Chapter 12
Appendix on Double Cosets

We now discuss a double coset decomposition for the symplectic group $GSp(2n, F)$, which in the case $n = 2$ was found by Schröder [81]. Let F be a local non-Archimedean field of residue characteristic not equal to 2, let $\mathfrak{o}_F$ be its ring of integers, and let π_F denote a prime element. Let $G(F) = GSp(2n, F) \subseteq Gl(2n, F)$ be the group of symplectic similitudes. Hence, $g \in G(F)$ iff $g' J g = \lambda(g) \cdot J$ for a scalar $\lambda(g) \in F^*$, where

$$J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$$

and where E denotes the unit matrix. Then $g \in G(F) \iff (g')^{-1} \in G(F) \iff g' \in G(F)$ and $J' = J^{-1} = -J \in G(F)$. Let $G(\mathfrak{o}_F) = GSp(2n, \mathfrak{o}_F)$ denote the group of all unimodular symplectic similitudes.

Centralizers. For $n = i + j$ and $i \leq j$ put

$$s \;=\; diag(E^{(i,i)}, -E^{(j,j)}, E^{(i,i)}, -E^{(j,j)}) \;\in\; G(F).$$

The connected component of the centralizer $H = (G_s)^0$ of s is a maximal connected reductive subgroup of G. $H(F)$ is isomorphic to the subgroup of all matrices (g_1, g_2) in $GSp(2i, F) \times GSp(2j, F)$ with similitude factor $\lambda(g_1) = \lambda(g_2)$

$$1 \to \; H(F) \;\to\; GSp(2i, F) \times GSp(2j, F) \;\to\; F^* \to 1.$$

The Matrices $g(e_1, \ldots, e_i)$. Let denote $g(e_1, \ldots, e_i)$ the upper triangular matrix

$$g(D) = \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}, \qquad S = \begin{pmatrix} 0^{(i,i)} & D \\ D' & 0^{(j,j)} \end{pmatrix},$$

defined by $D = \big(diag(\pi_F^{e_1}, \ldots, \pi_F^{e_i})\,,\ 0^{(i,j-i)}\big)$, where we assume $e_\nu \in \mathbb{Z}$ and

$$e_1 \leq e_2 \ldots \leq e_i \leq \infty.$$

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Theorem 12.1. *The matrices* $g(e_1, \dots, e_i)$*, for* $e_1 \leq e_2 \cdots \leq e_i \leq \infty$ *and* $e_\nu < 0$ *or* $e_\nu = \infty$ *for all* $\nu = 1, \dots, i$*, define a system of representatives for the double cosets*

$$H(F) \setminus G(F) \,/G(\mathfrak{o}_F).$$

Remark 12.1. An alternative choice would have been $g(e_1, \dots, e_i)$ with $e_1 \leq \dots \leq e_i \leq 0$. Using this representatives one obtains the following corollary.

Corollary 12.1. *Let* T *be the diagonal torus in* H *or in* G*. Then there exists an element* $r \in G(F)$ *such that the set of conjugates* $\{trt^{-1} \mid t \in T(F)\}$ *of* r *contains a complete set of representatives of* $H(F) \setminus G(F)/G(\mathfrak{o}_F)$*.*

For instance, one can choose $r = g(0, \dots, 0) \in G(\mathfrak{o}_F)$. For $\mathbf{D} = diag(D, E, D^{-1}, E) \in T(F)$ and $D = diag(\pi_F^{e_1}, \dots, \pi^{e_i})$ then $\mathbf{D} r \mathbf{D}^{-1} = g(e_1, \dots, e_i)$.

The proof of the theorem requires some preparation. In the following we always assume that D satisfies $e_1 \leq \cdots \leq e_\nu$ and $e_{\nu+1} = \cdots = e_i = \infty$ for some $\nu \leq i$:

1. **The parabolic subgroups P_s.** There is a parabolic subgroup $P = P_s$ of G with Levi component L in $H = H_s$. Let $P = L \cdot N$, where N is the unipotent radical. Then Iwasawa decomposition $G(F) = L(F) \cdot N(F) \cdot G(\mathfrak{o}_F)$ allows us to choose representatives $g(M, N, U, V) \in N(F)$ of the form

$$g(M, N, U, V) = \begin{pmatrix} E & M & U & N \\ 0 & E & * & V \\ 0 & 0 & E & 0 \\ 0 & 0 & -M' & E \end{pmatrix}.$$

   Notice $g(0, 0, 0, V) g(M, N, U, 0) = g(M, N, U, V)$ and $V = V' = V^{(j,j)}$ is symmetric. Since $g(0, 0, 0, V) \in H(F)$ we can assume $V = 0$ and therefore write $g(M, N, *) = g(M, N, *, 0)$ for the representative

$$g(M, N, *) = \begin{pmatrix} E & M & * & N \\ 0 & E & N' & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & -M' & E \end{pmatrix}.$$

   For the moment $M, N \in Hom_F(F^j, F^i)$ are still arbitrary.
2. Notice $g(M_1, N_1, U_1) \cdot g(M_2, N_2, U_2) = g(M_1 + M_2, N_1 + N_2, U_1 + U_2 + M_1 \cdot N_2' - N_1 \cdot M_2')$; hence, $g(M, N, U) \cdot g(0, 0, \tilde{U}) = g(0, 0, \tilde{U}) \cdot g(M, N, U) = g(M, N, \tilde{U} + U)$ and $g(0, 0, U) \in H(F)$. $S = U - M \cdot N'$ is symmetric. For symmetric $S = S'$ now $g(0, 0, S) \in H(F)$. Hence, we can choose the representatives in the form

$$g(M, N) = g(M, N, M \cdot N').$$

Since $g(M_1, N_1)g(M_2, N_2) = g(M_1+M_2, N_1+N_2, M_1N_1'+M_2N_2'+M_1N_2'-N_1M_2') = g(M_1+M_2, N_1+N_2, (M_1+M_2)(N_1+N_2)' - M_2N_1' - N_1M_2') = g(0, 0, -M_2N_1' - N_1M_2') \cdot g(M_1+M_2, N_1+N_2)$

$$H(F)\cdot g(M_1, N_1)g(M_2, N_2)\cdot G(\mathfrak{o}_F) \;=\; H(F)\cdot g(M_1+M_2, N_1+N_2)\cdot G(\mathfrak{o}_F).$$

Hence, $(M, N) \in Hom_F(F^{2j}, F^i)$ can be modified within the double coset by adding an arbitrary element from $Hom_{\mathfrak{o}_F}(\mathfrak{o}_F^{2j}, \mathfrak{o}_F^i)$.

