That is, we have  $\alpha(n) \geq \pi/2$ , for all even  $n \geq 8$ .

We confirm three weaker versions of this statement. In Section 4, we show that if we enforce acute rotation angles, *two* tours instead of one will certainly suffice.

**Theorem 2.** *Let  $n \geq 8$  be even.*

(i) *Every set of  $n$  points in the plane can be partitioned into two even parts, each of which admits an acute spanning tour. Given the  $n$  points, the two tours can be computed in  $O(n)$  time.*

(ii) *Every set of  $n$  points in the plane can be partitioned into two parts of sizes  $2\lfloor \frac{n}{4} \rfloor$  and  $2\lceil \frac{n}{4} \rceil$ , each of which admits an acute spanning tour. Given the  $n$  points, the two tours can be computed in  $O(n^{4/3} \log^{1+\varepsilon} n)$  time, for every  $\varepsilon > 0$ .*

In Section 5, we prove the existence of an acute tour in the special case when the points are in convex position.

**Theorem 3.** *Every even set  $S$  of  $n$  points in the plane in convex position, with  $n \geq 12$ , admits an acute spanning tour. Given the  $n$  points, such a tour can be computed in  $O(n)$  time.*

A much stronger statement holds for *random* point sets, uniformly selected from a not necessarily connected region.

**Theorem 4.** *Let  $B$  be an open region in the plane bounded by finitely many rectifiable Jordan curves and let  $S$  be a set of  $n$  points, randomly and uniformly selected from  $B$ . Then, for any  $\varepsilon > 0$ , the point set  $S$  almost surely admits a spanning tour with no rotation angle larger than  $\varepsilon$ , as  $n$  tends to infinity.*

The last result easily generalizes to higher dimensions.

*Related problems and results.* Various angle conditions imposed on *geometric graphs* (that is, graphs with straight-line edges) drawn on a fixed vertex set have been studied in [2,3,4,5]. For instance, sharpening an earlier bound of Bárány,

Pór, and Valtr [5], Kynčl [11] proved that any point set admits a (possibly self-intersecting) Hamiltonian path, in which each rotation angle is at least  $\pi/6$ . This result conjectured by Fekete and Woeginger [10] cannot be improved.

Aichholzer et al. [2] studied similar questions for planar geometric graphs. Among other results, they showed that any point set in general position in the plane admits a non-intersecting Hamiltonian (spanning) path with the property that each rotation angle is at most  $3\pi/4$ . They also conjectured that this value can be replaced by  $\pi/2$ . Arkin et al. introduced the notion of *reflexivity* of a point set, as the minimum number of reflex vertices in a polygonalization (i.e., simple polygon) of the set [4]. They gave estimates for the maximum reflexivity of an  $n$ -element point set. Recently, Ackerman et al. have made further progress on this problem [1].

## 2 Balanced Partitions

It is well known (see, e.g. [8], Section 6.6) that every region (every continuous probability measure) in the plane can be cut into *four* parts of equal area (measure) by two orthogonal lines. This statement immediately implies:

**Lemma 1.** *Given a set  $S$  of  $n \geq 8$  points in the plane ( $n$  even), one can always find two orthogonal lines  $\ell_1, \ell_2$  and a partition  $S = S_1 \cup S_2 \cup S_3 \cup S_4$  with  $|S_1| = |S_3| = \lfloor \frac{n}{4} \rfloor$ ,  $|S_2| = |S_4| = \lceil \frac{n}{4} \rceil$  such that  $S_1$  and  $S_3$  belong to two opposite closed quadrants determined by  $\ell_1$  and  $\ell_2$ , and  $S_2$  and  $S_4$  belong to the other two opposite quadrants.*

