A Remark on Quantum Ergodicity for CAT Maps

J. Bourgain

Institute for Advanced Study, School of Mathematics, Princeton, NJ 08540, USA
bourgain@math.ias.edu

1 Introduction

The purpose of this Note is to give an affirmative answer to a question raised in the paper of P. Kurlberg and Z. Rudnick [K-R]. We first briefly recall the background (see [K-R2]). Given $A \in SL_2(\mathbb{Z})$, consider the automorphism of the torus $\mathbb{T}^2 : x \mapsto Ax$.

Given $f \in \mathcal{C}^\infty(\mathbb{T}^2)$, the classical evolution defined by A is $f \mapsto f \circ A$. The quantization is obtained as follows. Let $N \in \mathbb{Z}_+$ be a large integer and consider the Hilbert space $\mathcal{H}_N = L^2(\mathbb{Z}_N)$, $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ with inner product

$$\langle \phi, \psi \rangle = \frac{1}{N} \sum_{x \in \mathbb{Z}_N} \phi(x)\overline{\psi(x)}.$$

The basic observables are given by the operators $T_N(n), n = (n_1, n_2) \in \mathbb{Z}^2$ defined as follows

$$\big(T_N(n)\phi\big)(x) = e^{i\pi \frac{n_1 n_2}{N}} e^{2\pi i \frac{n_2 x}{N}} \phi(x + n_1). \tag{1.1}$$

Writing $f(x) = \sum_{n\in\mathbb{Z}^2} \hat{f}(n) e^{2\pi i n x}$, $f \in C^\infty(\mathbb{T}^2)$, its quantization is then defined by

$$Op_N(f) = \sum_{n\in\mathbb{Z}^2} \hat{f}(n) T_N(n). \tag{1.2}$$

Assume further that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfies

$$ab \equiv cd \equiv 0 \pmod 2.$$

One may then assign to A a unitary operator $U_N(A)$ called quantum propagator or quantized cat map, which satisfies the 'exact' Egorov theorem

$$U_N(A)^* Op_N(f) U_N(A) = Op_N(f \circ A). \tag{1.3}$$

We are concerned with the eigenfunctions of $U_N(A)$ which play the role of energy eigenstates.

It is shown in [K-R] that for N taken in a subsequence $\mathcal{N} \subset \mathbb{Z}_+$ of asymptotic density one, we have for all $f \in \mathcal{C}^\infty(\mathbb{T}^2)$

$$\max_{\psi} \left| \langle Op_N(f)\psi, \psi\rangle - \int_{\mathbb{T}^2} f \right| \underset{N\in\mathcal{N}}{\overrightarrow{N\to\infty}} 0 \tag{1.4}$$

where the maximum is taken over all normalized eigenfunctions ψ of $U_N(A)$.

The quantization of the cat map described above was proposed by Hannay and Berry [H-B]. A few comments at this point. In the context of cat maps, Schnirelman's general theorem when the classical dynamics is ergodic (which is the case when $A \in SL_2(\mathbb{Z})$ is hyperbolic) takes the following form. Let $f \in C^\infty(\mathbb{T}^2)$. If $\{\psi_j\}$ is an arbitrary orthonormal basis of $\mathcal{H}_N$ consisting of eigenfunctions of $U_N(A)$, there is a subset $J(N) \subset \{1, \dots, N\}$ such that $\frac{\#J(N)}{N} \to 1$ and for $j \in J(N)$

$$\langle Op_N(f)\psi_j, \psi_j\rangle \to \int_{\mathbb{T}^2} f \text{ when } N \to \infty. \tag{1.5}$$

Hence the [K-R] result (1.4) goes beyond (1.5), since they obtain a statement valid for all eigenfunctions of $U_N(A)$.