3. For $A = A^{(i,i)}$ and $B = B^{(i,i)}$ we later consider the special cases

$$M = (A, 0), \qquad N = (B, 0).$$

We then simply write $g(A, B)$ or $g(A, B, *)$ instead of $g(M, N)$ and $g(M, N, *)$, respectively. The formulas above are valid with A, B in place of M, N. For

$$A = \begin{pmatrix} A_0 & 0 \\ A_1 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix}$$

and $k \times k$-matrices A_0, B_0, and $k < i$ and integral matrices A_1, A_2, B_1 step (2) allows us to replace the matrices A_1, A_2 by zero and B_1 by the unit matrix, without changing the double coset.

4. Next, for $U_i \in Gl(i, \mathfrak{o}_F)$ we obtain equivalent representatives $g(M, N, *)$ and $g(U_i \cdot M, U_i \cdot N, *)$ by conjugation with $diag(U_i, E, (U_i')^{-1}, E)$.
5. On the $i \times 2j$-matrices in $Hom_F(F^{2j}, F^i)$ the elements $g \in Sp(2j, \mathfrak{o}_F)$ act by multiplication from the right

$$(\tilde{M}, \tilde{N}) = (M, N) \cdot g^{-1}.$$

$g(M, N, *)$ and $g(\tilde{M}, \tilde{N}, *)$ define the same double coset. It suffices to show this for generators g of $Sp(2j, \mathfrak{o}_F)$. For this notice $w_j \cdot g(M, N, 0) \cdot w_j^{-1} = g(N, -M, *)$ and $u_T \cdot g(M, N, 0) \cdot u_T^{-1} = g(M, N - MT, *)$ for the generators (see [28], Satz A.5.4)

$$w_j = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & 0 & 0 & E \\ 0 & 0 & E & 0 \\ 0 & -E & 0 & 0 \end{pmatrix}$$

and

$$u_V = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & V \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \end{pmatrix}.$$

For integral symmetric V these are contained in the intersection of $G(\mathfrak{o}_F)$ and $H(F)$. Hence, we may choose our representatives (A, B) in

$$Gl(i, \mathfrak{o}_F) \setminus Hom_F(F^{2j}, F^i) \,/Sp(2j, \mathfrak{o}_F),$$

where these, in addition, may be modified by elements from $Hom_{\mathfrak{o}_F}(\mathfrak{o}_F^{2j}, \mathfrak{o}_F^i)$.

12.1 Reduction to Standard Type

We say (M,N) are of *standard* type if

$$(M,N) = \Big((A,0),(B,0)\Big)$$

for an $i \times i$-diagonal matrix $B = diag(\pi_F^{e_1},\ldots,\pi_F^{e_i})$ and a nilpotent $i \times i$-lower triangular matrix A such that:

(a) $e_1 \leq e_2 \leq \ldots \leq e_i \leq 0$.
(b) $B^{-1} \cdot A$ is an integral matrix.

The Reduction. We now construct elements in $Sp(2j,\mathfrak{o}_F) \times Gl(i,\mathfrak{o}_F)$ which transform a given $(M,N) \in Hom_F(F^{2j},F^i)$ into standard type. For this temporarily replace (M,N) by $(N,-M)$ (using conjugation by w_j as in step (5)), and then replace the resulting matrix by its transpose in

$$Hom_F(F^i,F^{2j}).$$

By this $Sp(2j,\mathfrak{o}_F)$ now acts from the left and $Gl(i,\mathfrak{o}_F)$ acts from the right. Our argument now proceeds using induction. Start with an arbitrary matrix in $Hom_F(F^i,F^{2j})$. We say it is of *weak* r-standard type if it is of the form

$$\begin{pmatrix} B_r & * \\ 0 & * \\ -A'_r & * \\ 0 & * \end{pmatrix},$$

where $r \leq i \leq j$ and $B_r = diag(\pi_F^{e_1},\ldots,\pi_F^{e_r})$ and $e_1 \leq \ldots \leq e_r \leq \infty$, such that A'_r is a strict upper triangular $r \times r$-matrix such that:

(a) π^{e_r} divides the greatest common divisor (gcd) π^e of all entries of the matrix denoted by a star.
(b) π^{e_ν} divides all entries of the νth column for $1 \leq \nu \leq r$.

If, in addition, the shape is

$$\begin{pmatrix} B_r & 0 \\ 0 & * \\ -A'_r & * \\ 0 & * \end{pmatrix}$$

we say the matrix is *partially* of r-standard type.

By elimination of the right upper block a representative of weak partial r-standard type can be transformed to become partially of r-standard type. Use right multiplication with some element in $Gl(i,\mathfrak{o}_F)$ to clear the first r rows of the dotted area by adding columns. This does not change condition (b), since $e_1 \leq \ldots e_r$

and e_r is less than or equal to the gcd of the remaining columns (beginning from $r+1$). Since we add multiples of $\pi_F^{e-e_\nu}\lambda, \lambda \in \mathfrak{o}_F$ times the νth column $(\nu \leq r)$, we add terms in $\pi_F^e \mathfrak{o}_F$ as follows from condition (b). Therefore, the gcd of the back columns will not be changed by this procedure.

The Induction Step. For a matrix partially of $(r-1)$-standard type consider the columns beginning from the rth column. By right multiplication with a permutation matrix in $Gl(i, \mathfrak{o}_F)$ one can achieve the gcd π_F^e of all these columns already being the gcd of the entries of the rth column vector $v \in F^{2j}$. The first $(r-1)$-entries of v are zero since the matrix we started with was partially of $(r-1)$-standard type, and since the permutations of columns beginning from the rth column do not change the property such that the upper entries of these columns vanish.

Now our modifications will only involve multiplications with elements in $G(\mathfrak{o}_F)$ from the left. This changes the columns beginning from the $(r+1)$th. In particular, this may destroy the property that the first $(r-1)$-coordinates of these columns vanish. The given matrix is of the form

$$\begin{pmatrix} B_{r-1} & 0 & * \\ 0 & * & * \\ -A'_{r-1} & * & * \\ 0 & * & * \end{pmatrix}$$

such that the gcd π_F^e of the "middle" rth column divides the gcd of all columns beginning from the $(r+1)$th. This property is preserved under multiplication with substitutions from $G(\mathfrak{o}_F)$. Hence, in principle, we can concentrate on the first r columns since it is enough to bring our representative into a form of weak partial r-standard type. We therefore temporarily ignore all columns beginning from the $(r+1)$th column.

A suitable symplectic transformation of an embedded $Sp(2(j-r+1), \mathfrak{o}_F)$ by multiplication from the left allows us to make all coordinates of v be zero, except the rth and the $(j+1), \ldots, (j+r-1)$th coordinate entries. By this the first $(r-1)$-columns of our representative will not be changed. In addition we can achieve the rth coordinate entry of v being a power π_F^f of the prime element. For this notice that the unimodular symplectic matrices act transitively on primitive vectors ([28], Hilfssatz A.5.2).