**Proof.** By a standard compactness argument, it is sufficient to prove this statement for point sets  $S$  in general position, in the sense that no 3 points of  $S$  are on a line, no 3 determine a right angle, and no two segments spanned by 4 points are orthogonal to each other. Choose a very small  $\varepsilon > 0$  and replace each point  $p \in S$  by a disk of radius  $\varepsilon$  around  $p$ . Applying the above mentioned result from [8] to the union of these  $n$  disks, we obtain two orthogonal lines that meet the requirements of the lemma.  $\square$

**Lemma 2.** *Given a set  $S$  of  $n$  points in the plane ( $n$  even), there exist three concurrent lines such that the angle between any two of them is  $\pi/3$ , and there is a partition  $S = S_1 \cup \dots \cup S_6$  with  $|S_1| = |S_4|$ ,  $|S_2| = |S_5|$ , and  $|S_3| = |S_6|$ , such that  $S_i$  is contained in the  $i$ -th closed angular region (wedge) determined by the lines, in counterclockwise order.*

**Proof.** Just like before, by compactness, it is sufficient to prove the statement for point sets in general position. This time, it is convenient to assume that no 3 points of  $S$  determine an angle which is an integer multiple of  $\pi/3$ , and there are no 2 pairs of points such that the angle between their connecting lines is an integer multiple of  $\pi/3$ .

Choose again a very small  $\varepsilon > 0$  and replace each point  $p \in S$  by a disk  $D_p$  of radius  $\varepsilon$  centered at  $p$ . Approximate very closely the union of these disks by

a continuous measure  $\mu$  which is strictly positive on every Jordan region in the plane and for which  $\mu(\mathbb{R}^2) = n$  and  $|\mu(D_p) - 1| < \varepsilon$  for every  $p \in S$ .

We say that a line  $\ell$  is a *bisecting* line with respect to the continuous measure  $\mu$  if the measures of both half-planes bounded by  $\ell$  are equal to  $n/2$ . Clearly, there is a unique bisecting line parallel to every direction, and this line changes continuously as the direction varies. Choose *three* bisecting lines  $\ell_1, \ell_2, \ell_3$  such that the angle between any two of them is  $\pi/3$ . By changing the direction of  $\ell_1$ , we can achieve that these lines pass through the same point. Indeed, as we turn  $\ell_1$  by  $\pi/3$ , the crossing point of the other two lines moves from one side of  $\ell_1$  to the other. Therefore, there is an intermediate position in which the three lines pass through the same point.

An easy case analysis shows that if  $\varepsilon$  was sufficiently small, then either no  $\ell_i$  intersects any disk  $D_p$  or there is one  $\ell_i$  that intersects two  $D_p$ 's and the others do not intersect any. In the former case, the lines satisfy the conditions in the lemma, in the latter one, they can be slightly perturbed so as to meet the requirements.  $\square$

Given a set  $S$  of  $n$  points in *general position* in the plane (i.e., no three points are collinear), a line passing through two elements of  $S$  is called a *halving line* if there are  $\lfloor (n-2)/2 \rfloor$  points on one of its sides and  $\lceil (n-2)/2 \rceil$  points on the other [12]. The number of halving lines of an  $n$ -element point set in the plane is bounded from above by  $O(n^{4/3})$ , as was established by Dey [9]. It is also known that the set of halving lines can be computed in  $O(n^{4/3} \log^{1+\varepsilon} n)$  time [6], for every  $\varepsilon > 0$ .

**Remark.** Starting with an arbitrary halving line  $\ell$  and following the rotation scheme described in [12], one can enumerate all halving lines for  $S$ . Using this approach, one obtains algorithmic proofs of Lemmas 1 and 2 that run in  $O(n^{4/3} \log^{1+\varepsilon} n)$  time, for every  $\varepsilon > 0$ .