Previously, the only result providing an infinite set $\mathcal{N}$ of integers N (primes) satisfying (1.4) was due to Degli-Esposti, Graffi and Isola [D-G-I], conditional to GRH. The precise form of the [K-R] result is as follows (using previous notations)

$$\sum_{j=1}^{N} \left| \langle Op_N(f)\psi_j, \psi_j\rangle - \int_{\mathbb{T}^2} f \right|^4 \ll \frac{N(\log N)^{14}}{o(A,N)^2} \tag{1.6}$$

where $o(A, N)$ denotes the order of A mod N. (See [K-R], Theorem 2.) In order to derive (1.4) from (1.6), one needs to ensure that $o(A, N) \gg N^{1/2}$ for $N \in \mathcal{N}$. Verifying this property for sequence $\mathcal{N}$ of asymptotic density 1 is in fact a significant part of the [K-R] paper (the issue is related to the classical Gauss–Artin problem.) It is shown in [K-R] one may ensure for $N \in \mathcal{N}$ of asymptotic density 1, that

$$o(A, N) \gg N^{1/2} \exp\left((\log N)^\delta\right) \tag{1.7}$$

for some $\delta > 0$.

The authors raise the question how to get results when $o(A, N)$ is smaller than $N^{1/2}$. We will show here how to settle this problem using the new exponential sum bounds obtained in [BGK], [B], [B-C] for multiplicative subgroups G of finite fields and their products. These results provide nontrivial estimates even when G is very small.

They will allow us to deal with the case when $o(A, N) \gg N^\varepsilon$ (say for N prime) for an arbitrary small given $\varepsilon > 0$. Unlike a stronger statement such as (1.7), the generic validity of this last condition is essentially obvious to verify. Our results are stated in Proposition 2 (prime modulus) and Theorem 3 (arbitrary modulus). Note that in (3.1) below the discrepancy is estimated as $N^{-\delta}$, which is better than the bound obtained in [K-R].

The results of importance for what follows are the following

Theorem 1 (see [BGK] if $f = 1$ and [B-C] if $f > 1$). *Let $G < \mathbb{F}_{p^f}^*$ be of order t such that*

$$t > p^{\varepsilon f} \tag{1.8}$$

and

$$\max_{\substack{r|f \\ r<f}} (t, p^r - 1) < t^{1-\varepsilon} \tag{1.9}$$

where $\varepsilon > 0$ is an arbitrarily small given constant.

Then

$$\max_{\mathcal{X} \neq \mathcal{X}_0} \left| \sum_{x \in G} \mathcal{X}(x) \right| < Ct^{1-\delta} \tag{1.10}$$

where $\mathcal{X}$ runs over the nontrivial additive characters of $\mathbb{F}_{p^f}$, thus $\mathcal{X}(x) = e\big(\frac{1}{p} Tr(ax)\big), a \in \mathbb{F}_{p^f}^*$, and $\delta = \delta(\varepsilon) > 0$.

In the application below, $f = 2$.

Also needed is the following exponential sum bound in $\mathbb{F}_p \times \mathbb{F}_p$, obtained in [B].

Theorem 2 ([B]). *Let $G < \mathbb{F}_p^* \times \mathbb{F}_p^*$ be generated by $(\theta_1, \theta_2) \in \mathbb{F}_p^* \times \mathbb{F}_p^*$ satisfying*

$$O(\theta_1) > p^\varepsilon \tag{1.11}$$

$$O(\theta_2) > p^\varepsilon \tag{1.12}$$

$$O(\theta_1 \theta_2^{-1}) > p^\varepsilon \tag{1.13}$$

with $\varepsilon > 0$ a given arbitrary constant. We denote here $O(\theta)$ the multiplicative order of $\theta \in \mathbb{F}_p^*$.

There is $\delta = \delta(\varepsilon) > 0$ such that

$$\max_{(a_1,a_2) \neq (0,0)} \left| \sum_{x \in G} e_p(a_1 x_1 + a_2 x_2) \right| < C|G|^{1-\delta}. \tag{1.14}$$

Acknowledgement. The author is grateful to Z. Rudnick for his comments on an earlier version of this account.