After this the matrix is almost of weak partial r-standard type, being of the form

$$\begin{pmatrix} B_r & * \\ 0 & * \\ -A'_r & * \\ 0 & * \end{pmatrix}$$

such that (a) is satisfied. We are done if the rth coordinate entry π_F^f of the rth column is equal to π_F^e. If it is not, then $e < f$. Then there exists ν with $1 \leq \nu < r$ such that the gcd of the rth column is realized at the $(j+\nu)$th coordinate entry. It then remains to bring the gcd of column v to the "top." Left multiplication by a symplectic unimodular substitution – on the standard basis w_i of F^{2j} given by $w_\mu \mapsto w_\mu$ for $\mu \neq j+\nu, j+r$ and by $w_{j+\nu} \mapsto w_{j+\nu}+w_r$ and $w_{j+r} \mapsto w_{j+r}+w_\nu$ – has no effect on the lower half, i.e., A'_r will not be changed. Also the zero blocks on the left side will not be changed. The matrix B_r, on the other hand, will be modified. Since the rth line of $-A'_r$ is zero, only the last line of B_r will be changed – in fact by addition of the νth line of $-A'_r$. Let $x_1, \ldots, x_r$ denote the new entries. For example, $x_r = \pi_F^{e_r} + \pi_F^{e} = \varepsilon \cdot \pi^e$ ($\varepsilon \in \mathfrak{o}_F^*$).

Next the modified B_r will again be diagonalized by left multiplication by a unimodular symplectic matrix of the form $diag(U, E, (U')^{-1}, E)$, where

$$U = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & 0 & \ldots & 1 & 0 \\ y_1 & y_2 & & y_{r-1} & \varepsilon^{-1} \end{pmatrix}.$$

This transforms $-A'_r$ into $-(U')^{-1} \cdot A'_r = -A'_r$ (the rth column of A'_r is zero) and transforms B_r into $U \cdot B_r$. For suitable y_ν the matrix $U \cdot B_r$ will become a diagonal matrix with the entries $diag(\pi_F^{e_1}, \ldots, \pi_F^{e_{r-1}}, \pi_F^{e})$, provided $y_\nu \cdot \pi_F^{\varepsilon_\nu} = -\varepsilon^{-1} x_\nu$ holds. By condition (b) of the matrix of partial $(r-1)$-standard type we started with, such y_ν can be chosen in $\mathfrak{o}_F$. This implies $U \in Gl(r, \mathfrak{o}_F)$ and $diag(U, E, (U')^{-1}, E) \in G(\mathfrak{o}_F)$. This shows that our new matrix is now of weak partial r-standard type such that $e_r = e$, and it is a representative in the double coset of the matrix we started from. This completes the proof of the induction step.

Iterating this i times, we can get a matrix of partial i-standard type. Reverse transposition and reverse conjugation by w_j therefore gives an equivalent matrix replacing (M, N), which now is almost of standard type. It is of the form $(M, N) = \big((A, 0), (B, 0)\big)$ for $B = diag(\pi_F^{e_1}, \ldots, \pi_F^{e_i})$ and a lower triangular matrix A, whose diagonal is zero, and such that $e_1 \leq e_2 \leq \ldots \leq e_i \leq \infty$. Choose k to be maximal such that $e_k < 0$. By step (3) we can assume without restriction of generality $e_\nu = 0$ for $\nu > k$. Then B^{-1} is defined, and $B^{-1}A$ is an integral matrix. So we have a matrix of standard type.

Summary. *There exist representatives of the double cosets $H(F) \setminus G(F)/G(\mathfrak{o}_F)$ of the form $g = g((A, 0), (B, 0))$, such that:*

- $B = diag(\pi_F^{e_1}, \ldots, \pi_F^{e_i})$ *is a diagonal invertible $i \times i$-matrix with* $e_1 \leq \ldots e_i \leq 0$.
- *A is a lower triangular matrix.*
- *B^{-1} has integral entries.*
- *The lower $i \times i$-triangular matrix $B^{-1} \cdot A$ has integral entries.*
- *$B^{-1} \cdot A'$ is an $i \times i$-matrix with integral entries.*

12.2 The Quadratic Embedding

The matrix $\Lambda_s = s \cdot J = J \cdot s$ is skew-symmetric

$$\Lambda_s \;=\; \begin{pmatrix} 0 & 0 & E & 0 \\ 0 & 0 & 0 & -E \\ -E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \end{pmatrix}.$$

For $g \in G(F)$ the conditions $g \in H(F)$ and $\lambda(g)^{-1} g' \cdot \Lambda_s \cdot g \;=\; \Lambda_s$ are equivalent; since $\lambda(g)^{-1} \cdot (g')^{-1} = JgJ^{-1}$ and $J^{-1}\Lambda_s = s$, the first equation is equivalent to $s \cdot g = g \cdot s$.

Consequence. *$Elm(H(F) \cdot g) \;= \lambda(g)^{-1} g' \cdot \Lambda_s \cdot g$ defines an injection*

$$Elm: \; H(F) \setminus G(F) \;\hookrightarrow\; \Lambda^2(F^{2n})$$

of the cosets $H(F) \setminus G(F)$ into the vector space $\Lambda^2(F^{2n})$ of skew-symmetric $2n$-matrices.

Remark 12.2. The quadratic form $q(\Lambda) \;=\; Trace(\Lambda \cdot J \cdot \Lambda \cdot J)$ defines a nondegenerate symmetric bilinear form on $\Lambda^2(F^{2n})$ such that $q(\lambda(g)^{-1} g' \cdot \Lambda \cdot g) = q(\Lambda)$ holds for all $g \in G(F)$.

Notation. We write $Elm(A, B)$ for the matrix $Elm(g(A, B, AB'))$. Then $Elm(A, B)$ is a skew-symmetric matrix contained in the symplectic group $Sp(2n, F)$.

By definition Λ_s and g are both contained in $G(F) \;=\; GSp(2n, F)$. In all that follows, we may therefore restrict ourselves to the case $i \;=\; j$ since $Elm(A, B)$ is in $Sp(2i, F) \times Sp(2(j - i), F)$, and its "component" is in $Sp(2(j - i), F)$ is $J = J^{(j-i,j-i)}$.

Assumption. For simplicity of notation we therefore assume from now on $j = i$, without restriction of generality.

Then

$$Elm(A, B) = \begin{pmatrix} 0 & X \\ -X' & \mathcal{A} \end{pmatrix}$$

defined by $n \times n$-block matrices $X = \begin{pmatrix} E & 0 \\ 2A' - E \end{pmatrix}$ and $\mathcal{A} = \begin{pmatrix} 2(B \cdot A' - A \cdot B') & -2 \cdot B \\ 2 \cdot B' & 0 \end{pmatrix}$.

Remark 12.3. The skew-symmetric matrix $\mathcal{A}$ is invertible since B is invertible.