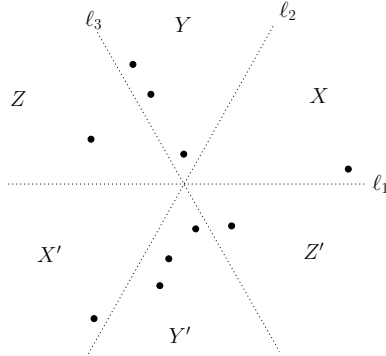
### 3 Making a Tour with Rotation Angles at Most $2\pi/3$

In this section, we prove Theorem 1. As we mentioned in the Introduction, for small even values of  $n$ , namely for  $n = 4$  and  $n = 6$ , we need to allow rotation angles as large as  $2\pi/3$ . Here we show that this value suffices for all even  $n$ .

Let  $\ell_1, \ell_2, \ell_3$  be three concurrent lines satisfying the conditions of Lemma 2. They divide the plane into six wedges.

Let  $X, Y, Z, X', Y', Z'$  denote the six wedges in counterclockwise order, labeled as in Fig. 3. Note that the angle between the  $x$ -axis and any edge  $p_{i-1}p_i$  of a tour with  $p_{i-1} \in X$  and  $p_i \in X'$ , say, belongs to the interval  $[0, \pi/3]$ . A piece  $p_{i-1}p_i p_{i+1}$  of a tour is of the form  $XX'X$ , say, if  $p_{i-1}, p_{i+1} \in X$  and  $p_i \in X'$ .

**Observation 1.** *Consider a piece of a tour, which is of the form  $XQX$ , where  $Q = Y', X'$ , or  $Z$ . Then the rotation angle at the middle point of this piece, which belongs to  $Q$ , is at most  $2\pi/3$ . The same holds for any other piece consisting of two edges, which starts and ends in the same wedge, and whose middle point belongs to one of the three opposite wedges.*

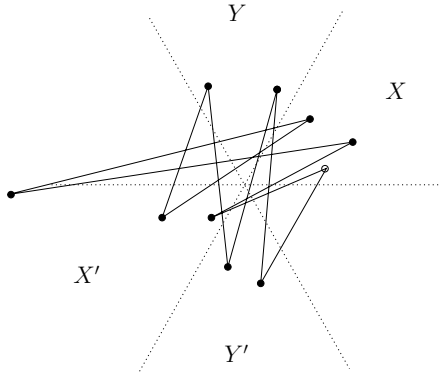


**Fig. 3.** Three concurrent bisecting lines of  $S$ :  $\ell_1, \ell_2, \ell_3$ , at angles  $0, \pi/3$ , and  $2\pi/3$

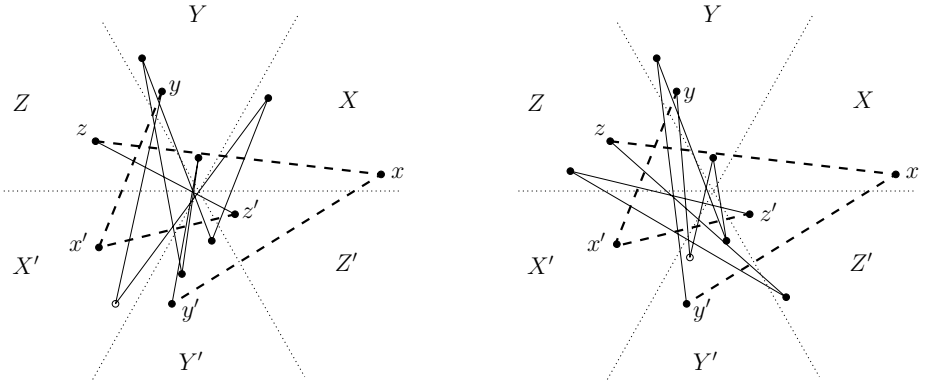
**Observation 2.** Consider a piece of a tour, which is of the form  $XX'Y$  or  $XX'Z'$ . Then the rotation angle at the middle point of this piece, which belongs to  $X'$ , is at most  $2\pi/3$ . The same holds for any other piece of the form  $X'XZ, X'XY', YY'X, YY'Z, Y'YX', Y'YZ', ZZ'Y, ZZ'X', Z'ZX, Z'ZY'$ .

