Analytical Design of 2-D Narrow Bandstop FIR Filters

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Abstract. Novel approach in the design of 2-D extremely narrow bandstop FIR filters is presented. The completely analytical design method is based on the 1-D optimal bandstop FIR filters. The 1-D FIR optimal bandstop filters are based on Zolotarev polynomials. Closed form formulas for the design of the filters are presented. One example demonstrates the design procedure. One application of the 2-D FIR filter with extremely narrow stop bands is presented.

1 Introduction

Two-dimensional narrow bandstop FIR filters play important role in the image and video enhancement/restoration tasks. They are frequently used in order to remove a single frequency component from the spectrum of the signal. The image filtering can be accomplished by both the nonlinear [1], [4], [7], [8], [9], [10] and linear [5], [12], [13], [14], [15] filters. In our paper we are concerned with completely analytical design of 2-D bandstop FIR filters with extremely narrow circularly symmetrical stop bands. The design of the 2-D narrow bandstop FIR filters is based on the 1-D optimal narrow bandstop FIR filters [14]. We introduce the degree formula which relates the degree of the generating polynomial, the length of the filter, the notch frequency, the width of the stopbands and the attenuation in the passbands. Based on the expansion of the generating polynomials into the Chebyshev polynomials, the recursive formula for the direct computation of the impulse response coefficients is presented. The design procedure is recursive one and it does not require any FFT algorithm or any iterative technique.

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2 Polynomial Equiripple Approximation

Let us denote $H(z)$ the transfer function of a 1-D FIR filter with the impulse response $h(m)$ of the length N as

$$H(z) = \sum_{m=0}^{N-1} h(m) z^{-m} \,. \tag{1}$$

Assuming an odd length $N = 2n+1$ and even symmetry of the impulse response $h(m)$ we can write the transfer function of the bandstop FIR filter

$$H(z) = z^{-n} \left[h(0) + \sum_{m=1}^{n} 2h(m)\, T_m(w) \right] = z^{-n} \left[h(0) + \sum_{m=1}^{n} 2h(m)\, T_m(\cos \omega T) \right] \tag{2}$$

where $T_m(w)$ is Chebyshev polynomial of the first kind and $w = (z+z^{-1})/2$. The

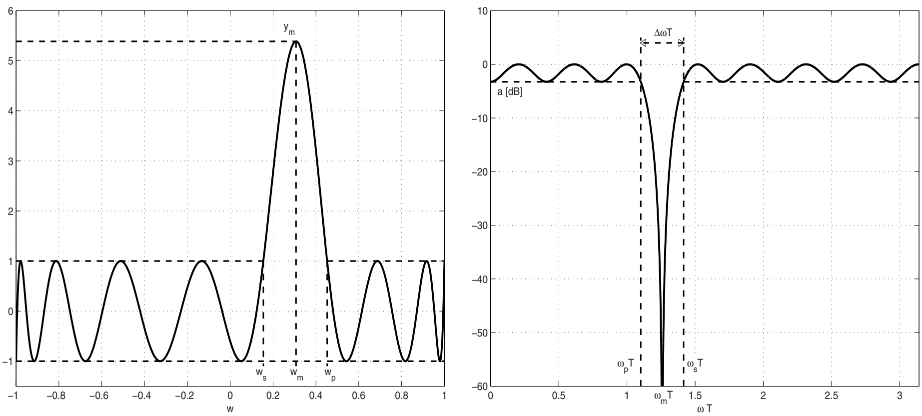

Fig. 1. Zolotarev polynomial $Z_{6,9}(w)$ with $\kappa = 0.6966$, $w_s = 0.1543$, $w_m = 0.3071$, $w_p = 0.4523$, $y_m = 5.3864$ and corresponding amplitude frequency response $20 \log |H(e^{j\omega T})|$ [dB] with parameters $\omega_p T = 0.3506\,\pi$, $\omega_m T = 0.4006\,\pi$, $\omega_s T = 0.4507\,\pi$, $\Delta\omega T = 0.1001\,\pi$ and $a = -3.2634$ dB