So there are matrices Z and $\tilde{\mathcal{A}}$ such that

$$\begin{pmatrix} 0 & X \\ -X' & \mathcal{A} \end{pmatrix} \;=\; \begin{pmatrix} E & Z \\ 0 & E \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{A}} & 0 \\ 0 & \mathcal{A} \end{pmatrix} \begin{pmatrix} E & 0 \\ Z' & E \end{pmatrix}.$$

Notice $X = Z \cdot \mathcal{A}$, $\tilde{\mathcal{A}} = -Z \cdot \mathcal{A} \cdot Z'$, and $Z = X \cdot \mathcal{A}^{-1}$, $X' = -\mathcal{A} \cdot Z'$, $\tilde{\mathcal{A}} = Z \cdot X' = X \cdot \mathcal{A}^{-1} \cdot X'$.

Corollary 12.2. *The $(n \times n)$-matrix Z is symmetric. Hence,*

$$g(Z) = \begin{pmatrix} E & 0 \\ Z & E \end{pmatrix} \in G(F).$$

Proof. $Z = X \cdot \mathcal{A}^{-1}$ satisfies $Z = Z'$ if $-\mathcal{A}' \cdot Z \cdot \mathcal{A} = \mathcal{A} \cdot X$ is symmetric. Since

$$\mathcal{A} \cdot X = \begin{pmatrix} 2 \cdot (B \cdot A' - A \cdot B') & -2 \cdot B \\ 2 \cdot B' & 0 \end{pmatrix} \begin{pmatrix} E & 0 \\ 2 \cdot A' & -E \end{pmatrix}$$

$$= \begin{pmatrix} 2B \cdot A' - 2A \cdot B' - 4B \cdot A' & 2B \\ 2B' & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \cdot (B \cdot A' + A \cdot B') & 2 \cdot B \\ 2 \cdot B' & 0 \end{pmatrix}$$

is symmetric, Z is also symmetric. □

It follows that

Fact. $g(Z)' \cdot Elm(A, B) \cdot g(Z) = \begin{pmatrix} \tilde{\mathcal{A}} & 0 \\ 0 & \mathcal{A} \end{pmatrix}$, *where* $\tilde{\mathcal{A}} = (\mathcal{A}')^{-1} = -\mathcal{A}^{-1}$ *and*

$$\mathcal{A} = \begin{pmatrix} 2(BA' - AB') & -2 \cdot B \\ 2 \cdot B' & 0 \end{pmatrix} = -\mathcal{A}'.$$

Formula for Z. $\mathcal{A}$ is invertible by assumption. Since $Elm(A, B)$ and $g(Z)$, and hence also $g(Z)'$, are symplectic matrices, we have $\tilde{\mathcal{A}} = (\mathcal{A}')^{-1}$. Notice that

$$\begin{pmatrix} E & -A \\ 0 & E \end{pmatrix} \begin{pmatrix} 0 & -2 \cdot B \\ 2 \cdot B' & 0 \end{pmatrix} \begin{pmatrix} E & 0 \\ -A' & E \end{pmatrix}$$

$$= \begin{pmatrix} -2A \cdot B' & -2 \cdot B \\ 2 \cdot B' & 0 \end{pmatrix} \begin{pmatrix} E & 0 \\ -A' & E \end{pmatrix}$$

$$= \begin{pmatrix} -2(A \cdot B' - B \cdot A') & -2 \cdot B \\ 2 \cdot B' & 0 \end{pmatrix} = \mathcal{A}.$$

Hence,

$$-2 \cdot Z = -2 \cdot X \cdot \mathcal{A}^{-1} = \begin{pmatrix} E & 0 \\ 2 \cdot A' & -E \end{pmatrix} \begin{pmatrix} E & 0 \\ A' & E \end{pmatrix} \begin{pmatrix} 0 & B \\ -B' & 0 \end{pmatrix}^{-1} \begin{pmatrix} E & A \\ 0 & E \end{pmatrix}$$

$$= \begin{pmatrix} E & 0 \\ A' & -E \end{pmatrix} \begin{pmatrix} 0 & -(B')^{-1} \\ B^{-1} & 0 \end{pmatrix} \begin{pmatrix} E & A \\ 0 & E \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -(B')^{-1} \\ -B^{-1} & -A' \cdot (B')^{-1} \end{pmatrix} \begin{pmatrix} E & A \\ 0 & E \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -(B')^{-1} \\ -B^{-1} & -B^{-1} \cdot A - A' \cdot (B')^{-1} \end{pmatrix}.$$

Since we have shown that we can assume the representative to be of standard type, the matrices B^{-1} and $B^{-1}A$ are integral; hence, Z is also integral. Therefore, we have

Fact. *The symplectic matrix $g(Z)$ is contained in $G(\mathfrak{o}_F)$.*

The injection Elm already defined induces an injection elm

$$\boxed{elm: \; H(F) \setminus G(F)/G(\mathfrak{o}_F) \; \hookrightarrow \; \Lambda^2(F^{2n})/G(\mathfrak{o}_F)}\,.$$

A Consequence. Suppose (M, N) is of standard type. Consider the double coset of $g(M, N) = g(A, B)$. Its image $elm(A, B)$ in $\Lambda^2(F^{2n})/G(\mathfrak{o}_F)$ is represented by the symplectic block matrix

$$diag(\tilde{\mathcal{A}}, \mathcal{A}) \; = \; diag(-\mathcal{A}^{-1}, \mathcal{A}).$$

12.3 Elementary Divisors

We dispose over another obvious map

$$\Lambda^2(F^{2n})/G(\mathfrak{o}_F) \; \to \; \Lambda^2(F^{2n})/\big(Gl(2n, \mathfrak{o}_F) \times \mathfrak{o}_F^*\big).$$

Here $(h, \varepsilon) \in Gl(2n, \mathfrak{o}_F) \times \mathfrak{o}_F^*$ acts on $\Lambda^2(F^{2n})$ by $\Lambda \mapsto \varepsilon \cdot h' \cdot \Lambda \cdot h$. For this we may consider the general case $i \leq j$, and we then claim

Lemma 12.1. *The composed map*

$$\mathcal{L}: \; H(F) \setminus G(F) \,/G(\mathfrak{o}_F) \quad \longrightarrow \quad \Lambda^2(F^{2n})/\Big(Gl(2n, \mathfrak{o}_F) \times \mathfrak{o}_F^*\Big),$$

which maps $H(F)gG(\mathfrak{o}_F)$ to the orbit of $\lambda(g)^{-1}g'\Lambda_s g$, is an injection.

We say two skew-symmetric invertible matrices in $\Lambda^2(F^m)$ are equivalent if there exists a unimodular matrix h in $Gl(m, \mathfrak{o}_F)$ such that $\Lambda_1 = h' \cdot \Lambda_2 \cdot h$. Concerning the orbits (right side of the map in the last lemma) recall the result of Frobenius:

(A) Λ_1 and Λ_2 are equivalent if and only if they have the same elementary divisors (understood in the usual sense).