**Proof of Theorem 1.** We distinguish two cases:

*Case 1.* There are at most two nonempty double wedges. If all points are contained in a unique double wedge, say  $XX'$  then, by Observation 1, they can be connected by an acute tour of the form  $(XX')^*$ . The tours starts in  $X$ , ends in  $X'$ , and alternates between the wedges  $X$  and  $X'$  until all points in  $X \cup X'$  are exhausted. Assume now that there are exactly two nonempty double wedges,  $XX'$  and  $YY'$ , say, and refer to Fig. 4. Consider a spanning tour of the form  $(XX')^*(YY')^*$ , where  $(XX')^*$  and  $(YY')^*$  are point sequences that alternate between the corresponding opposite wedges until all points in those wedges are



**Fig. 4.** Case 1: points in two double wedges. A tour of the form  $XX'XX'XX'YY'YY'$  is shown; its starting vertex in  $X$  is drawn as an empty circle.



**Fig. 5.** Case 2: points in three double wedges. Left: a tour of the form  $X'XY'YY'Yy'xz z'x'y$  is shown; its starting vertex in  $X'$  is drawn as an empty circle. Right: a tour of the form  $Y'YY'Yy'xz Z'Zz'x'y$  is shown; its starting vertex in  $Y'$  is drawn as an empty circle.

exhausted. By Observations 1 and 2, at each vertex of this tour the rotation angle is at most  $2\pi/3$ .

*Case 2.* There are exactly three nonempty double wedges; refer to Fig. 5.

Arbitrarily pick one point from each wedge:  $x \in X$ ,  $y \in Y$ ,  $z \in Z$ ,  $x' \in X'$ ,  $y' \in Y'$ ,  $z' \in Z'$ . Consider the two triangles  $\Delta xzy'$  and  $\Delta yx'z'$ . The sum of the interior angles of the two triangles is obviously  $2\pi$ . By averaging, there is one pair of points lying in opposite wedges, say  $x$  and  $x'$ , whose angles sum up to at most  $2\pi/3$ . Thus, each of these angles is at most  $2\pi/3$ :  $\angle zxy' \leq 2\pi/3$ , and  $\angle yx'z' \leq 2\pi/3$ .

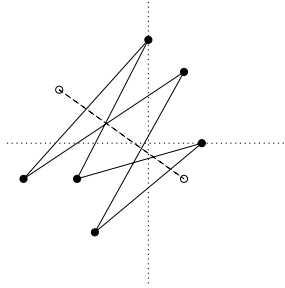
If  $|X \cap S| = |X' \cap S| \geq 2$ , consider a spanning tour of the form  $(X'X)^+(Y'Y)^+y'xz(Z'Z)^+z'x'y$ . Here  $(X'X)^+$  denotes a nonempty alternating path between the wedges  $X'$  and  $X$ , that starts in  $X'$ , ends in  $X$ , and involves all points except  $x$  and  $x'$ . The notations  $(Y'Y)^+$  and  $(Z'Z)^+$  are used analogously. An example is depicted in Fig. 5 (left). By Observations 1 and 2, and by our choice of  $x, y, z, x', y', z'$ , all rotation angles along this tour are at most  $2\pi/3$ , as required.

If  $|X \cap S| = |X' \cap S| = 1$ , consider a spanning tour  $(Y'Y)^+y'xz(Z'Z)^+z'x'y$ ; see Fig. 5 (right). The arguments justifying that all rotation angles are at most  $2\pi/3$  are the same as before.

The proof of Theorem 1 is now complete.

