10. $\mathbf{A_r(X)}$

Throughout this section we will assume that $\Phi_X : X \to S$ is weakly prepared.

Definition 10.1. *Suppose that $r \ge 2$. $\overline{A}_r(X)$ holds if*

1. $\nu(p) \le r$ *if $p \in X$ is a 1 point or a 2 point.*
2. *If $p \in X$ is a 1 point and $\nu(p) = r$, then $\gamma(p) = r$.*
3. *If $p \in X$ is a 2 point and $\nu(p) = r$, then $\tau(p) > 0$.*
4. $\nu(p) \le r - 1$ *if $p \in X$ is a 3 point*

Definition 10.2. *Suppose that $r \ge 2$. $A_r(X)$ holds if*

1. $\overline{A}_r(X)$ *holds.*
2. $\overline{S}_r(X)$ *is a union of nonsingular curves and isolated points.*
3. $\overline{S}_r(X) \cap (X - \overline{B}_2(X))$ *is smooth.*
4. $\overline{S}_r(X)$ *makes SNCs with $\overline{B}_2(X)$ on the open set $X - B_3(X)$.*
5. *The curves in $\overline{S}_r(X)$ passing through a 3 point $q \in X$ have distinct tangent directions at q. (They are however, allowed to be tangent to a 2 curve).*

Lemma 10.3. *Suppose that X satisfies $\overline{A}_r(X)$ with $r \ge 2$. Then there exists a sequence of quadratic transforms $X_1 \to X$ such that $A_r(X_1)$ holds.*

Proof. Let $\pi : X_1 \to X$ be a sequence of quadratic transforms so that the strict transform of $\overline{S}_r(X)$ makes SNCs with $\overline{B}_2(X)$. Then $\overline{A}_r(X_1)$ holds by Theorems 7.1 and 7.3, and $A_r(X_1)$ holds by Lemma 7.9 and Theorem 7.8. ☐

Definition 10.4. *Suppose that $A_r(X)$ holds. A weakly permissible monoidal transform $\pi : X_1 \to X$ is called permissible if π is the blow-up of a point, a 2 curve or a curve C containing a 1 point such that $C \cup \overline{S}_r(X)$ makes SNCs with $\overline{B}_2(X)$ at all points of C.*

Remark 10.5.
1. *If $A_r(X)$ holds and $\pi : X_1 \to X$ is a permissible monoidal transform, then the strict transform of $\overline{S}_r(X)$ on X_1 makes SNCs with $\overline{B}_2(X_1)$ at 1 and 2 points, and has distinct tangent directions at 3 points.*
2. *If $\pi : X_1 \to X$ is a quadratic transform centered at a point $p \in X$ with $\nu(p) = r$ and $A_r(X)$ holds, then $A_r(X_1)$ holds.*
3. *If $A_r(X)$ holds and all 3 points q of X satisfy $\nu(q) \le r - 2$, then $\overline{S}_r(X)$ makes SNCs with $\overline{B}_2(X)$.*

The Remark follows from Lemmas 7.9 and 7.7, and the observation that the strict transforms of nonsingular curves with distinct tangent directions at a point p intersect the exceptional fiber of the blow-up of p transversally in distinct points.

S.D. Cutkosky: LNM 1786, p. 109, 2002.