2 The Prime Case

Considering first the case with $N = p$ prime, we show the following

Proposition 1. *For all $\varepsilon > 0$, there is $\delta > 0$ such that if $o(A, N) > N^\varepsilon$, then, assuming n and nA linearly independent mod N, we have*

$$\max_\psi \big|\langle T_N(n)\psi, \psi\rangle\big| < 2N^{-\delta} \tag{2.1}$$

with the maximum taken over the normalized eigenfunctions ψ of $U_N(A)$.

Proof. Denote $t = o(A, N)$. Since $U_N(A)$ is unitary, write for $j = 1, \dots, t$

$$\begin{aligned}\langle T_N(n)\psi, \psi\rangle &= \langle T_N(n)U_N(A)^j\psi, U_N(A)^j\psi\rangle \\ &= \frac{1}{t}\sum_{j=1}^{t}\langle U_N(A)^{-j}T_N(n)U_N(A)^j\psi, \psi\rangle.\end{aligned} \tag{2.2}$$

By Egorov's theorem (1.3), we have

$$U_N(A)^{-1}T_N(n)U_N(A) = T_N(nA) \tag{2.3}$$

and iterating

$$U_N(A)^{-j}T_N(n)U_N(A)^j = T_N(nA^j).$$

Hence from (2.2)

$$|\langle T_N(n)\psi, \psi\rangle| \le \|D(n)\| \tag{2.4}$$

where $D = D(n)$ is following operator on $\mathcal{H}_N$

$$D = \frac{1}{t}\sum_{j=1}^{t} T_N(nA^j) \tag{2.5}$$

and $\|\ \|$ stands for the operator norm.

Take a (sufficiently large) positive integer ℓ (to be specified) and estimate

$$\|D\|^{4\ell} \le \text{ trace } (DD^*)^{2\ell}. \tag{2.6}$$

Recall the following properties (see [K-R])

$$T_N(m)^* = T_N(-m) \tag{2.7}$$

and

$$T_N(m)T_N(n) = e_N\left(\frac{\omega(m,n)}{2}\right)T_N(m+n) \tag{2.8}$$

with

$$\omega(m,n) = m_1n_2 - m_2n_1.$$

Expanding (2.6) using (2.7)–(2.8) gives

$$(DD^*)^{2\ell} = \frac{1}{t^{4\ell}}\sum_{j_1,\dots,j_{4\ell}=1}^{t}\gamma_{j_1\dots j_{4\ell}}T_N\big(n(A^{j_1} - A^{j_2}\cdots - A^{j_{4\ell}})\big) \tag{2.9}$$

where $|\gamma_{j_1} \dots j_{4\ell}| = 1$.

Next

$$\text{trace } T_N(n) = \begin{cases} N \text{ if } n = (0,0) \bmod N \\ 0 \text{ otherwise.} \end{cases} \tag{2.10}$$

It follows now from (2.9), (2.10) that

$$(2.6) \leq t^{-4\ell} N.\#\big\{(j_1, \dots, j_{4\ell}) \in \{1, \dots, t\}^{4\ell} \,\big|\, n(A^{j_1} - \dots - A^{j_{4\ell}}) \equiv 0 \mod N\big\}. \tag{2.11}$$

The issue becomes now to estimate (2.11).

Recall that $N = p$ (prime).

Following [K-R], let K be the real quadratic field containing the eigenvalues of A (which are units) and O its maximal order. Let $\mathcal{P}$ be a prime of K lying above p and consider the residue class field $= O/\mathcal{P}$. If p splits, $K_p \simeq \mathbb{F}_p$ and if p is inert, $K_p \simeq \mathbb{F}_{p^2}$. Diagonalizing A over K_p, we obtain $A' = \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right)$ and $n' = (n'_1, n'_2)$ in the eigenvector basis. Also $n'_1 \neq 0, n'_2 \neq 0$ in K_p as a consequence of the linear independence assumption for n and $nA \bmod p$. Our problem is therefore reduced to estimating the number (†) of solutions in $(j_1, \dots, j_{4\ell}) \in \{1, \dots, t\}^{4\ell}$ of the system of equations

$$\begin{cases} \sum_{s=1}^{4\ell} (-1)^s \varepsilon^{j_s} = 0 & (2.12) \\ \sum_{s=1}^{4\ell} (-1)^s \varepsilon^{-j_s} = 0 & (2.12') \end{cases}$$

in K_p. Here $\varepsilon \in K_p^*$ is of order t.