## 4 Covering by Two Acute Tours

**Proof of Theorem 2. (i)** Take a horizontal line  $\ell$  and a partition of our point set  $S = S^+ \cup S^-$  into two subsets, each of size  $n/2$ , such that  $S^+$  and  $S^-$  are in the closed half-planes above and below  $\ell$ , respectively. If some points of  $S$  lie on



**Fig. 6.** Even set covered by two tours with 6 and 2 points, respectively;  $a = 3$ , and  $b = 1$  (A double-edge counts as a tour.)

$\ell$ , we can include them in either of these sets so as to satisfy the condition. Next, take a vertical line  $\ell'$  which gives rise to another equipartition of  $S$ . Assume for simplicity that  $\ell$  and  $\ell'$  coincide with the  $x$  and  $y$  coordinate axes. See Fig. 6, for an illustration.

Thus, we obtain a partition  $S = S_1 \cup S_2 \cup S_3 \cup S_4$  such that all points of  $S_i$  belong to the  $i$ -th closed quadrant determined by the axes (enumerated in the counterclockwise order),  $|S_1| = |S_3| = a$ , and  $|S_2| = |S_4| = b$  for some integers  $a$  and  $b$  with  $a + b = n/2$ . Connect now all elements of  $S_1 \cup S_3$  by a tour of length  $2a$  alternating between  $S_1$  and  $S_3$ . Similarly, connect the elements of  $S_2 \cup S_4$  by an alternating tour of length  $2b$ . Obviously, both tours are acute. The above procedure can be performed in linear time, using any linear time selection algorithm [7].

(ii) Find two orthogonal lines and a partition  $S = S_1 \cup S_2 \cup S_3 \cup S_4$  satisfying the conditions of Lemma 1. Using the notation of the proof of part (i), now we have  $a = \lfloor \frac{n}{4} \rfloor$  and  $b = \lceil \frac{n}{4} \rceil$ . As above, we obtain two acute tours, of lengths  $2\lfloor \frac{n}{4} \rfloor$  and  $2\lceil \frac{n}{4} \rceil$ , respectively. This completes the the proof of part (ii) of Theorem 1.  $\square$

By keeping only the larger tour, Theorem 2 immediately implies

**Corollary 1.** *For any even  $n$ , every  $n$ -element point set in the plane admits an acute even tour covering at least half of its elements.*

## 5 Acute Tours for Points in Convex Position

Throughout this section, let  $S$  denote a set of  $n \geq 8$  points in the plane, in convex position and let  $S = S_1 \cup S_2 \cup S_3 \cup S_4$  be a partition satisfying the conditions in Lemma 1. A 3-edge path (on 4 points) is called a *hook* if the rotation angles at its two intermediate vertices are acute.

**Lemma 3.** *Let  $P = \{p_1, p_2, p_3, p_4\}$  be the vertex set of a convex quadrilateral, with  $p_i \in S_i$ ,  $i = 1, 2, 3, 4$ . Then at least one of the following two conditions is satisfied.*

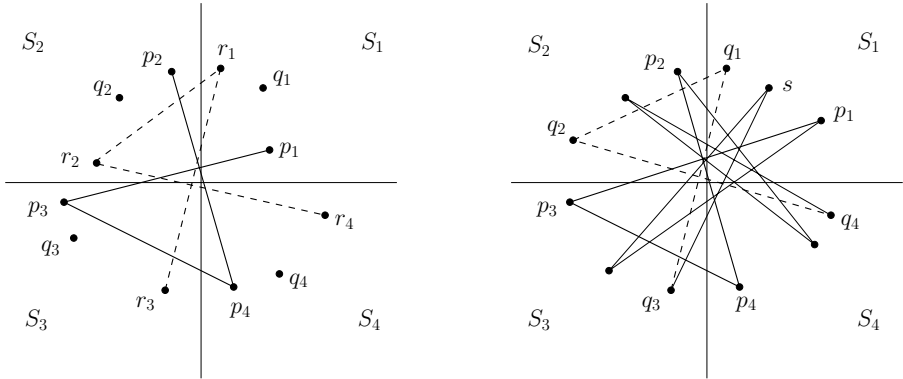


- (i)  $p_1p_3p_4p_2$  and  $p_3p_1p_2p_4$  are hooks, or
- (ii)  $p_1p_3p_2p_4$  and  $p_3p_1p_4p_2$  are hooks.