(B) The product of the first k-elementary divisors (in the usual sense) is the gcd of all $k \times k$-minors.
(C) $\varepsilon \cdot \Lambda$ and Λ are equivalent for any $\varepsilon \in \mathfrak{o}_F^*$.

Hence, the orbits $\Lambda^2(F^m)/\big(Gl(m,\mathfrak{o}_F) \times \mathfrak{o}_F^*\big)$ are described by the elementary divisors.

Proof of Lemma 12.1. Without restriction of generality we again assume $i = j$. Then the skew-symmetric $(n \times n)$-matrix $\mathcal{A}$ can be brought into the following Frobenius standard form by a suitable unimodular transformation $U \in Gl(n, \mathfrak{o}_F)$:

$$U' \cdot \mathcal{A} \cdot U \ = diag\left(\begin{pmatrix} 0 & \pi_F^{a_1} \\ -\pi_F^{a_1} & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \pi_F^{a_i} \\ -\pi_F^{a_i} & 0 \end{pmatrix}\right),$$

where $a_1 \le \ldots \le a_i$. These symplectic elementary divisors are determined by the elementary divisors of the matrix $U'\mathcal{A}U$ (in the usual sense), which are $\pi_F^{a_1}, \pi_F^{a_1}, \pi_F^{a_2}, \pi_F^{a_2}, \cdots$.

The diagonalizing matrix U defines

$$g = diag\big((U')^{-1}, U\big) \ \in \ Sp(2n, \mathfrak{o}_F) \subseteq Gl(2n, \mathfrak{o}_F).$$

The symplectic $2n \times 2n$-matrix $diag(\tilde{\mathcal{A}}, \mathcal{A}) = diag((\mathcal{A}')^{-1}, \mathcal{A})$ will be transformed by $g \in G(\mathfrak{o}_F)$ into the "symplectic normal form"

$$diag\left(\begin{pmatrix} 0 & \pi_F^{-a_1} \\ -\pi_F^{-a_1} & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \pi_F^{-a_i} \\ -\pi_F^{-a_i} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \pi_F^{a_1} \\ -\pi_F^{a_1} & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \pi_F^{a_i} \\ -\pi_F^{a_i} & 0 \end{pmatrix}\right).$$

This symplectic normal form defines the same coset in $\Lambda^2(F^{2n})/G(\mathfrak{o}_F)$ as the matrices $diag(\tilde{\mathcal{A}}, \mathcal{A})$ and $elm(M, N)$.

Claim 12.1. $a_i \le -a_i$. In other words, the exponents of the elementary divisors of $diag(\tilde{\mathcal{A}}, \mathcal{A})$, in increasing order, are the numbers

$$a_1, a_1, a_2, a_2, \ldots, a_i, a_i, -a_i, -a_i, \ldots, -a_1, -a_1.$$

(In the general case $j > i$ there are $n - 2i$ additional zeros in the middle.) Hence, the elementary divisors of $diag(\tilde{\mathcal{A}}, \mathcal{A})$ uniquely determine the exponents $a_1 \le a_2 \le \cdots \le a_i$ of the symplectic Frobenius normal form of $\mathcal{A}$, as defined above. This immediately implies the lemma, provided the claim $a_i \le 0$ holds. To show this claim, notice $\pi_F^{-a_i} = det(\mathcal{A})^{-1} \cdot gcd(\Lambda^{n-1}(\mathcal{A}))$ and $det(\mathcal{A})^{-1} \cdot \Lambda^{2i-1}(\mathcal{A}) = \mathcal{A}^{-1}$. Hence, $\pi_F^{-a_i} = gcd(\mathcal{A}^{-1})$ is the first elementary divisor of $\mathcal{A}^{-1}$. Thus, to prove the claim, it suffices to show that $\mathcal{A}^{-1}$ is an integral matrix. Since

$$\mathcal{A} = -2 \begin{pmatrix} B & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} G & E \\ -E & 0 \end{pmatrix} \begin{pmatrix} B' & 0 \\ 0 & E \end{pmatrix}$$

for the matrix $G = B^{-1}A - (B^{-1}A)'$, we get

$$\mathcal{A}^{-1} = -\frac{1}{2}\begin{pmatrix}(B')^{-1} & 0\\ 0 & E\end{pmatrix}\begin{pmatrix}0 & -E\\ E & G\end{pmatrix}\begin{pmatrix}B^{-1} & 0\\ 0 & E\end{pmatrix}.$$

Since B^{-1} and G are integral, $\mathcal{A}^{-1}$ is integral, which proves the lemma. □

Proof of Theorem 12.1. By Lemma 12.1 it suffices to show that for $(e_1, \dots, e_i)$, subject to the conditions stated in Theorem 12.1, the elementary divisors of the matrices

$$Elm\Big(g(e_1,\dots,e_i)\Big) = \begin{pmatrix} 0 & 0 & E & 0\\ 0 & 0 & 0 & -E\\ -E & 0 & 0 & -2\cdot D\\ 0 & E & 2\cdot D' & 0\end{pmatrix}$$

determine $(e_1, \dots, e_i)$ uniquely such that every possible constellation of elementary divisors – as determined above – is realized by some $Elm(g(e_1, \dots, e_i))$. This, however, is rather obvious. The elementary divisors of $Elm(g(e_1, \dots, e_i))$ are $\pi_F^{e_1}, \pi_F^{e_1}, \dots, \pi_F^{e_r}, \pi_F^{e_r}, \dots,$, where $r \le i$ is chosen to be maximal such that $e_r < 0$. The following elementary divisors are pairs of 1 and then followed by the inverse numbers $\pi_F^{-e_r}, \dots, \pi_F^{-e_1}$ (in fact notice it is enough to consider minors in the right lower $n \times n$-block). This implies that the representatives $g(e_1, \dots, e_i)$ uniquely represent the double cosets $H(F)\backslash G(F)/G(\mathfrak{o}_F)$, which proves the theorem. □