1-D equiripple narrow bandstop FIR filter is based on the Zolotarev polynomial $Z_{p,q}(w)$ which approximates constant value in equiripple Chebyshev sense in the two disjoint intervals as shown in Fig. 1. The notation $Z_{p,q}(w)$ emphasizes that p counts the number of zeros right from the maximum w_m and q corresponds to the number of zeros left from the maximum w_m. Zolotarev derived the general solution of this approximation problem in terms of Jacobi's elliptic functions

$$Z_{p,q}(w) = \frac{(-1)^p}{2} \left[\left(\frac{H(u - \frac{p}{n}\, \mathbf{K}(\kappa))}{H(u + \frac{p}{n}\, \mathbf{K}(\kappa))} \right)^n + \left(\frac{H(u + \frac{p}{n}\, \mathbf{K}(\kappa))}{H(u - \frac{p}{n}\, \mathbf{K}(\kappa))} \right)^n \right] \tag{3}$$

where u is expressed by the incomplete elliptical integral of the first kind

$$u = F\left(\mathrm{sn}\left(\frac{p}{n}\ \mathbf{K}(\kappa)|\kappa\right)\sqrt{\frac{1+w}{w+2\,\mathrm{sn}^2\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)-1}}|\kappa\right) . \qquad (4)$$

The function $H\left(u \pm \frac{p}{n}\ \mathbf{K}(\kappa)\right)$ is the Jacobi's Eta function, $\mathrm{sn}(u|\kappa)$, $\mathrm{cn}(u|\kappa)$, $\mathrm{dn}(u|\kappa)$ are Jacobi's elliptic functions, $\mathbf{K}(\kappa)$ is the quarter-period given by the complete elliptic integral of the first kind, $F(\phi|\kappa)$ is the incomplete elliptic integral of the first kind and κ is the Jacobi's elliptic modulus. The degree of the Zolotarev polynomial is $n = p + q$. A comprehensive treatise of the Zolotarev polynomials was published in [14]. It includes the analytical solution of the coefficients of Zolotarev polynomials, the algebraic evaluation of the Jacobi's Zeta function $\mathrm{Z}(\frac{p}{n}\mathbf{K}(\kappa)|\,\kappa)$ and of the elliptic integral of the third kind $\Pi(\sigma_m, \frac{p}{n}\mathbf{K}(\kappa)|\,\kappa)$ of the discrete argument. The position of the maximum value $y_m = Z_{p,q}(w_m)$ is

$$w_m = 1 - 2\,\mathrm{sn}^2\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right) + 2\,\frac{\mathrm{sn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)\mathrm{cn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)}{\mathrm{dn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)}\mathrm{Z}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right) . \qquad (5)$$

The maximum value y_m useful for the normalization of the Zolotarev polynomial is given as

$$y_m = \cosh 2n\left(\sigma_m \mathrm{Z}(\frac{p}{n}\mathbf{K}(\kappa)|\kappa) - \Pi(\sigma_m, \frac{p}{n}\mathbf{K}(\kappa)|\kappa)\right) . \qquad (6)$$

The degree of the Zolotarev polynomial $Z_{p,q}(w)$ expresses the *degree equation*

$$n \geq \frac{\ln(y_m + \sqrt{y_m^2 - 1})}{2\sigma_m \mathrm{Z}(\frac{p}{n}\mathbf{K}(\kappa)|\kappa) - 2\Pi(\sigma_m, \frac{p}{n}\mathbf{K}(\kappa)|\kappa)} . \qquad (7)$$

The auxiliary parameter σ_m is given by the formula

$$\sigma_m = F\left(\arcsin\left(\frac{1}{\kappa\ \mathrm{sn}\left(\frac{p}{n}\ \mathbf{K}(\kappa)|\kappa\right)}\sqrt{\frac{w_m - w_s}{w_m + 1}}\right)|\kappa\right) \qquad (8)$$

where $F(\Phi|\kappa)$ is the incomplete elliptical integral of the first kind. The recursive algorithm for the evaluation of the coefficients $a(m)$ of the Zolotarev polynomial based on the expansion into Chebyshev polynomials of the first kind

$$Z_{p,q}(w) = \sum_{m=0}^{n} a(m) T_m(w) \qquad (9)$$

was derived and presented in [14]. It is summarized in Table 1. The impulse response coefficients $h(m)$ of the 1-D equiripple bandstop FIR filter are obtained by the normalization of the coefficients $a(m)$ as follows

$$h(n) = \frac{y_m - a(0)}{y_m + 1}\ ,\ h(n \pm m) = -\frac{a(m)}{2(y_m + 1)}\ ,\ m = 1 \ldots n\ . \qquad (10)$$