**Proof.** At least one of the two angles defined by the diagonals  $p_1p_3$  and  $p_2p_4$  is larger or equal to  $\pi/2$ . Let  $x$  denote the crossing point of these diagonals. If  $\angle p_1xp_2 \geq \pi/2$ , then the two 3-edge paths  $p_1p_3p_4p_2$  and  $p_3p_1p_2p_4$  are hooks, while if  $\angle p_2xp_3 \geq \pi/2$ , then  $p_1p_3p_2p_4$  and  $p_3p_1p_4p_2$  are hooks.  $\square$

We say that a convex quadrilateral  $P$ , as in Lemma 3, is of *type 1* if  $\angle p_1xp_2 \geq \pi/2$ , and of *type 2*, otherwise (i.e., if  $\angle p_2xp_3 > \pi/2$ ).

**Lemma 4.** Let  $P = \{p_1, p_2, p_3, p_4\}$ ,  $Q = \{q_1, q_2, q_3, q_4\}$ , and  $R = \{r_1, r_2, r_3, r_4\}$  be three vertex-disjoint convex quadrilaterals with  $p_i, q_i, r_i \in S_i$ , for  $i = 1, 2, 3, 4$ . Then there exist two hooks induced by two of these quadrilaterals such that the two endpoints of the first one and the two endpoints of the second one lie in different parts of the partition  $S_1 \cup S_2 \cup S_3 \cup S_4$ . Two such hooks are called *opposite*. (See Fig. 7 (left).)



**Fig. 7.** Left:  $p_1p_3p_4p_2$  and  $r_3r_1r_2r_4$  are two opposite hooks. Right: an acute tour of  $S$  of the form  $(S_1S_3)^+p_1p_3p_4p_2(S_4S_2)^+q_4q_2q_1q_3$ , starting at  $s \in S_1$ .

**Proof.** By the pigeonhole principle, two out of the three quadrilaterals, say  $P$  and  $Q$ , must have the same type. By Lemma 3, one can find a hook in each of them such that their endpoints are all in different parts of the partition, i.e., two opposite hooks.  $\square$

**Proof of Theorem 3.** Consider a partition  $S = S_1 \cup S_2 \cup S_3 \cup S_4$  satisfying the conditions in Lemma 1. Since  $|S| \geq 12$ , we have  $|S_i| \geq 3$ . Pick 3 points from each  $S_i$ , and using these points construct three vertex-disjoint convex quadrilaterals,  $P$ ,  $Q$ , and  $R$ . By Lemma 4, two of these quadrilaterals,  $P$  and  $Q$ , say, determine opposite vertex-disjoint hooks. Suppose without loss of generality that  $P$  and  $Q$  are of type 1, and these two hooks are  $p_1p_3p_4p_2$  and  $q_4q_2q_1q_3$ , where  $p_i, q_i \in S_i$ ,  $i = 1, 2, 3, 4$ . See Fig. 7(right).

Let  $(S_i S_j)^+$  denote a polygonal path starting in  $S_i$ , ending in  $S_j$ , alternating between  $S_i$  and  $S_j$ , and exhausting all points of  $S_i \cup S_j$ , except for  $p_i, p_j, q_i, q_j$ . The following tour is acute:  $(S_1 S_3)^+ p_1 p_3 p_4 p_2 (S_4 S_2)^+ q_4 q_2 q_1 q_3$ , and this completes the proof.  $\square$

## 6 Random Point Sets

We first verify Theorem 4 for centrally symmetric convex bodies, and then in its full generality.