Case 1: The Split Case. Thus $K_p = \mathbb{F}_p$. Apply Theorem 2 with $\theta_1 = \varepsilon, \theta_2 = \varepsilon^{-1}$ for which $0(\theta_1) = 0(\theta_2) = t > p^\varepsilon$ and $0(\theta_1\theta_2^{-1}) = 0(\varepsilon^2) > \frac{t}{2} > \frac{1}{2}p^\varepsilon$. Hence (1.11) holds for some $\delta_1 = \delta_1(\varepsilon) > 0$.

Estimate by the circle method

$$\begin{aligned} (\dagger) &= \frac{1}{p^2} \sum_{0 \leq a_1, a_2 < p} \Big| \sum_{j=1}^{t} e_p(a_1\varepsilon^j + a_2\varepsilon^{-j}) \Big|^{4\ell} \\ &< \frac{1}{p^2} t^{4\ell} + \max_{(a_1,a_2) \neq (0,0)} \Big| \sum_{j=1}^{t} e_p(a_1\varepsilon^j + a_2\varepsilon^{-j}) \Big|^{4\ell} \\ &< \frac{1}{p^2} t^{4\ell} + C^\ell t^{(1-\delta_1)4\ell} \\ &< t^{4\ell}(p^{-2} + C^\ell p^{-4\varepsilon\delta_1\ell}). \end{aligned} \tag{2.13}$$

Taking

$$\ell > \frac{1}{\varepsilon\delta_1} \tag{2.14}$$

it follows that (for p large enough)

$$(\dagger) < 2t^{4\ell}p^{-2}. \tag{2.15}$$

Case 2: The Inert Case. Then $K_p \approx \mathbb{F}_{p^2}$. Let $G = \{\varepsilon^j | 0 \le j < t\} < K_p^*$. We have to distinguish 2 further subcases.

Assume first that $t = |G|$ satisfies

$$(t, p-1) < t^{1-\frac{\varepsilon}{2}} \tag{2.16}$$

so that condition (1.6) of Theorem 1 is fulfilled.

Then (1.7) holds with $\delta = \delta_1 = \delta_1(\varepsilon)$. By the circle method, we obtain again

$$\begin{aligned}(\dagger) &= \frac{1}{p^2}\sum_{\mathcal{X}}\Big|\sum_{x\in G}\mathcal{X}(x)\Big|^{4\ell}\\ &< \frac{t^{4\ell}}{p^2} + \max_{\mathcal{X}\neq\mathcal{X}_0}\Big|\sum_{x\in G}\mathcal{X}(x)\Big|^{4\ell}\\ &< t^{4\ell}\big(p^{-2} + Cp^{-4\ell\varepsilon\delta_1}\big)\\ &< 2t^{4\ell}p^{-2}\end{aligned} \tag{2.16$'$}$$

for a choice of ℓ as in (2.14).

Next, suppose (2.16) violated. Then $t = t_1t_2$ where

$$t_1 | p-1 \text{ and } t_2 < t^{\varepsilon/2}.$$

Replace G by $G_1 = G^{t_2} < \mathbb{F}_p^*$ generated by $\varepsilon_1 = \varepsilon^{t_2}$ of order t_1 in $\mathbb{F}_p^*$, $t_1 > p^{\varepsilon/2}$.