3 Analytical Design of 2-D Narrow Bandstop FIR Filter

The goal of the design of the 2-D narrow bandstop FIR filter is to obtain the 2-D impulse response $h(m,n)$ of the filter satisfying the specified notch frequency $\omega_{m1}T$, width of the bandstop $\Delta\omega_1 T$, the attenuation in the passbands a_1 [dB] in the direction ω_1 and the specified values $\omega_{m2}T$, $\Delta\omega_2 T$, a_2 [dB] in the direction ω_2. The design procedure is as follows :

1. For the specified values $\omega_{m1}T$, $\Delta\omega_1 T$ and a_1 [dB] (Fig. 1) in the direction ω_1 design the 1-D FIR narrow bandpass filter. The design procedure consists of the following steps :
   a) Evaluate the Jacobi's elliptic modulus κ
   $$\kappa = \sqrt{1 - \frac{1}{\tan^2(\varphi_s)\tan^2(\varphi_p)}} \tag{11}$$
   for the auxiliary parameters φ_s and φ_p
   $$\varphi_s = \frac{\omega_{m1} + \Delta\omega_1/2}{2}T \ , \ \ \varphi_p = \frac{\pi - (\omega_{m1} - \Delta\omega_1/2)}{2}T \ . \tag{12}$$
   b) Calculate the rational values $\dfrac{p}{n} = \dfrac{F(\varphi_s|\kappa)}{\mathbf{K}(\kappa)}$ and $\dfrac{q}{n} = \dfrac{F(\varphi_p|\kappa)}{\mathbf{K}(\kappa)}$.
   c) Determine the required maximum value y_m
   $$y_m = \frac{2}{1 - 10^{0.05\, a_1[dB]}} - 1\,. \tag{13}$$
   d) Using the degree equation (7) calculate and round up the minimum degree n required to satisfy the filter specification. For the algebraic evaluation of the Jacobi's Zeta function $\mathrm{Z}(\frac{p}{n}\mathbf{K}(\kappa)|\,\kappa)$ and the elliptic integral of the third kind $\Pi(\sigma_m, \frac{p}{n}\mathbf{K}(\kappa)|\,\kappa)$ in the degree equation (7) use the algebraical procedure [14].
   e) Calculate the integer values $p = \left[n\dfrac{F(\varphi_s|\kappa)}{\mathbf{K}(\kappa)}\right]$ and $q = \left[n\dfrac{F(\varphi_p|\kappa)}{\mathbf{K}(\kappa)}\right]$.
   The brackets [ ] stand for the rounding operation.
   f) For the integer values p, q and the elliptic modulus κ evaluate the coefficients $a(m)$ (9) of the Zolotarev polynomial $Z_{p,q}(w)$ using recursive algorithm summarized in Tab. 1.
   g) From the coefficients $a(m)$ calculate the M coefficients of the impulse response $h_1(m)$ of the 1-D equiripple bandpass FIR filter using (10).
2. Repeat the first step for the design of the 1-D FIR equiripple narrow bandpass filter in the direction ω_2 specified by $\omega_{m2}T$, $\Delta\omega_2 T$ and a_2 [dB] resulting in the impulse response $h_2(n)$ of the length N coefficients.
3. From the 1-D impulse responses
   $$h_1(m)\,, \ \ m = 0, \ \ldots\ , M - 1 \quad , \qquad h_2(n)\,, \ \ n = 0, \ \ldots\ , N - 1 \tag{14}$$

compose the 2-D impulse responses $h_1(m,n)$ and $h_2(m,n)$ by the zero padding. The non-zero coefficients are

$$\begin{aligned} h_1(\frac{M-1}{2},n) &= h_1(m), \quad m=0,\dots,M-1 \\ h_2(m,\frac{N-1}{2}) &= h_2(n), \quad n=0,\dots,N-1 \ . \end{aligned} \tag{15}$$