Remark 12.4. In fact we have now also determined the image of the map $\mathcal{L}$. It consists of all orbits which contain a matrix in Frobenius normal form with exponents which satisfy

$$a_1 \le \dots \le a_i \le -a_i \le \dots - a_1.$$

12.4 The Compact Open Groups

Now fix some representative $g(D)$ as in Theorem 12.1. For simplicity assume $i = j$. Recall $D = D'$. Then

$$H_D \;=\; H(F) \;\cap\; g(D)G(\mathfrak{o}_F)g(D)^{-1}$$

is a compact open subgroup of $H(F)$. For

$$h \;=\; \begin{pmatrix}\alpha_1 & 0 & \beta_1 & 0\\ 0 & \alpha_2 & 0 & \beta_2\\ \gamma_1 & 0 & \delta_1 & 0\\ 0 & \gamma_2 & 0 & \delta_2\end{pmatrix}$$

in $H(F)$ we have the symplectic conditions $\alpha_i'\delta_i - \gamma_i'\beta_i = \lambda \cdot E, \lambda \in \mathfrak{o}_F^*$, $\alpha_i'\gamma_i = \gamma_i'\alpha_i$, $\gamma_i\delta_i' = \delta_i'\beta_i$. Furthermore, h is contained in H_D if and only if

$$g(D)^{-1}\cdot h\cdot g(D) \;=\; \begin{pmatrix} \alpha_1 & -D\gamma_2 & -D\gamma_2 D'+\beta_1 & -D\delta_2+\alpha_1 D\\ -D'\gamma_1 & \alpha_2 & -D'\delta_1+\alpha_2 D' & -D'\gamma_1 D+\beta_2\\ \gamma_1 & 0 & \delta_1 & \gamma_1 D\\ 0 & \gamma_2 & \gamma_2 D' & \delta_2 \end{pmatrix}$$

is contained in $G(\mathfrak{o}_F)$. Then $\alpha_i, \gamma_i, \delta_i$ and $D'\gamma_1, D\gamma_2$ and $\gamma_2 D'$ and $\gamma_1 D$ are integral, and $det(h)\in\mathfrak{o}_F^*$ (first integrality conditions). Furthermore, we have the four congruence conditions (*) modulo integral matrices:

$$\beta_1\equiv D\gamma_2 D', \qquad \beta_2\equiv D'\gamma_1 D,$$

$$D\delta_2\equiv\alpha_1 D, \qquad D'\delta_1\equiv\alpha_2 D'.$$

Since D^{-1} is integral, and hence $D^{-1}\beta_1, \beta_1(D')^{-1}, (D')^{-1}\beta_2, \beta_2 D^{-1}$ and $D^{-1}\alpha_1 D$, $D'\delta_1(D')^{-1}$, $D\delta_2 D^{-1}$, $(D')^{-1}\alpha_2 D'$ are necessarily integral (second integrality conditions). We reformulate the integrality conditions by introducing the integral skew-symmetric matrix

$$\Lambda_D = \begin{pmatrix} 0 & D^{-1}\\ -D^{-1} & 0\end{pmatrix}.$$

Define

$$GSp(\Lambda_D) = \Big\{ h\in Gl(F^{2i}) \;\mid\; h'\Lambda_D h = \lambda\cdot\Lambda_D,\ \lambda\in F^* \Big\}.$$

Notice $diag(E,-E)\in GSp(\Lambda_D)$ and $J\in GSp(\Lambda_D)$; hence,

$$I = \begin{pmatrix} 0 & E\\ E & 0\end{pmatrix} \in GSp(\Lambda_D),$$

and, therefore, $g=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in GSp(\Lambda_D) \iff g^I = IgI = \left(\begin{smallmatrix} d & c\\ b & a\end{smallmatrix}\right)\in GSp(\Lambda_D)$.

Also notice that $g_k\in GSp(\Lambda_D)$ holds for the two matrices ($k=1,2$)

$$g_k := \begin{pmatrix} a_k & b_k\\ c_k & d_k\end{pmatrix} = \begin{pmatrix} 0 & D\\ E & 0\end{pmatrix}\begin{pmatrix} \alpha_k & \beta_k\\ \gamma_k & \delta_k\end{pmatrix}\begin{pmatrix} 0 & D\\ E & 0\end{pmatrix}^{-1} = \begin{pmatrix} D\delta_k D^{-1} & D\gamma_k\\ \beta_k D^{-1} & \alpha_k\end{pmatrix}.$$

All the integrality conditions stated above when put together express the fact that both matrices g_k and $diag(D,D)^{-1}g_k diag(D,D)$ are integral matrices (for $k=1,2$) with equal similitude factor in $\mathfrak{o}_F^*$. If $\Gamma=(\mathfrak{o}_F)^{2i}$ denotes the standard lattice in F^{2i}, then Λ_D defines a skew-symmetric pairing $\langle .,.\rangle_D$ on Γ by $\langle v,w\rangle_D = v'\Lambda_D w$. The dual lattice $\Gamma^*=\{w\in F^{2i}\mid \langle w,\Gamma\rangle_D\in\mathfrak{o}_F\}$ is $\Gamma^* = diag(D,D)(\Gamma)$. The matrices in $GSp(\Lambda_D)$, which preserve the lattice Γ, define a compact open subgroup $Aut(\Gamma,\Lambda_D)\subseteq GSp(\Lambda_D)$. Each $g\in Aut(\Gamma,\Lambda_D)$ preserves the dual lattice Γ^*. Hence, $g(\Gamma^*)=\Gamma^*$ or $diag(D,D)^{-1}\cdot g\cdot diag(D,D)\in Aut(\Gamma,\Lambda_{D^{-1}})$. This shows that the first and second integrality conditions are equivalent to

$$g_1, g_2 \in Aut(\Gamma,\Lambda_D).$$

This defines an injective homomorphism

$$H_D \hookrightarrow Aut(\Gamma, \Lambda_D) \times Aut(\Gamma, \Lambda_D),$$

induced from the injection of $H(F)$ into $GSp(\Lambda_D) \times GSp(\Lambda_D)$, which is defined by

$$h \mapsto (k_1, k_2) = (g_1, g_2^I).$$

Now $\Gamma \subseteq \Gamma^*$ since Λ_D is integral. Hence, we get a homomorphism

$$1 \to K_D \to Aut(\Gamma, \Lambda_D) \to Aut(\Gamma^*/\Gamma)$$

with kernel, say, K_D. Obviously $g \in K_D \iff (g - id)diag(D, D)$ is integral. For $\mathcal{G} = M_{i,i}(\mathfrak{o}_F) \cdot D^{-1} \subseteq M_{i,i}(\mathfrak{o}_F)$ the above four congruences (*) are equivalent to $b_1 \equiv c_2$, $c_1 \equiv b_2$, $d_1 \equiv a_2$, and $a_1 \equiv d_2$ modulo $\mathcal{G}$. In other words, the conditions (*) mean $(g_1 - g_2^I)diag(D, D)$ is integral, or $(id - g_1^{-1}g_2^I)diag(D, D)$ is integral. In other words, we get the condition

$$g_1 \equiv g_2^I \; mod \; K_D,$$

or $k_1 \equiv k_2$ mod K_D. Hence, H_D is isomorphic to the group of all $(k_1, k_2) \in Aut(\Gamma, \Lambda_D)^2$ such that $k_1 \equiv k_2$ mod K_D. Since K_D is a normal subgroup of $Aut(\Gamma, \Lambda_D)$, this proves that H_D is isomorphic to the semidirect product $K_D \lhd Aut(\Gamma, \Lambda_D)$

$$1 \to K_D \to H_D \to Aut(\Gamma, \Lambda_D) \to 1.$$

Example 12.1. For $D = d{\cdot}E$ we have $GSp(\Lambda_D) = GSp(2i, F)$, and $Aut(\Gamma, \Lambda_D) = G(\mathfrak{o}_F)$ such that K_D is the principal congruence subgroup of level d.