**Lemma 5.** *Let  $B$  be a centrally symmetric convex body in the plane and let  $S$  be a set of  $n$  points, randomly and uniformly selected from  $B$ . Then, for any  $\varepsilon > 0$ ,  $S$  almost surely admits a spanning tour with no rotation angle larger than  $\varepsilon$ , as  $n$  tends to infinity.*

**Proof.** Let  $\varepsilon$  be fixed, and let  $o$  denote the center of  $B$ . Assume without loss of generality that  $\text{area}(B) = 1$ . Any chord through  $o$  divides the area of  $B$  into two equal parts. Therefore, there is a positive constant  $\delta = \delta(B, \varepsilon)$ , depending only on  $B$  and  $\varepsilon$ , such that for every wedge  $W$  with angle at most  $\pi - \frac{\varepsilon}{2}$  and apex at  $o$ , we have that  $\text{area}(W \cap B) \leq 1/2 - \delta$ . Let  $m = \lceil n/2 \rceil$ .

Let  $p_1, p_2, \dots, p_n$  be  $n$  random points, independently and uniformly selected from  $B$ , listed in their circular order of visibility from  $o$ . The indices are taken modulo  $n$ , so that  $p_{n+1} = p_1$ . Note that almost surely all points  $p_i$  are distinct and different from  $o$ .

If  $n$  is odd, consider the spanning tour  $C = p_1 p_{m+1} p_2 p_{m+2} \dots p_m p_1$ . For every  $i$ , almost surely we have

$$\pi - \frac{\varepsilon}{2} \leq \angle p_i o p_{m+i-1} \leq \pi + \frac{\varepsilon}{2},$$

and

$$\pi - \frac{\varepsilon}{2} \leq \angle p_i o p_{m+i} \leq \pi + \frac{\varepsilon}{2}.$$

Therefore, we almost surely have  $\angle p_{m+i-1} p_i p_{m+i} \leq \varepsilon$ , for every  $i$ , and the tour  $C$  meets the requirements.

If  $n$  is even, we choose two odd numbers  $n_1, n_2$  with  $n_1 + n_2 = n$  such that  $0 \leq n_1 - n_2 \leq 2$ . That is,  $n_1$  is  $m$  or  $m+1$  while  $n_2$  is  $m$  or  $m-1$ . Connect the points  $p_i$  by two disjoint cycles,  $C_1$  and  $C_2$ , of length  $n_1$  and  $n_2$ , with property that (1) in the cyclic order around  $o$ , the points  $p_1, p_2, \dots$  belong alternately to  $C_1$  and  $C_2$ , as much as possible; and (2) every edge of  $C_1$  and  $C_2$  connects two points,  $p_i$  and  $p_j$ , with  $|j - i - m| \leq 3 \pmod{n}$ . We distinguish two cases.

*Case 1.*  $n_1 = n_2 = m$ . Let

$$C_1 = p_1 p_{2+m} p_3 p_{4+m} p_5 \dots p_{n-1} p_m,$$

$$C_2 = p_2 p_{3+m} p_4 p_{5+m} p_6 \dots p_n p_{1+m}.$$

Switching between these two cycles at two points, we can combine them into a single spanning tour  $C$ , as follows.

$$C = p_1 p_{2+m} p_3 p_{4+m} p_5 \cdots p_{n-1} p_m p_2 p_{3+m} p_4 p_{5+m} p_6 \cdots p_n p_{1+m}.$$

It remains true that  $|j - i - m| \leq 3 \pmod n$  for every edge  $p_i p_j$  of  $C$ , so that almost surely all rotation angles of  $C$  will be smaller than  $\varepsilon$ .