4. The 2-D impulse response $h_{BP}(m,n)$ of the dimension $M \times N$ of the narrow bandpass FIR filter is given by the 2-D linear discrete convolution

$$h_{BP}(m,n) = h_1(m,n) ** h_2(m,n) \ . \tag{16}$$

5. The impulse response $h(m,n)$ of the final 2-D bandstop FIR filter is

$$\begin{aligned} h(m,n) = -h_{BP}(m,n) \ \text{for} \quad m &\neq \frac{M-1}{2} \\ n &\neq \frac{N-1}{2} \\ h(\frac{M-1}{2},\frac{N-1}{2}) = 1 - h_{BP}(\frac{M-1}{2},\frac{N-1}{2}) \ . \end{aligned} \tag{17}$$

4 Example

Design the 2-D bandstop FIR filter specified in the direction ω_1 by the notch frequency $\omega_{m1}T = 0.4\,\pi$, width of the passbands $\Delta\omega_1 T = 0.1\,\pi$ for the attenuation in the passbands $a_1 = -1$ dB and in the direction ω_2 by the values $\omega_{m2}T = 0.6\,\pi$, $\Delta\omega_2 T = 0.1\,\pi$ for $a_2 = -1$ dB.
Using our recursive design procedure we obtain the two 1-D equiripple narrow band FIR filters with the impulse responses $h_1(m)$, $h_2(n)$ (step 1 and 2 in Sec. 3). The impulse responses $h_1(m)$, $h_2(n)$ of the length $M = N = 41$ coefficients are summarized in Table 2. Their amplitude frequency responses are shown in Fig. 2. The impulse responses $h_1(m)$, $h_2(n)$ are used for the design of the 2-D bandstop FIR filter (step 3, 4 and 5 in Sec. 3). The impulse response $h(m,n)$ of the 2-D narrow bandstop FIR filter consists of 41×41 coefficients. The amplitude frequency response $20\log|H(e^{j\omega_1}, e^{j\omega_2})|$ of the 2-D narrow bandstop FIR filter with its contours is shown in Fig. 3.

5 Application of the 2-D Narrow Bandstop FIR Filter

The narrow 2-D bandstop FIR filters were successfully applied for the removal of the unwanted frequency components in the spectrum of the image. Here we present the enhancement of the rastered newspaper picture. The notch frequencies $\omega_1 = 0.32\pi$, $\omega_2 = 0.42\pi$ to be removed were obtained by the evaluation of the spectrum of the input image. The impulse response $h(m,n)$ of the applied filter exhibits 37×37 coefficients. The input and processed image are shown in Fig. 4. The attenuation of the disturbing raster is apparent.

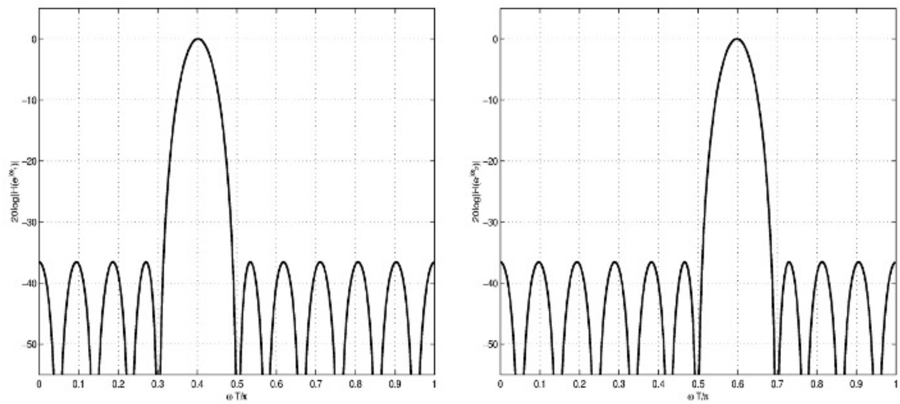

Fig. 2. Amplitude frequency responses $20\log|H(e^{j\omega_1})|$ and $20\log|H(e^{j\omega_2})|$