12.5 The Twisted Group $\tilde{H}$

Whereas in the last section we considered $i \leq j$, we now have to restrict ourselves to the special case $n = i + j$, where $i = j$. In this special case the normalizer N_s of the subgroup $H = G_s$ of $G = GSp(2n)$ is not connected. This now allows us to define Galois twists $\tilde{H}$ of the group H considered in the last section. For $i = j$ the centralizer in the adjoint group of the element s (defined at the beginning of this chapter) is nontrivial. The element

$$w = \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & E \\ 0 & 0 & E & 0 \end{pmatrix} \in G(\mathfrak{o}_F)$$

in G_s generates $N_s/G_s \cong \mathbb{Z}/2\mathbb{Z}$. We have $ws = -sw$, $w^2 = id$, and $wJ = Jw$.

Suppose K/F is a quadratic field extension, and σ is the generator of the Galois group of this extension. Since $H^1(F, GSp(2n))$ is trivial, there exists an $g_0 \in GSp(2n,K)$ such that

$$g_0^{-1}\sigma(g_0) = w, \qquad \sigma(g_0) = g_0 \cdot w.$$

This condition determines the coset $G(F) \cdot g_0$ uniquely. Then the group $\tilde{H} = g_0 H g_0^{-1}$ is invariant under σ, and defines a form $\tilde{H}$ of H over F together with an embedding $\tilde{H} \hookrightarrow G$ defined over F. Notice for the norm-1-subgroup N^1 in $Res_{K/F}(\mathbb{G}_m)$

$$1 \to \tilde{H} \to Res_{K/F}(GSp(2i)) \to N^1 \to 1,$$

where the morphism on the right is $g \mapsto (\sigma(\lambda(g))/\lambda(g)$. Let us make some choice, i.e., $g_0 = diag(1, \frac{\alpha^2}{2}, \alpha^2, 2) \cdot diag(U, (U')^{-1})$ for $U = \left(\begin{smallmatrix} 1/2 & 1/2 \\ -\alpha & \alpha \end{smallmatrix}\right)$, where $K = F(\alpha)$ and $A^{-1} = \alpha^2 \in F^*$. We can assume that the valuation is $v_K(\alpha^2) = 0$ or -1 depending on whether K/F is unramified or not since the residue characteristic is different from 2. Then $\tilde{H}$ becomes the subgroup of all the matrices η defined on page 239.

The Map $\tilde{\mathcal{L}}$. Now consider the commutative diagram

$$\begin{array}{ccc}
\tilde{H}(F) \backslash G(F)/G(\mathfrak{o}_F) & \longrightarrow & g_0 H(K) g_0^{-1} \backslash G(K)/G(\mathfrak{o}_K) \\
\Big\downarrow {\scriptstyle \tilde{\mathcal{L}}} & & \Big\downarrow {\scriptstyle L(g_0^{-1})} \\
\Lambda^2(K^{2n})/(Gl(2n,\mathfrak{o}_K) \times \mathfrak{o}_K^*) & \xleftarrow{\ \mathcal{L}\ } & H(K) \backslash G(K)/G(\mathfrak{o}_K)
\end{array}$$

where the upper map is induced by the scalar extension maps, the right vertical bijection is $\tilde{H}(K)gG(\mathfrak{o}_K) \mapsto H(K)g_0^{-1}gG(\mathfrak{o}_K)$, and the lower horizontal injective map is defined as in Lemma 12.1, but now for the field K instead of F. The left vertical map is the composition of the other maps $\tilde{\mathcal{L}}(\tilde{H}(F)gG(\mathfrak{o}_F)) = \mathcal{L}(H(K)g_0^{-1}gG(\mathfrak{o}_K)) =$ orbit of $\lambda(g_0^{-1}g)^{-1} \cdot (g_0^{-1}g)'\Lambda_s(g_0^{-1}g) =$ orbit of $\lambda(g)^{-1} \cdot g'\tilde{\Lambda}_s g$ for

$$\tilde{\Lambda}_s = \begin{pmatrix} 0 & 0 & 0 & -\alpha^{-1}E \\ 0 & 0 & -\alpha E & 0 \\ 0 & \alpha E & 0 & 0 \\ \alpha^{-1}E & 0 & 0 & 0 \end{pmatrix}.$$

The Image of $\tilde{\mathcal{L}}$. The F-rational element $diag(E, A \cdot E, E, E) \times 1$ transforms $\tilde{\Lambda}_s$ to $-\alpha^{-1} \cdot J$ within its orbits. Hence, the image of $\tilde{\mathcal{L}}$ is contained in the image of the $Gl(2n,F) \times F^*$ orbit of the matrix $-\alpha^{-1} \cdot J$. Therefore,

$$image(\tilde{\mathcal{L}}) \subseteq \big(v_K(\alpha), \ldots, v_K(\alpha)\big) + v_K(F^*)^n \subseteq v_K(K^*)^n,$$

considered as a subset of the n exponents, which define the Frobenius normal form of the skew-symmetric $2n \times 2n$-matrix. On the other hand, this image is contained in the image of the lower horizontal map $\mathcal{L}$, which is the set $\{(a_1, \ldots, a_i, -a_i, \ldots, -a_1) \in v_K(K^*)^n \mid a_1 \leq \cdots a_\nu \leq -a_\nu \leq \cdots - a_1\}$.

We claim that $image(\tilde{\mathcal{L}})$ is the intersection of these two sets. We show that every element of this intersection is some $\tilde{\mathcal{L}}(\tilde{H}(F)\tilde{g}(D)G(\mathfrak{o}_F))$. Notice

$$\tilde{\mathcal{L}}(\tilde{H}(F)\tilde{g}(D)G(\mathfrak{o}_F)) = \text{orbit of} \begin{pmatrix} E & T \\ 0 & E \end{pmatrix}' \tilde{\Lambda}_s \begin{pmatrix} E & T \\ 0 & E \end{pmatrix} = \text{orbit of} \begin{pmatrix} 0 & M \\ -M' & TM - M'T \end{pmatrix}$$

for $M = \left(\begin{smallmatrix} 0 & M_0^{-1} \\ M_0 & 0 \end{smallmatrix}\right)$, $M_0 = -\alpha E$. For $T = \left(\begin{smallmatrix} A^{-1} \cdot D & 0 \\ 0 & 0 \end{smallmatrix}\right)$ this gives the orbit of

$$\begin{pmatrix} 0 & 0 & 0 & -\alpha^{-1}E \\ 0 & 0 & -\alpha E & 0 \\ 0 & \alpha E & 0 & -\alpha D \\ \alpha^{-1}E & 0 & \alpha D & 0 \end{pmatrix}.$$

Obviously for $e = v_K(\alpha) \in \{0, -\frac{1}{2}\}$ and $D = A^{-1} \cdot diag(\pi_F^{e_1}, \cdots, \pi_F^{e_i})$ as in Theorem 12.1 the first $2i$-elementary divisors of this matrix are $(e + e_1, e + e_1, e + e_2, e + e_2, \ldots, e + e_i, e + e_i, *, \ldots, *)$ (arises from the lower right block, since αD gives rise to the minors with the highest order denominator). This suffices to prove the claim with the representatives $\tilde{g}(D) = \left(\begin{smallmatrix} E & T \\ 0 & E \end{smallmatrix}\right)$, where $T = \left(\begin{smallmatrix} A^{-1} \cdot D & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for D as above.