*Case 2.*  $n_1 = m + 1, n_2 = m - 1$ . Let

$$C_1 = p_1 p_{2+m} p_3 p_{4+m} p_5 \cdots p_n p_{m+1},$$

$$C_2 = p_2 p_{3+m} p_4 p_{5+m} p_6 \cdots p_{n-1} p_m.$$

We can combine them into a single spanning tour  $C$ , as follows.

$$C = p_1 p_{2+m} p_3 p_{4+m} p_5 \cdots p_n p_{m+1} p_2 p_{3+m} p_4 p_{5+m} p_6 \cdots p_{n-1} p_m.$$

It remains true that  $|j - i - m| \leq 3 \pmod n$  for every edge  $p_i p_j$  of  $C$ , so that almost surely all rotation angles of  $C$  will be smaller than  $\varepsilon$ . □

To prove Theorem 4 in its full generality, we need the following technical lemma. Its proof is very similar to that of Lemma 5. The minor modifications are left to the reader.

**Lemma 6.** *Let  $B$  be a centrally symmetric convex set in the plane with nonempty interior. Let  $o$  denote the center of  $B$ , let  $\varepsilon > 0$  be fixed, let  $s$  and  $t$  be two points of  $B$ , and let  $S'$  be a set of at most  $\varepsilon n/4$  points not belonging to  $B$ .*

*Then, for any set  $S$  of  $n$  points randomly and uniformly selected from  $B$ , the set  $S \cup S'$  almost surely admits a spanning path satisfying the following conditions, as  $n \rightarrow \infty$ :*

- (i) *all of its turning angles are at most  $\varepsilon$ ;*
- (ii) *its first two points are  $p_1$  and  $p_2$  such that  $\angle op_1 p_2 \leq \varepsilon/3, \angle sop_1 \leq \varepsilon/3$ ;*
- (iii) *its last two points are  $q_2$  and  $q_1$  such that  $\angle oq_1 q_2 \leq \varepsilon/3, \angle toq_1 \leq \varepsilon/3$ .*

**Proof of Theorem 4.** Assume without loss of generality that  $\text{area}(B) = 1$ . Consider a square lattice of minimum distance  $\delta$ , for some  $\delta > 0$  to be specified later. Let  $A = A(\delta)$  denote the total area of all cells (lattice squares of side length  $\delta$ ) completely contained in  $B$ , and let  $A' = A'(\delta)$  denote the total area of all those cells that intersect  $B$ , but are not completely contained in it. Obviously,  $A + A' \geq 1$ . Since the boundary of  $B$  is the union of finitely many rectifiable curves, we have

$$\lim_{\delta \rightarrow 0} A = 1, \quad \limsup_{\delta \rightarrow 0} \frac{A'}{\delta} < \infty.$$

Therefore, we can choose  $\delta > 0$  so that  $A' \leq \varepsilon/6$ .

Let  $X_1, X_2, \dots, X_m$  denote the cells completely contained in  $B$ , in some arbitrary order, and let  $o_i$  denote the center of  $X_i$ . For any  $1 \leq i \leq m$ , let  $s_i$  be a point on the line  $o_i o_{i-1}$  such that  $o_i$  belongs to the segment  $s_i o_{i-1}$ . Analogously,

let  $t_i$  be a point on the line  $o_i o_{i+1}$  such that  $o_i$  belongs to the segment  $t_i o_{i+1}$ . Here the indices are taken modulo  $m$ .

Let  $S$  be a set of  $n$  points in  $B$ , selected independently, randomly, and uniformly. Let  $S_i = S \cap X_i$ , for  $1 \leq i \leq m$ , and let  $S' = S \setminus \bigcup_{i=1}^m S_i$ . Divide  $S'$  into  $m$  almost equal parts,  $S'_1, S'_2, \dots, S'_m$  with  $||S'_i| - |S'_j|| \leq 1$ , for any  $i, j = 1, \dots, m$ .

For each  $1 \leq i \leq m$ , apply Lemma 6 with  $S_i, S'_i, s_i$ , and  $t_i$ , to obtain a spanning path  $P_i$ . The spanning tour  $P_1 P_2 \dots P_m$  obtained by the concatenation of these paths now meets the requirements.  $\square$

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