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Epilogue

Our aim in this text was not to give a definite treatment of the spectral and regularity properties of Fokker-Planck operators or Witten Laplacians. We tried instead to give an account of how the known techniques from partial differential equations and spectral theory can be applied for their analysis, while completing or referring to existing and sometimes recent results. We hope that this synthetic text will help the researchers in Partial Differential Equations, Probability theory or Mathematical Physics for further developments in this field, which happened to be and is still very active.

During the publishing process of this text some new results have been obtained. The accurate asymptotics of the exponentially small eigenvalues presented in Chapter 16 have been proved in a quite general framework in [HelKlNi] and [HelNi2]. An accurate description of the spectrum and pseudospectrum of a semiclassical Fokker-Planck operator has been given by Hérau-Sjöstrand-Stolk in [HerSjSt]. A work in preparation by F. Hérau deals with the return to the equilibrium for some nonlinear Fokker-Planck equation arising in kinetic theory.

When writing the final version of this text, we heard also about the recent work of Bismut [Bi2, Bi3, Bi4, Bi5] (and even more recently about his collaboration with Lebeau [Leb]). The so called "hypoelliptic Laplacian" that he introduces in order to compute geometrical invariants and which acts in the cotangent bundle (phase-space), looks like what we have called here the Fokker-Planck operator with a partial diffusion only in the momentum variable. The structures exhibited by Bismut bring a new point of view and may suggest new questions in analysis.

Besides the mathematical questions that we addressed in this text, other developments are possible towards more involved models: general drift-diffusion operators, chains of anharmonic oscillators, other kinetic equations. Our point of view was to restrict our attention to the simplest models which already exhibit a very rich structure. For further information on related problems or other issues, we refer the reader to [Ris], [EckHai1], [Re-BeTh3] or [Vi2].