Galois Descent. From the last argument it follows that every $g \in G(F)$ can be written in the form $g = h\tilde{g}(D)k_0^{-1}$ for some $h \in \tilde{H}(K)$, some $k_0 \in G(\mathfrak{o}_K)$, and some D as in Theorem 12.1. Then $\sigma(h)\tilde{g}(D)\sigma(k^{-1}) = h\tilde{g}(D)k^{-1}$ implies $\tilde{g}(D)^{-1}h^{-1}\sigma(h)\tilde{g}(D) = k^{-1}\sigma(k)$ or

$$b(\sigma) = k^{-1}\sigma(k) \in H_D(K) := \Big(\tilde{g}(D)^{-1}\tilde{H}(K)\tilde{g}(D)\Big) \cap G(\mathfrak{o}_K).$$

Suppose the 1-cocycle $b(\sigma) = k^{-1}\sigma(k) \in H_D(K)$ is a 1-coboundary $b(\sigma) = y^{-1}\sigma(y)$ for some $y \in H_D(K)$. Then $\tilde{k} = yk^{-1} \in G(\mathfrak{o}_F)$ and $g = (h\tilde{g}(D)y^{-1}\tilde{g}(D)^{-1}) \cdot \tilde{g}(D) \cdot (yk^{-1}) = \tilde{h} \cdot \tilde{g}(D) \cdot \tilde{k}$. Since $g, \tilde{k} \in G(F)$ and $\tilde{g}(D) \in G(F)$, $\tilde{h} \in \tilde{H}(F)$ and

$$g \in \tilde{H}(F) \cdot \tilde{g}(D) \cdot G(\mathfrak{o}_F).$$

The Obstruction. For $n = 2$ the group $H_D(K)$ is isomorphic to the group $Gl(2, R)^0$, where R is the ring $\mathfrak{o}_F \otimes_{\mathfrak{o}_F} \mathfrak{o}_K(i)$ (see page 248). Here σ acts by its natural action on the first factor, and is trivial on the second factor. By Shapiro's lemma the class $[b(\sigma)]$ in the cohomology $H^1(\langle\sigma\rangle, H_D(K))$ is trivial if this holds for its image in the quotient $H^1(\langle\sigma\rangle, Gl(2, \mathfrak{o}_K/(\pi_F^i))^0$. Now it is easy to show that the fiber over the trivial element under the reduction map

$$H^1\Big(\langle\sigma\rangle,\ Gl(2, \mathfrak{o}_K/(\pi_F^i))^0\Big) \to H^1\Big(\langle\sigma\rangle,\ Gl(2, \mathfrak{o}_K/(\pi_K))^0\Big)$$

is trivial. This is easily shown by induction on i. If K/F is unramified, the cohomology set $H^1\Big(\langle\sigma\rangle,\ Gl(2, \mathfrak{o}_K/(\pi_K))^0\Big)$ is trivial. If K/F is ramified, this is not the case. But in the ramified case, for the vanishing of the obstruction classes $[b(\sigma)]$

it is therefore sufficient that the image of the 1-cocycle $b(\sigma)$ is trivial in the quotient group $Gl(2, \mathfrak{o}_K/(\pi_K))^0$ of $H_D(K)$. However, this follows from Lemma 7.5 since the reduction of $b(\sigma) = k^{-1}\sigma(k) \in G(\mathfrak{o}_K)$ in $G(\mathfrak{o}_K/(\pi_K))$ is trivial. Notice $k^{-1}\sigma(k) = k^{-1}k = 1$ in $G(\mathfrak{o}_K/(\pi_K))$.

Injectivity of $\tilde{\mathcal{L}}$. In general, if all the above-defined cohomology obstructions $[b(\sigma)]$ in $H^1(\langle\sigma\rangle, H_D(K))$ are trivial, we obtain

$$G(F) = \bigcup_D \tilde{H}(F) \cdot \tilde{g}(D) \cdot G(\mathfrak{o}_F).$$

Furthermore, since $\tilde{\mathcal{L}}(H(F) \cdot \tilde{g}(D_1) \cdot G(\mathfrak{o}_F)) = \tilde{\mathcal{L}}(H(F) \cdot \tilde{g}(D_2) \cdot G(\mathfrak{o}_F))$ implies $D_1 = D_2$ as shown above, we even conclude

Theorem 12.2. *The matrices $\left(\begin{smallmatrix} E & T \\ 0 & E \end{smallmatrix}\right)$ for $T = \left(\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $D = A^{-1} \cdot diag(\pi_F^{e_1}, \cdots, \pi_F^{e_i})$, where $e_1 \leq \cdots \leq e_i$ and $e_\nu < 0$ or $e_\nu = \infty$ for $\nu = 1, \ldots, i$, define inequivalent representatives and for $n = 2$ a full system of representatives for the double cosets*

$$\tilde{H}(F) \backslash G(F) / G(\mathfrak{o}_F).$$

In general, consider the matrix group $H' = Res_{K/F}(GSp(\Lambda_D))^0$. For an integral extension ring $\mathfrak{O}$ of $\mathfrak{o}_F$ with fraction field L consider the subset of $H'(L) = GSp(\Lambda_D)^0(L \otimes_F K)$ defined by all block matrices $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ for which $A, B, C,$ and D are matrices of the form

$$X \otimes 1 + Y \cdot D^{-1} \otimes \sqrt{A}^{-1},$$

such that X and Y are $i \times i$-matrices with coefficients in $\mathfrak{O}$. In fact, this subset defines a subgroup. For $\mathfrak{O} = \mathfrak{o}_K$ this group is isomorphic to $H_D(K)$, and σ acts on these matrices by $\sigma(X \otimes 1 + YD^{-1} \otimes \sqrt{A}^{-1}) = \sigma(X) \otimes 1 + \sigma(Y)D^{-1} \otimes \sqrt{A}^{-1}$, via its natural action on $\mathfrak{O} = \mathfrak{o}_K$. Notice that the coefficients of the matrices g are in $\mathfrak{O} \otimes_{\mathfrak{o}_F} \mathfrak{o}_K \subseteq L \otimes_F K$.