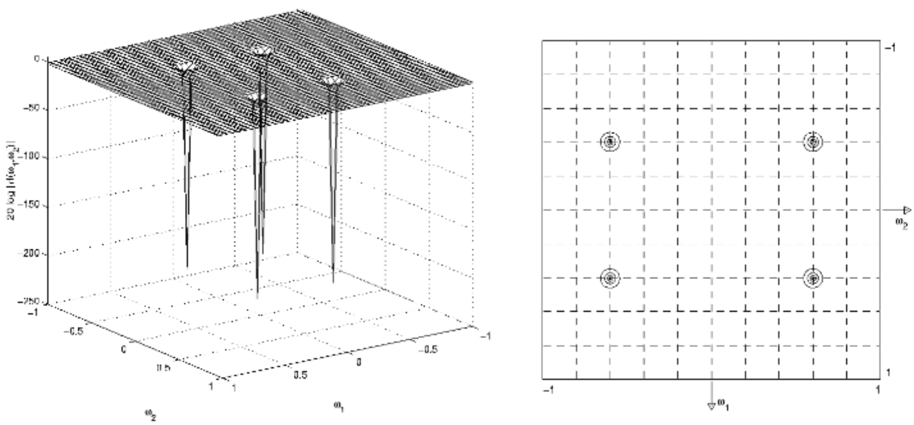

Fig. 3. Amplitude frequency response $|H(e^{j\omega_1}, e^{j\omega_2})|$ with contours

Fig. 4. Input and filtered image

Table 1. Recursive algorithm for the evaluation of the coefficients $a(m)$ of the Zolotarev polynomials $Z_{p,q}(w)$

given

p, q, κ

initialisation

$$n = p + q$$

$$w_p = 2\,\mathrm{sn}^2\left(\frac{q}{n}\mathbf{K}(\kappa)|\kappa\right) - 1$$

$$w_s = 1 - 2\,\mathrm{sn}^2\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)$$

$$w_a = \frac{w_p + w_s}{2}$$

$$w_m = w_s + 2\,\frac{\mathrm{sn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)\mathrm{cn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)}{\mathrm{dn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)}\mathrm{Z}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)$$

$$\alpha(n) = 1$$

$$\alpha(n+1) = \alpha(n+2) = \alpha(n+3) = \alpha(n+4) = \alpha(n+5) = 0$$

body

(*for* $m = n+2$ *to* *3*)

$$8c(1) = n^2 - (m+3)^2$$

$$4c(2) = (2m+5)(m+2)(w_m - w_a) + 3w_m[n^2 - (m+2)^2]$$

$$2c(3) = \frac{3}{4}[n^2 - (m+1)^2] + 3w_m[n^2 w_m - (m+1)^2 w_a] - (m+1)(m+2)(w_p w_s - w_m w_a)$$

$$c(4) = \frac{3}{2}(n^2 - m^2) + m^2(w_m - w_a) + w_m(n^2 w_m^2 - m^2 w_p w_s)$$

$$2c(5) = \frac{3}{4}[n^2 - (m-1)^2] + 3w_m[n^2 w_m - (m-1)^2 w_a] - (m-1)(m-2)(w_p w_s - w_m w_a)$$

$$4c(6) = (2m-5)(m-2)(w_m - w_a) + 3w_m[n^2 - (m-2)^2]$$

$$8c(7) = n^2 - (m-3)^2$$

$$\alpha(m-3) = \frac{1}{c(7)}\sum_{\mu=1}^{6} c(\mu)\alpha(m+4-\mu)$$

(*end loop* *on* m)

normalisation

$$s(n) = \frac{\alpha(0)}{2} + \sum_{m=1}^{n}\alpha(m)$$

$$a(0) = (-1)^p\,\frac{\alpha(0)}{2s(n)}$$

(*for* $m = 1$ *to* n)

$$a(m) = (-1)^p\,\frac{\alpha(m)}{s(n)}$$

(*end loop* *on* m)

Table 2. Coefficients of the Impulse Responses

m,n		$h_1(m)$	$h_2(n)$	m,n		$h_1(m)$	$h_2(n)$
0	40	0.008036	0.008036	11	29	0.020208	-0.020208
1	39	0.003713	-0.003713	12	28	-0.047824	-0.047824
2	38	-0.008856	-0.008856	13	27	-0.055411	0.055411
3	37	-0.013403	0.013403	14	26	0.019492	0.019492
4	36	0.004187	0.004187	15	25	0.075345	-0.075345
5	35	0.023801	-0.023801	16	24	0.026236	0.026236
6	34	0.011174	0.011174	17	23	-0.065318	0.065318
7	33	-0.025726	0.025726	18	22	-0.068525	-0.068525
8	32	-0.033363	-0.033363	19	21	0.025845	-0.025845
9	31	0.010947	-0.010947	20		0.093129	0.093129
10	30	0.050326	0.050326				

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