Write $j \in \{0, 1, \dots, t-1\}$ in the form $j = j_1t_2 + j_2$ with $j_1 \in \{0, 1, \dots, t_1 - 1\}$ and $j_2 \in \{0, 1, \dots, t_2 - 1\}$. Estimate

$$(\dagger) = \frac{1}{p^4}\sum_{a_1,a_2\in\mathbb{F}_{p^2}}\Big|\sum_{j=0}^{t-1} e_p\big(Tr(a_1\varepsilon^j) + Tr(a_2\varepsilon^{-j})\big)\Big|^{4\ell}$$

and by Hölder's inequality

$$p^{-4}t_2^{4\ell-1}\sum_{a_1,a_2\in\mathbb{F}_{p^2}}\sum_{j_2=0}^{t_2-1}\Big|\sum_{j_1=0}^{t_1-1} e_p\big(Tr(a_1\varepsilon^{j_2})\varepsilon_1^{j_1} + Tr(a_2\varepsilon^{-j_2})\varepsilon_1^{-j_1}\big)\Big|^{4\ell} \tag{2.17}$$

the inner sum in (2.17) is again estimated by Theorem 2. Thus for some $\delta_1 = \delta\big(\frac{\varepsilon}{2}\big) > 0$

$$\Big|\sum_{j_1=0}^{t_1-1} e_p(b_1\varepsilon_1^{j_1} + b_2\varepsilon_1^{-j_1})\Big| < Ct_1^{1-\delta_1} \tag{2.18}$$

for $(b_1, b_2) \in \mathbb{F}_p \times \mathbb{F}_p$, $(b_1, b_2) \neq (0, 0)$.

Therefore clearly

$$
\begin{aligned}
(2.17) &\le p^{-4} t_1^{4\ell} t_2^{4\ell-1} \\
&\cdot \Big| \big\{ (a_1, a_2, j) \in \mathbb{F}_{p^2} \times \mathbb{F}_{p^2} \times \{0, 1, \dots, t_2 - 1\} \mid Tr(a_1 \varepsilon^j) = Tr(a_2 \varepsilon^{-j}) = 0 \big\} \Big| \\
&\quad + C t_2^{4\ell} t_1^{4\ell(1-\delta_1)} \\
&\le p^{-4} t_1^{4\ell} t_2^{4\ell-1} t_2 p^2 + C t_2^{4\ell} t_1^{4\ell(1-\delta_1)} \\
&\le t^{4\ell} (p^{-2} + C p^{-2\varepsilon\delta_1\ell}). \qquad (2.19)
\end{aligned}
$$

Taking $\ell > \frac{1}{\varepsilon\delta_1}$, we obtain again that

$$
(\dagger) < 2p^{-2} t^{4\ell}. \qquad (2.20)
$$

Thus (2.20) holds provided we take $\ell = \ell(\varepsilon)$ large enough, and gives the bound on the number of solutions of (2.12), (2.12′).

Returning to (2.11), we conclude that

$$
(2.6) < \frac{2}{N}
$$

hence

$$
\|D\| < 2N^{-1/4\ell}. \qquad (2.21)
$$

This proves (2.1).

Remark. As observed in [K-R], the condition of linear independence mod N of n and nA ($n \in \mathbb{Z}^2$ being fixed, $n \neq (0,0)$) is automatically satisfied for N a sufficiently large prime. Indeed, since A does not have rational eigenvectors, $\det(n, nA) \in \mathbb{Z}\backslash\{0\}$ for all $n \in \mathbb{Z}^2\backslash\{0\}$.

If $o(A,p) = t$, necessarily $p \mid \det(A^t - 1)$, where $\det(A^t - 1) \in \mathbb{Z}\backslash\{0\}$. Therefore a prime $p < T$ for which $o(A,p) < T^\varepsilon$ necessarily divides

$$
B = \prod_{1 < t < T^\varepsilon} \det(A^t - 1). \qquad (2.22)
$$

The number of these primes is at most $\log |B| < C.T^{2\varepsilon}$.

In view of Proposition 1, this shows the following

Proposition 2. *For all $\varepsilon > 0$, there is $\delta > 0$ and a sequence $\mathcal{S} = \mathcal{S}_\varepsilon$ of primes such that*

$$
\#\{N \in \mathcal{S} \mid N < T\} < CT^\varepsilon \qquad (2.23)
$$

and for all $n \in \mathbb{Z}^2\backslash\{(0,0)\}$

$$
\max_\psi |\langle T_N(n)\psi, \psi\rangle| < N^{-\delta} \qquad (2.24)
$$

if N is a sufficiently large prime, $N \notin \mathcal{S}$.
(The maximum taken over all normalized eigenfunctions ψ of $U_N(A)$.)

Hence, for $f \in \mathcal{C}^\infty(\mathbb{T}^2)$

$$
\max_\psi \left| \langle Op_N(f)\psi, \psi\rangle - \int_{\mathbb{T}^2} f \right| < N^{-\delta} \qquad (2.25)
$$

for N a sufficiently large prime outside $\mathcal{S}$.

3 The Case of General Modulus

We may now establish the following

Theorem 3. *There is a density 1 sequence $\mathcal{N}$ of integers N and $\delta > 0$ such that for all observables $f \in \mathcal{C}^\infty(\mathbb{T}^2)$, we have*

$$\max_\psi \left| \langle Op_N(f)\psi, \psi\rangle - \int_{\mathbb{T}^2} f \right| \ll C_f N^{-\delta} \text{ for } N \in \mathcal{N} \tag{3.1}$$

where the maximum is taken over all normalized eigenfunctions ψ of U_A.

Remark. Compared with [K-R], see in particular the combination of Corollary 9 and Theorem 17 in [K-R], what we get more is an $N^{-\delta}$ estimate rather than $1/\exp(\log N)^\delta$ for some $\delta > 0$.

The main ingredient is the improvement for N prime obtained in previous section.

Proof of Theorem 3. Fix a small positive number $\tau > 0$ (to be specified). Given a positive integer N, write $N = N_1^2 N_2$ with N_2 square-free. Since

$$\big|\{T < N < 2T \mid N_1 > T^\tau\}\big| < \sum_{T^\tau < N_1 \leq T^{\frac{1}{2}}} \frac{T}{N_1^2} < T^{1-\tau} \tag{3.2}$$

we may restrict ourselves to integers N with square-free part $N_2 > N^{1-2\tau}$.

Next, we require that for any prime divisor p of N, $p > \sqrt{\log N}$, we have

$$o(A,p) > p^{\frac{1}{3}}. \tag{3.3}$$

As pointed out in the previous section, this property is satisfied for all primes $2^k \leq p < 2^{k+1}$ except $2^{\frac{2}{3}k}$ of them. Our requirement (3.3) will therefore exclude from $[T, 2T]$ at most

$$\sum_{2T \gg 2^k > \sqrt{\log T}} 2^{\frac{2}{3}k} \frac{T}{2^k} \ll T(\log T)^{-1/6} \tag{3.4}$$

integers, which again leads to a density zero sequence. Given N as above, write $N = N_1^2 N_0 N'$ where $N_1 < N^\tau, N_0 < [\sqrt{\log N}]! < N^\tau$ and N' is a simple product of primes $p > \sqrt{\log N}$ for which (3.3) holds. Returning to the proof of Proposition 1, we estimate (2.11)

$$t^{-4\ell} N \big|\{(j_1, \dots, j_{4\ell}) \in \{1, \dots, t\}^{4\ell} \mid n(A^{j_1} - \dots - A^{j_{4\ell}}) \equiv 0 (\operatorname{mod} N)\}\big| \tag{3.5}$$

(up to this point no primality of N was involved).

For $M \in \mathbb{Z}_+$, denote $Mat_2(M)$ the 2×2 matrices over $\mathbb{Z}/M\mathbb{Z}$ and G_M its multiplicative subgroup $\{A^j \mid 0 \leq j < o(A, M)\}$.

With previous decomposition of N, the map

$$G_N \to G_{N_1^2} \times G_{N_0} \times \prod_{p|N'} G_p$$

is injective. Defining

$$Q_M = \left|\left\{(\alpha_1, \dots, \alpha_{4\ell}) \in G_M^{4\ell} \mid n(\alpha_1 - \dots - \alpha_{4\ell}) \equiv 0 (\mathrm{mod}\, M)\right\}\right| \tag{3.6}$$

the last factor in (3.5) equals Q_N. Obviously

$$Q_N \le Q_{N_1^2} \cdot Q_{N_0} \cdot \prod_{p|N'} Q_p. \tag{3.7}$$

Take $p|N'$ not dividing $\nu_n = \det(n, nA)$, so that n and nA are independent $\mathrm{mod}\, p$. Since (3.3) holds, the estimate (2.20) on (†) in the proof of Proposition 1 gives

$$Q_p < 2p^{-2}|G_p|^{4\ell} \tag{3.8}$$

where $\ell = \ell(\frac{1}{3})$ is some integer in particular independent of the choice of τ.

From (3.7), (3.8)

$$\begin{aligned} Q_M &< (N_1^2 N_0 \nu_n)^{16\ell} \prod_{\substack{p|N' \\ (p,\nu_n)=1}} \frac{2o(A,p)^{4\ell}}{p^2} \\ &< \frac{(N_1^2 N_0 \nu_n)^{16\ell+2}}{N^2} \left(\exp \frac{\log N}{\log\log N}\right) \left[\prod_{p|N_2} o(A,p)\right]^{4\ell} \\ &< C_A |n|^{40\ell} N^{60\tau\ell-2} \left[\prod_{p|N_2} o(A,p)\right]^{4\ell} \end{aligned} \tag{3.9}$$

(N_2 = square free part of N).

At this point, recall Proposition 11 of [K-R]. It asserts that we may minorate

$$o(A,N) > c_A \frac{\prod_{p|N_2} o(A,p)}{\exp(3(\log\log N)^4)} \tag{3.10}$$

by further exclusion of N outside a density zero sequence

Substituting (3.10) in (3.9) gives,

$$\begin{aligned} Q_N &< C_A |n|^{40\ell} N^{60\tau\ell-2} \exp\big(13\ell(\log\log N)^4\big) o(A,N)^{4\ell} \\ &< C_A |n|^{40\ell} N^{61\tau\ell-2} o(A,N)^{4\ell}. \end{aligned} \tag{3.11}$$

Hence, from the argument in the initial part of the proof of Proposition 1

$$|\langle T_N(n)\psi, \psi\rangle| < C_A |n|^{10} N^{61\tau - \frac{1}{4\ell}}. \tag{3.12}$$

Choosing τ small enough, the claim easily follows.

References

[B] Bourgain, J.: Mordell's estimate revisited. JAMS

[B-C] Bourgain, J., Chang, M.: A Gauss sum estimate in arbitrary finite fields. CRASP

[BGK] Bourgain, J., Glibichuk, A., Konyagin, S.: Estimates for the number of sums and products and for exponential sums in fields of prime order. J. London Math. Soc., to appear

[H-B] Hannay, J.H., Berry, M.V.: Quantization of linear maps on a torus - Fresnel diffraction by a periodic grating. Physica D, **1**, 267–291 (1980)

[D-G-I] Degli Esposti, E., Graffi, S., Isola, S.: Classical limit of the quantized hyperbolic toral automorphisms. Commun. Math. Phys., **167**, 471–507 (1995)

[K-R] Kurlberg, P., Rudnick, Z.: On quantum ergodicity for linear maps of the torus. Comm. Math. Phys., **222**, no 1, 201–227 (2001)

[K-R2] Kurlberg, P., Rudnick, Z.: On the distribution of the matrix elements for the quantum cat maps. Annals of Math., **161**, 489–507 (2003)